Communications in Combinatorics and Optimization
Vol. 5 No. 1, 2020 pp.41-54
DOI: 10.22049/CCO.2019.26347.1099

CCO
Commun. Comb. Optim.

On the edge geodetic and edge geodetic domination numbers of a graph

Vladimir Samodivkin

Department of Mathematics, University of Architecture, Civil Engineering and Geodesy
Sofia, Bulgaria
vl.samodivkin@gmail.com

Received: 27 September 2018; Accepted: 19 July 2019
Published Online: 21 July 2019

Abstract: In this paper, we study both concepts of geodetic dominating and edge geodetic dominating sets and derive some tight upper bounds on the edge geodetic and the edge geodetic domination numbers. We also obtain attainable upper bounds on the maximum number of elements in a partition of a vertex set of a connected graph into geodetic sets, edge geodetic sets, geodetic dominating sets and edge geodetic dominating sets, respectively.

Keywords: domination number, (edge) geodetic number, (edge) geodetic domination number

AMS Subject classification: 05C69, 05C12

1. Introduction and preliminaries

By a graph, we mean a finite, undirected graph with neither loops nor multiple edges. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. In a graph G, for a subset $S \subseteq V(G)$ the *subgraph induced* by S is the graph $\langle S \rangle$ with vertex set S and edge set $\{xy \in E(G) \mid x, y \in S\}$. We write K_n for the *complete graph* of order n, $K_{m,n}$ for the *complete bipartite graph* with partite sets of order m and n, P_n for the *path* on n vertrices, and C_m for the *cycle* of length m. For any vertex x of a graph G, $N_G(x)$ denotes the set of all neighbors of x in G, $N_G[x] = N_G(x) \cup \{x\}$ and the degree of x is $deg(x, G) = |N_G(x)|$. The *minimum* and *maximum* degrees of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a subset $A \subseteq V(G)$, let $N_G[A] = \cup_{x \in A} N_G[x]$. If $e = uv$ is an edge of a graph G with $deg(u) = 1$ and $deg(v) > 1$, then we call e a *pendant edge*, u a *leaf* and v a *support vertex*. Let $L(G)$ and S(G) be the sets of all leaves and all support vertices of a graph G, respectively. The *corona* $cor(G)$ of a graph G is constructed from G, where for each vertex $v \in V(G)$, a new vertex v' and a pendant edge vv' are added.

For two graphs G_1 and G_2, the *Cartesian product* '$G_1 \square G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$ and $(x_1, x_2)(y_1, y_2) \in E(G_1 \square G_2)$ if and only if $x_1 y_1 \in E(G_1)$ and $x_2 = y_2$, or $x_2 y_2 \in E(G_2)$ and $x_1 = y_1$.
Let $\mathcal{P}$ be a property of vertex subsets of a graph G and let $\mu(G)$ denote the minimum cardinality of sets with property $\mathcal{P}$. Any set with property $\mathcal{P}$ of cardinality $\mu(G)$ is called a *μ-set* of G.
A vertex in a graph G dominates itself and its neighbors. A set of vertices D in a graph G is a *dominating set* if each vertex of G is dominated by some vertex of D. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. Cockayne and Hedetniemi [3] defined the *domatic number* $d(G)$ of a graph G to be the maximum number of elements in a partition of $V(G)$ into dominating sets. A dominating set D of G is called an *efficient dominating set* (an ED-*set*) of G if the distance between any two vertices in D is at least three. Not all graphs have ED-sets. If G has an ED-set, then any ED-set is a γ-set of G [1]. For a comprehensive introduction to the theory of domination in graphs we refer the reader to Haynes et al. [10].
The *distance* $d_G(u, v)$ between two vertices u and v in G is the length of a shortest $u - v$ path or ∞, if no such path exists. The *diameter* of G, denoted by $diam(G)$, is the maximum distance between two vertices in G. A $u - v$ path of length $d_G(u, v)$ is called a $u - v$ *geodesic*. Any geodesic of length $diam(G)$ is called a *diametral path*. Two vertices u and v of G are called *antipodal* if $d(u, v) = diam(G)$. A vertex w is said to lie on a $u - v$ geodesic P if w is an internal vertex of P. The *closed interval* $I_G[u, v]$ consists of u, v and all vertices lying on some $u - v$ geodesic of G. For a set S of vertices, let the interval $I_G[S]$ of S be the union of the intervals $I_G[u, v]$ over all pairs of vertices u and v in S. Note that a vertex w belongs to $I_G[u, v]$ if and only if $d_G(u, v) = d_G(u, w) + d_G(w, v)$. A set $S \subseteq V(G)$ is a *geodetic set* of G if $I_G[S] = V(G)$. Harary et al. [9] define the *geodetic number* $g(G)$ of a graph G as the minimum cardinality of a geodetic set. It is NP-complete to decide for a given chordal or chordal bipartite graph G and a given integer k whether G has a geodetic set of cardinality at most k ([4]).
Let G be a connected graph and $u, v \in V(G)$. The set $I^e_G[u, v]$ consists of all edges of G lying in any $u - v$ geodesic in G. If $S \subseteq V(G)$, then the set $I^e_G[S]$ denotes the union of all $I^e_G[u, v]$, where $u, v \in S$. A subset S of $V(G)$ is an *edge geodetic set* of G if $I^e_G[S] = E(G)$. Clearly, any edge geodetic set of a graph is geodetic. The *edge geodetic number* of G, denoted by $g_e(G)$, is the minimum cardinality of an edge geodetic set of G (Santhakumaran and John [16]).
A subset S of $V(G)$ is a *geodetic dominating set* (a GD-*set* for short) in G if S is both a geodetic set and a dominating set. The minimum cardinality of a geodetic dominating set of G is its the *geodetic domination number*, and is denoted by $\gamma g(G)$. The study of the geodetic domination was initiated by Escuardo et al. [5] in 2011. Some other interesting results can also be found in Hansberg and Volkmann [8].
The notion of *edge geodetic domination* was introduced by Arul Paul Sudhahar et al. [19]. A set of vertices S of a graph G is an *edge geodetic dominating set* (an EGD-*set*) if it is both an edge geodetic set and a dominating set of G. The minimum

cardinality among all the EGD sets of G is called the *edge geodetic domination number* (EGD-number) and is denoted by $\gamma g_e(G)$.
The paper is organized as follows. The next section consists of some known results which will be useful in proving our main results. In Section 3, we give some general results and sharp bounds for the geodetic domination number and the edge geodetic domination number. To present our results obtained in Section 4 we need the following definitions. For every connected n-order graph G we define the *geodetic partitionable number* (*edge geodetic partitionable number, geodetic domatic number, edge geodetic domatic number*, respectively), denoted $gp(G)$ ($g_ep(G)$, $gd(G)$, $g_ed(G)$, respectively), to be the maximum number of elements in a partition of $V(G)$ into geodetic sets (edge geodetic sets, geodetic dominating sets, edge geodetic dominating sets, respectively). In Section 4 we give upper bounds on these four parameters and present some families of graphs, that achieve these bounds.

2. Known results

A vertex v is an *extreme vertex* of a graph G if the subgraph induced by its neighbours is complete. The set of all extreme vertices of G is denoted by $Ext(G)$. A vertex v in a connected graph G is said to be a *semi-extreme vertex* if it has a neighbor, say u, with $N[v] \subseteq N[u]$. The set of all semi-extreme vertices of G is denoted by $Se(G)$.

Theorem A. Let G be a connected n-order graph, $n \geq 2$.

(i) [9] $2 \leq g(G) \leq n$ and $g(G) = n$ if and only if $G = K_n$.

(ii) [16] $2 \leq g(G) \leq g_e(G) \leq n$.

(iii) [9] Each geodetic set of G contains all extreme vertices of G.

(iv) [18] Each semi-extreme vertex of G is contained in every edge geodetic set of G. The equality $g_e(G) = n$ holds if and only if all vertices of G are semi-extreme.

(v) [5] $2 \leq \max\{g(G), \gamma(G)\} \leq \gamma g(G) \leq n$.

(vi) [19] $2 \leq g_e(G) \leq \gamma g_e(G) \leq n$ and $\gamma g(G) \leq \gamma g_e(G)$.

(vii) [17] If $S = \{u, v\}$ is an edge geodetic set of a connected graph G, then u and v are antipodal vertices of G.

A *vertex cover* of a graph is a set of vertices such that each edge of the graph is incident to at least one vertex of the set. The *vertex cover number* of G, denoted by $\alpha(G)$, is the minimum cardinality among all vertex covers of G.

Theorem B. [14] Let G be a connected graph with $\delta(G) \geq 2$. Then $\gamma(G) = \alpha(G)$ if and only if G is bipartite such that for every pair x and y of distance 2 of the smaller partite set, there exist at least two common neighbors of x and y of degree 2.

A subset X of $V(G)$ is called *independent* if its vertices are mutually non-adjacent. The *independence number* $\beta_0(G)$ is the largest cardinality among all independent sets of a graph G.

Theorem C. [7] Let G be a graph without isolated vertices. A subset I of $V(G)$ is independent if and only if $V(G) - I$ is a vertex cover of G. In particular, $\beta_0(G) = |V(G)| - \alpha(G)$.

Theorem D. Let G be a connected n-order graph, $n \geq 2$.

(i) [12] $\gamma(G) \leq n/2$;

(ii) [6, 13] $\gamma(G) = n/2$ if and only if either $G = C_4$ or G is the corona $cor(H)$ for any connected graph H.

Theorem E. [3] If G is a connected n-order graph, then $d(G) \leq \delta(G)+1$ and $d(G)\gamma(G) \leq n$.

3. Edge geodetic domination

We begin with a result on graphs G with $g_e(G) = 2$.

Proposition 1. *If G is a graph with $g_e(G) = 2$, then G is bipartite.*

Proof. Let $S = \{u, v\}$ be a g_e-set of G, P a $u-v$ geodesic of length k, and $N^i(u)$ the set of all vertices at distance i from u, $i = 0, 1, .., k$. Clearly $N^0(u), N^1(u), .., N^k(u)$ form a partition of $V(G)$. Since S is an edge geodetic set of G, each $N^i(u)$ is independent. But then the union of all $N^i(u)$ with i even and the union of all $N^i(u)$ with i odd are both independent and form a partition of $V(G)$. Thus, G is bipartite. □

Corollary 1. *If G is a connected graph having a cycle of odd length, then $g_e(G) \geq 3$.*

A characterization of all connected graphs whose edge geodetic domination number is 2 follows.

Theorem 1. *Let G be a connected n-order graph, $n \geq 2$. Then $\gamma g_e(G) = 2$ if and only if either $G = K_2$ or $G = K_{2,n-2}, n \geq 3$, or $G = (V_1, V_2; E)$ is a bipartite graph having vertices $v_1 \in V_1$ and $v_2 \in V_2$ such that $N(v_1) = V_2 - \{v_2\}$, $N(v_2) = V_1 - \{v_1\}$, and no vertex in $V(G) - \{v_1, v_2\}$ is a leaf.*

Proof. *Necessity*: Let $\gamma g_e(G) = 2$ and $S = \{v_1, v_2\}$ a γg_e-set of G. Then $G = (V_1, V_2; E)$ is bipartite (by Proposition 1) and $d(v_1, v_2) = diam(G)$ because of Theorem A(vii). Hence if $diam(G) \leq 2$, then clearly either $G = K_2$ or $G = K_{2,n-2}, n \geq 3$. Since $\{v_1, v_2\}$ is a dominating set of G, $diam(G) \leq 3$. So, let $d(v_1, v_2) = 3$. As S is

an edge geodetic set of G, clearly no vertex in $V(G)-S$ is a leaf, v_1 is adjacent to all vertices in $V_i-\{v_2\}$ and v_2 is adjacent to all vertices in $V_j-\{v_1\}$, where $\{i,j\}=\{1,2\}$.
Sufficiency: In all three cases $\{v_1,v_2\}$ is a γg_e-set of G. So, $\gamma g_e=2$. □

Let $\mu \in \{g, g_e, \gamma g, \gamma g_e\}$. We say that a graph G is a μ-*excellent* if each its vertex belongs to some μ-set. An n-*crown graph* $H_{n,n}$ is a graph obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching, $n \geq 3$. By the above theorem we immediately obtain the following characterization of γg_e-excellent graphs G with $\gamma g_e(G)=2$.

Corollary 2. *All γg_e-excellent graphs with the edge geodetic domination number equals 2 are $K_2, K_{2,2}=C_4$ and all crown graphs.*

Proposition 2. *Let G be an n-order connected graph, $n \geq 2$.*

(i) Let $g_e(G)<\gamma g_e(G)$. Then for each g_e-set S of G the set $V(G)-S$ contains the closed neighborhood of some vertex of G. In particular, $g_e(G) \leq \min\{n-\delta(G)-1, n-3\}$.

(ii) $\gamma g_e(G)=n$ if and only if $g_e(G)=n$.

(iii) If G has $k \geq 1$ vertices of degree $n-1$, then $\gamma g_e(G)=n$ when $k \geq 2$, and $\gamma g_e(G)=n-1$ when $k=1$.

Proof. (i) If S is a g_e-set of G, then since $g_e(G)<\gamma g_e(G)$, S is not a dominating set of G. Hence there is a vertex x_S with $N[x_S] \subseteq V(G)-S$. Clearly x_S is not a leaf and then $\gamma g_e(G)=|S| \leq n-deg(x_S)-1 \leq \min\{n-\delta(G)-1, n-3\}$.
(ii) If $g_e(G)=n$, then $\gamma g_e(G)=n$ because Theorem A(vi). So let $\gamma g_e(G)=n$. Then by (i), $g_e(G) \geq \gamma g_e(G)$ and by Theorem A(vi), the equality $\gamma g_e(G)=g_e(G)=n$ follows.
(iii) Let x be a vertex of degree $n-1$ in G. Hence each vertex of $V(G)-\{x\}$ is semi-extreme. By Theorem A(iv), (a) exactly one of $V(G)-\{x\}$ and $V(G)$ is a γg_e-set of G, and (b) $\gamma g_e(G)=n$ when $k \geq 2$. So, let x be the only vertex of G of degree $n-1$ and $xy \in E(G)$. Since $deg(y)<n-1$, there is $z \in V(G)-\{x,y\}$ non-adjacent to y. But then z,x,y is a $z-y$ geodesic that contains xy. Thus $\gamma g_e(G)=n-1$. □

Remark 1. The bound in Proposition 2 is tight. Indeed, (a) for the $(r+s+1)$-order graph G obtained from the stars $K_{1,r}$ and $K_{1,s}$ having a leaf in common, where $r,s \geq 2$, is fulfilled $\delta(G)=1$ and $g_e(G)=(r+s+1)-3$, and (b) for a graph H, having a cut-vertex x such that $deg(x)=r \geq 3$ and $H-x$ has r components each of which is a complete graph of order more than r, is fulfilled $\delta(H)=r$ and $g_e(H)=|V(H)|-r-1<|V(H)|-3$.

Theorem 2. *Let G be a connected n-order graph, $n \geq 2$. Then $2-n/2 \leq g_e(G)-\gamma(G) \leq \gamma g_e(G)-\gamma(G) \leq n-1$. Let $\mu \in \{g_e, \gamma g_e\}$. Then (a) $2-n/2=\mu(G)-\gamma(G)$ if and only if $G \in \{P_2, P_4, C_4\}$, and (b) $\mu(G)-\gamma(G)=n-1$ if and only if G has at least 2 vertices of degree $n-1$.*

Proof. By Theorems A and D we know that $2 \le \mu(G) \le n$ and $1 \le \gamma(G) \le n/2$. Hence $2-n/2 \le \mu(G)-\gamma(G) \le n-1$. First note that $\mu(G)-\gamma(G) = n-1$ if and only if both $\mu(G) = n$ and $\gamma(G) = 1$ hold, which by Proposition 2 is equivalent to G has at least 2 vertices of degree $n-1$. Second clearly $\mu(G) - \gamma(G) = 2 - n/2$ if and only if both $\mu(G) = 2$ and $\gamma(G) = n/2$ are valid, which is equivalent to $G \in \{P_2, P_4, C_4\}$, because of Theorem D(ii). □

Let $\mu, \nu \in \{\gamma, g, g_e, \gamma g, \gamma g_e\}$. In a graph G, $\mu(G)$ is *strongly equal* to $\nu(G)$, written $\mu(G) \equiv \nu(G)$, if each μ-set of G is a ν-set of G and vice versa.

Observation 3. If T is a nontrivial tree, then (a) the set of all leaves in T is the unique g-set of T and the unique g_e-set of T, (b) $g(T) \equiv g_e(T)$, (c) each γg-set of T is a γg_e-set of T and vice versa, and (d) $\gamma g(T) \equiv \gamma g_e(T)$.

Example 1. Let D be a γ-set of C_n. Clearly $\gamma g_e(C_4) = 2 = \gamma(C_4)$ and if $n = 3, 5$ then $\gamma g_e(C_n) = 3 > \gamma(G)$. Let $n \ge 6$. Then the distance between any 2 elements u and v of D is at most 3 whenever there is no other element of D belonging to the shortest $(u-v)$-paths in C_n. Hence each γ-set of C_n is a γg_e-set when $n \ge 6$. Thus, $\gamma g_e(G) \equiv \gamma(C_n) = \lceil n/3 \rceil$ for $n \ge 6$.

To continue, we need the following obvious observation.

Observation 4. Let G be a connected graph of order $n \ge 2$. Then $\gamma g_e(G) = \min\{|S \cup D| :$ S is an EG-set and D is a dominating set of $G\} \le \gamma(G) + g_e(G)$.

A realization result concerning the numbers $\gamma(G), g(G), g_e(G), \gamma g(G)$ and $\gamma g_e(G)$ follows.

Theorem 5. *Let a, b and c be positive integers. Then there is a connected graph G such that $\gamma(G) = a$, $g(G) \equiv g_e(G) = b$ and $\gamma g(G) \equiv \gamma g_e(G) = c$ if and only if either $a, b \ge 2$ and $\max\{a, b\} \le c \le a + b$, or $a = 1$ and $2 \le b = c$.*

Proof. $\Rightarrow$ Denote by $\mathcal{G}_{a,b,c}$ the class of all graphs G with $\gamma(G) = a$, $g(G) \equiv g_e(G) = b$ and $\gamma g(G) \equiv \gamma g_e(G) = c$. Clearly, $a \ge 1$. By Theorem A, $\max\{g(G), \gamma(G)\} \le \gamma g(G)$ and $\max\{\gamma g(G), g_e(G)\} \le \gamma g_e(G)$. Hence for any $G \in \mathcal{G}_{a,b,c}$ are fulfilled: $a \ge 1$, $b \ge 2$ and $\max\{a, b\} \le c$. On the other hand, Observation 4 implies $c \le a + b$. If $a \ge 2$, we are done. If $a = 1$, then $b = c = n-1$ or $b = c = n$ depending of the number of vertices of degree $n-1$ by Proposition 2.
$\Leftarrow$ *Case* 1: $a = 1$. Clearly, for the star $K_{1,b}$ with $b \ge 2$ we have $\gamma(K_{1,b}) = 1$. On the other hand, the set of all leaves is the unique μ-set of $K_{1,b}$, for each $\mu \in \{g, g_e, \gamma g, \gamma g_e\}$. Therefore, $K_{1,b} \in \mathcal{G}_{1,b,b}$.
Case 2: $2 \le a = b = c$. If T is a tree of order a, then obviously $cor(T) \in \mathcal{G}_{a,a,a}$.

Case 3: $2 \leq a < b = c$. Let $H_{a,b}$ be the graph obtained from $K_{1,b}$ by subdividing $a-1$ edges once. Then the vertex of maximum degree and all leaves nonadjacent to it form a γ-set of $H_{a,b}$. Hence $H_{a,b}$ is in $\mathcal{G}_{a,b,b}$, $a < b$, because of Observation 3.
Case 4: $2 \leq b < a = c$. Take a copy of $K_{1,b-1}$ with the leaves $x_1, x_2, ..., x_{b-1}$ and the support vetex x (if $b = 2$ then let $V(K_{1,1}) = \{x, x_1\}$). Subdivide the edges xx_i, $i = 1, 2, ..., b-1$. Obtain the graph $U_{b,c}$ by taking a copy of the path $P : w_0, w_1, .., w_{3(a-b)}$, and joining w_0 to x. Since $U_{b,c}$ is a tree, $U_{b,c}$ is in some $\mathcal{G}_{p,q,r}$ (by Observation 3). Since $U_{b,c}$ has b leaves, $q = b$. Finally, $N(x) \cup \{w_3, w_6, ..., w_{3(a-b)}\}$ is clearly a γ-set of G which is also a geodetic dominating set of $U_{b,c}$. Thus, $p = r = a = c$.
Case 5: $2 \leq \min\{a, b\}$ and $\max\{a, b\} < c \leq a + b - 1$. Take a copy of $K_{1,c-b}$ with the leaves $y_1, y_2, ..., y_{c-b}$ and the support vertex y when $c-b \geq 2$, and $V(K_{1,c-b}) = \{y, y_1\}$ if $c - b = 1$. Subdivide each edge once. Obtain the graph G_5 by taking a copy of $K_{1,c-1}$ (with the central vertex x and $x_1, x_2, .., x_{c-1}$ as leaves) and adding the edges $x_b y_1, x_{b+1} y_2, .., x_{c-1} y_{c-b}$. If $c < a + b - 1$, then let G_6 be a graph obtained from G_5 by subdividing each of the edges $xx_{c-a+1}, xx_{c-a+2}, .., xx_{b-1}$ once.
Let us first consider the case $c = a + b - 1$. Note that $S = \{y, x_1, x_2, .., x_{c-a}\}$ with $|S| = b$ is the unique g-set of G_5 and the unique g_e-set of G_5. Also it is obvious that each γg-set D of G_5 is the union of S and some $(c-b)$-cardinality set $\{z_1, z_2, ..., z_{c-b}\}$, where $z_i \in \{y_i, x_{b-1+i}\}$, $i = 1, 2, .., c-b$. Clearly, D is also a γg_e-set of G_5. It remains to note that $D' = N(y) \cup \{x\}$ is the unique γ-set of G_5 and $|D'| = a$.
Second, assume that $c < a + b - 1$. Now $S_1 = \{y, x_1, x_2, .., x_{b-1}\}$ with $|S_1| = b$ is the unique g-set of G_6 and the unique g_e-set of G_6. Note also that each γg-set D_1 of G_6 has the form $S_1 \cup \{z_1, z_2, ..., z_{c-b}\}$, where $z_i \in \{y_i, x_{b-1+i}\}$, $i = 1, 2, .., c - b$. It is obvious that D_1 is also a γg_e-set of G_6 and $|D_1| = c$. Finally, each γ-set D'' of G_6 has the form $N(y) \cup \{x\} \cup \{u_{c-a+1}, .., u_{b-1}\}$, where $u_i \in N[x_i]$, $i = c - a + 1, .., b - 1$. Clearly, $|D''| = a$.
Case 6: $2 \leq a, b$ and $a + b = c$. Let x and y be antipodal vertices of a copy of a cycle C_6. Add new vertices $x_1, x_2, ..., x_{b-1}$ and join each of them to the vertex x to obtain the graph H. Define the graph G as obtained from H by taking a copy of the path on $3(a-2)+1$ vertices $y_0, y_1, ..., y_{3(a-2)}$ and joining y_0 to the vertex y. Clearly, (a) the set $A = \{x, y, y_2, y_5, ..., y_{3(a-2)-1}\}$ is the unique γ-set of G and $|A| = a$, (b) the set $B = \{x_1, x_2, ..., x_{b-1}, y_{3(a-2)}\}$ has b elements and it is the unique g-set of G and the unique g_e-set of G. Let C be a γg-set of G. Then, it contains all vertices in B, necessarily. On the other hand, it is easy to observe that such a γg-set must have at least a vertices of $G - C$ in order to be a dominating set in G. Therefore, $|C| \geq a + b = c$. Moreover, $S_2 = B \cup \{x, y\} \cup \cup_{i=0}^{a-3} \{y_{3i}\}$ if $a \geq 3$, and $S_2 = B \cup \{x, y\}$ if $a = 2$ is a geodetic dominating set in G of cardinality c. So, $\gamma g(G) = c$. Finally, it is obvious that C is an EGD-set of G as well, which implies that $\gamma g(G) \equiv \gamma g_e(G)$. $\square$

Theorem 5 shows that the bound in Observation 4 is attainable.

Observation 6. Let G be a connected n-order graph, $n \geq 2$, S an edge geodetic set of G and F a dominating set of $G - N[S]$. Then $S \cup F$ is an EGD-set of G. In particular, $\gamma g_e(G) \leq g_e(G) + \gamma(G - N[S])$.

Remark 2. Clearly $\{x_1, x_n\}$ is the unique g_e-set of $P_n : x_1, x_2, .., x_n$ and if $2 \leq n \leq 4$ then $\{x_1, x_n\}$ is the unique γg_e-set of P_n. Let $n \geq 5$ and S be a γg_e-set of P_n. Then we can choose S so that $x_2, x_{n-1} \notin S$. Hence $S - \{x_1, x_n\}$ is a γ-set of $P_n - \{x_1, x_2, x_{n-1}, x_n\}$. Thus $\gamma g_e(P_n) = g_e(P_n) + \gamma(P_n - \{x_1, x_2, x_{n-1}, x_n\})$, which shows that the bound in Observation 6 is attainable and in addition $\gamma g_e(P_n) = 2 + \lceil (n-4)/3 \rceil = \lceil (n+2)/3 \rceil$.

In the next theorem we give some conditions under which $g_e(G) \leq \alpha(G)$ is valid for a graph G.

Theorem 7. *Let G be an n-order connected graph, $n \geq 2$.*

(i) If $F \subsetneq V(G)$ is a vertex cover of G which contains all extreme vertices of G, then F is a geodetic dominating set of G.

(ii) If G has no extreme vertices, then (a) each vertex cover of G is an GD-set of G and (b) $\gamma g(G) \leq \alpha(G) = n - \beta_0(G)$.

(iii) If U is a vertex cover of G which contains all semi-extreme vertices of G, then U is an EGD-set of G.

(iv) If G has no semi-extreme vertices then (a) each vertex cover of G is an EGD-set of G and (b) $\gamma g_e(G) \leq \alpha(G) = n - \beta_0(G)$.

Proof. (i) By Theorem C, $V(G) - F$ is independent. As F contains all extreme vertices of G, each vertex x in $V(G) - F$ has 2 nonadjacent neighbors both belonging in F, say they are y_x and z_x. But then F is a dominating set and y_x, x, z_x is geodesic. This shows that F is a geodetic dominating set of G.
(ii) The required immediately follows by (i) and Theorem C.
(iii) The set $V(G) - U$ is independent because of Theorem C. Hence U is a dominating set of G. Let xy be an arbitrary edge of G with $x \in U$ and $y \in V(G) - U$. Since all semi-extreme vertices are in U, the vertex y has a neighbor, say $z_y \in U$, which is nonadjacent to x. Since x, y, z_y is geodetic, U is an edge geodetic dominating set of G.
(iv) Immediately by (iii) and Theorem C. □

Corollary 3. *Let G be a connected n-order triangle-free graph with $\delta(G) \geq 2$. Then each vertex cover of G is an EGD-set of G. In particular, $\gamma g_e(G) \leq \alpha(G) = n - \beta_0(G)$.*

Corollary 4. *Let $G = (X, Y; E)$ be a connected bipartite graph. Then both $X \cup L(G)$ and $Y \cup L(G)$ are edge geodetic dominating sets of G. In particular, if $\delta(G) \geq 2$ then $\gamma g_e(G) \leq \min\{|X|, |Y|\}$.*

Two edges in a graph G are *independent* if they have no a common endpoint. A *matching* in G is a set of (pairwise) independent edges. A *maximal matching* is a matching M of a graph G with the property that if any edge not in M is added to M, it is no longer a matching.

Corollary 5. *Let G be a triangle-free graph with minimum degree $\delta(G) \geq 2$. Then $\gamma g_e(G) \leq 2\min\{|M| \mid M$ is a maximal matching of $G\}$.*

Proof. Let M be an arbitrary maximal matching of a graph G. The maximality of M shows that $V(G) - V(M)$ is an independent set, where $V(M)$ is the set of all vertices incident with an edge of M. By Theorem C, $V(M)$ is a vertex cover of G and the required follows immediately by Corollary 3. □

The *girth* of a graph G, denoted by $girth(G)$, is the length of a shortest cycle contained in G.

Theorem 8. *Let G be a connected graph with $\delta(G) \geq 2$ and girth at least 6. Then $g_e(G) \leq \gamma g_e(G) \equiv \gamma(G) < \alpha(G)$.*

Proof. Let D be an arbitrary minimum dominating set of G. Suppose that D is not edge geodetic. Then there is an edge $e = xy$, with $x \notin D$, which lies in no shortest path connecting two vertices of D. Since G has no leaves, x has at least one more neighbor, say z. Clearly, $yz \notin E(G)$ and at most one of y and z is in D.
Case 1. $y \in D$. Hence $z \notin D$ and there is a vertex $t \in D - \{y\}$ which is adjacent to z. Since the girth of G is at lest 6, the path y, x, z, t is a $y - t$ geodesic and $y, t \in D$. But this contradicts the choice of $e = xy$.
Case 2. $y \notin D$. Then there are different $u, v \in D$ such that $ux, vy \in E(G)$. As G has girth at lest 6, the path u, x, y, v is a $u - v$ geodesic and $u, v \in D$. Again a contradiction.
Thus D is an EGD-set of cardinality $\gamma(G)$. Since D was chosen arbitrarily, and $\gamma g_e(G) \geq \gamma(G)$, we immediately obtain $\gamma g_e(G) \equiv \gamma(G)$. Finally, Theorem B and $girth(G) \geq 6$ imply $\gamma(G) \neq \alpha(G)$. □

A graph G is a *geodetic graph* if for every pair of its vertices there is a unique path of minimum length between them [12]. The concept of geodetic graph is a natural generalization of a tree. A subgraph H of a graph G is called an *isometric subgraph* if for every two vertices of H, the distance between them in G equals the distance in H. Notice that an isometric subgraph is an induced subgraph.

Theorem 9. *Let G be a geodetic graph and H its isometric subgraph with no isolated vertices. Let F be an edge geodetic set of H. Then all the following hold.*

(i) *$S = (V(G) - V(H)) \cup F$ is an edge geodetic set of G. If F is a dominating set of H, then S is an edge geodetic dominating set of G.*

(ii) *$g_e(G) \leq |V(G)| - |V(H)| + g_e(H)$ and $\gamma g_e(G) \leq |V(G)| - |V(H)| + \gamma g_e(H)$.*

Proof. (i) Clearly, to prove that $S = (V(G) - V(H)) \cup F$ is an edge geodetic set of G, it is sufficient to show that each edge of G with one end in $V(H - F)$ and the other end in $G - V(H)$ belongs to some $a - b$ geodesic, where $a, b \in S$. So, let $v' \in V(H) - F$

be adjacent to $x \in V(G) - V(H)$. Let $P : v_0, v_1, .., v_d$ be a $v_0 - v_d$ geodesic in H, and hence in G, with $v' = v_i \in V(P)$ and $V(P) \cap F = \{v_0, v_d\}$. Assume that neither the path $P_1 : v_0, v_1, .., v_i = v', x$ nor the path $P_2 : x, v' = v_i, v_{i+1}, .., v_d$ is geodesic in G. Let $Q_1 : u_0 = v_0, u_1, .., u_k = x$ and $Q_2 : x = w_0, w_1, .., w_{l-1}, w_l = v_d$ be geodesics in G. Then $Q_1.Q_2 : u_0 = v_0, u_1, .., u_k = x = w_0, w_1, .., w_{l-1}, w_l = v_d$ is a $v_0 - v_d$ walk in G with length $|E(Q_1.Q_2)| = |E(Q_1)| + |E(Q_2)| \leq (E(P_1)| - 1) + (|E(P_2)| - 1) = |E(P)|$. Since P is $v_0 - v_d$ geodesic, $Q_1.Q_2$ is also $v_0 - v_d$ geodesic. Since $V(P) \neq V(Q_1.Q_2)$ and G is geodetic, we arrive at a contradiction. Hence at least one of P_1 and P_2 is geodesic in G. Thus, S is an edge geodetic set of G. The rest is obvious.
(ii) Let $\mu \in \{g_e, \gamma g_e\}$. Choose F to be a μ-set of H. Then by (i) we immediately obtain $\mu(G) \leq |V(G)| - |V(H)| + \mu(H)$. □

Corollary 6. *Let G be an n-order geodetic graph, $n \geq 2$, with diameter d. Then $\gamma g_e(G) \leq n - \lfloor 2d/3 \rfloor$ and ([16]) $g_e(G) \leq n - d + 1$.*

Proof. Let H be any diametral path in G. Clearly H is an isometric subgraph of G. Applying Theorem 9 to G and H we obtain $\mu(G) \leq n - (d+1) + \mu(H)$, where $\mu \in \{g_e, \gamma g_e\}$. But $g_e(H) = 2$ and $\gamma g_e(H) = \lceil (d+3)/3 \rceil$ (by Remark 2). Thus, the result immediately follows. □

Theorem 10. *Let G be a connected graph, $\gamma(G) \geq 2$, D an ED-set of G and let $G - D$ contain no semi-extreme vertices. Then D is a γg-set of G, $\gamma(G) = \gamma g(G)$ and $I^e_G[D] \supseteq E(G) - \{xy \in E(G) \mid x, y \in N(d)$ for some $d \in D\}$. In particular, if no element of D belongs to a triangle, then D is a γg_e-set of G and $\gamma(G) = \gamma g_e(G)$.*

Proof. Choose $x \in V(G) - D$ arbitrarily. Since D is an ED-set, x has exactly one neighbor in D, say y_x. Since x is not semi-extreme, choose arbitrarily $z \in N(x) - N(y_x)$. Denote by t the unique neighbor of z in D. Then y_x, x, z, t is a geodesic in G. Hence D is a geodetic set and as z was chosen arbitrarily, $I^e_G[D] \supseteq E(G) - \{xy \in E(G) \mid x, y \in N(d)$ for some $d \in D\}$. Finally, if no element of D belongs to a triangle, then $I^e_G[D] \supseteq E(G)$ implies D is an EGD-set of G. □

The next result follows immediately by the above theorem.

Corollary 7. *Let G be a connected graph without semi-extreme vertices. If all γ-sets of G are efficient dominating, then (a) $\gamma(G) \equiv \gamma g(G)$, and (b) if G has no triangles, then $\gamma(G) \equiv \gamma g(G) \equiv \gamma g_e(G)$.*

4. Vertex partitions

Here we present some initial results on the parameters $gd, g_e d, gp, g_e p$.

Proposition 3. *Let G be a connected n-order graph, $n \geq 2$. Then* *(i)* $g_e d(G) \leq gd(G) \leq \min\{gp(G), d(G)\} \leq \delta(G) + 1$ *and* $g_e d(G) \leq g_e p(G) \leq gp(G)$.

(ii) $g_e d(G)\gamma(G) \leq gd(G)\gamma(G) \leq n$, $gp(G)g(G) \leq n$ *and* $g_e p(G) g_e(G) \leq n$.

Proof. (i) Denote by $\mathscr{S}(G)$ ($\mathscr{S}_e(G)$, $\mathscr{D}(G)$, $\mathscr{S}\mathscr{D}(G)$, $\mathscr{S}_e\mathscr{D}(G)$, respectively) the family of all geodetic sets of G (the family of all EG-sets of G, the family of all dominating sets of G, the family of all GD-sets of G, the family of all EGD-sets of G, respectively). Clearly,

$$\mathscr{S}(G) \supseteq \mathscr{S}_e(G) \text{ and } \mathscr{S}\mathscr{D}(G) = \mathscr{S}(G) \cap \mathscr{D}(G) \supseteq \mathscr{S}_e(G) \cap \mathscr{D}(G) = \mathscr{S}_e\mathscr{D}(G),$$

which implies $g_e d(G) \leq gd(G) \leq gp(G)$, $g_e d(G) \leq gd(G) \leq d(G)$ and $g_e d(G) \leq g_e p(G) \leq gp(G)$. The rest immediately follows by Theorem E.
(ii) Let $\mathcal{P} = [U_1, U_2, .., U_k]$ be a partition of $V(G)$ into geodetic sets, or into EG-sets, or into GD-sets, or into EGD-sets. Then $n = |U_1| + |U_2| + ... + |U_k| \geq k \min\{|U_r| \mid 1 \leq r \leq k\}$, which implies the required inequality. □

The next example shows that there is a graph G for which all inequalities in Proposition 3 become equalities.

Example 2. Let $\mu \in \{gp, g_e p, d, gd, g_e d\}$ and consider a crown graph $H_{n,n}$, $n \geq 3$, with $V(H_{n,n}) = \{1, 2, .., n, 1', 2', .., n'\}$ and $E(H_{n,n}) = \{ij' \mid i \neq j\}$. Clearly, all ν-sets of $H_{n.n}$ are $S_i = \{i, i'\}$, $i = 1, 2, .., n$, where $\nu \in \{\gamma, g, g_e, \gamma g, \gamma g_e\}$. Hence $\nu(H_{n,n}) = 2$ and since all S_i form a partition of $V(H_{n,n})$, $\mu(H_{n,n}) = n = \delta(H_{n,n}) + 1$ and $\mu(H_{n,n})\nu(H_{n,n}) = |V(H_{n,n})|$.

Let us consider the Cartesian product $C_p \square C_q$ of cycles as a $p \times q$ array of vertices $\{x_{i,j} \mid 0 \leq i \leq p-1 \text{ and } 0 \leq j \leq q-1\}$, with an adjacency $N(x_{ij}) = \{x_{i,j-1}, x_{i,j+1}, x_{i-1,j}, x_{i+1,j}\}$, where the first subscript is taken modulo p and the second subscript is taken modulo q. We let $x \equiv_z y$ mean $x \equiv y \pmod{z}$.

Example 3. Note that $diam(C_{2m} \square C_{2n}) = m + n$, $x_{i,j}$ and $x_{i+m,j+n}$ are antipodal, $g(C_{2m} \square C_{2n}) = 2$ and all g-sets of $C_{2m} \square C_{2n}$ are $X_{i,j} = \{x_{i,j}, x_{i+m,j+n}\}$ ([2]). It is easy to see that each $X_{i,j}$ is also a g_e-set of $C_{2m} \square C_{2n}$. Clearly, all $X_{i,j}$ form a partition of $V(C_{2m} \square C_{2n})$. Therefore $gp(C_{2m} \square C_{2n}) = g_e p(C_{2m} \square C_{2n}) = 2mn$ and $g_e p(C_{2m} \square C_{2n}) g_e(C_{2m} \square C_{2n}) = gp(C_{2m} \square C_{2n}) g(C_{2m} \square C_{2n}) = |V(C_{2m} \square C_{2n})|$.

Example 4. Here we consider the graph $C_4 \square C_n$, $n \geq 4$. It is known that $\gamma(C_4 \square C_n) = n$ ([11]). Define the sets $D_i = \{x_{i,j} \mid j \equiv_2 0\} \cup \{x_{i+2,j} \mid j \equiv_2 1\}$, $i = 0, 1, 2, 3$. It is easy to see that all these sets are EGD-sets of cardinality n. Hence $n = \gamma(C_4 \square C_n) = \gamma g(C_4 \square C_n) = \gamma g_e(C_4 \square C_n)$. Since D_0, D_1, D_2, D_3 form a partition of $V(G)$, $d(C_4 \square C_n) = gd(C_4 \square C_n) = g_e d(C_4 \square C_n) = 4$. Therefore $d(C_4 \square C_n)\gamma(C_4 \square C_n) = gd(C_4 \square C_n)\gamma g(C_4 \square C_n) = g_e d(C_4 \square C_n)\gamma g_e(C_4 \square C_n) = |V(C_4 \square C_n)|$.

An *efficient domination partition* (or an *ED-partition*) of a graph G is a partition of $V(G)$ into ED-sets. A graph G is said to be an *efficient domination partitionable graph* (or an EDP-*graph*) if G has an ED-partition.

Theorem 11. *[15] Let G be a graph of order n. Then the following assertions are equivalent.*

(i) G is an EDP-graph.

(ii) G is regular and $d(G) = \delta(G) + 1$.

(iii) $n = \gamma(G)(\Delta(G) + 1)$ and $n = d(G)\gamma(G)$.

If G is an EDP-graph, then each its γ-set is efficient dominating.

Theorem 12. *Let G be an n-order connected EDP-graph without semi-extreme vertices, $n \geq 3$.*

(i) Then $\gamma(G) \equiv \gamma g(G)$ and $d(G) = gd(G)$.

(ii) If G has no triangles, then $\gamma(G) \equiv \gamma g(G) \equiv \gamma g_e(G)$ and $d(G) = gd(G) = g_e d(G)$.

Proof. By Theorem 11, all γ-sets of G are efficient dominating. Now Corollary 7 immediately implies $\gamma(G) \equiv \gamma g(G)$ and if G has no triangles then $\gamma(G) \equiv \gamma g(G) \equiv \gamma g_e(G)$. Since G is an EDP-graph, $d(G) = gd(G)$ and if there is no triangles in G, $d(G) = gd(G) = g_e d(G)$. □

Denote by $\mathbb{Z}_n = \{0, 1, ..., n-1\}$ the additive group of order n. The *generalized Petersen graph* $P(n, k)$, where $n \geq 3$ and $k \in \mathbb{Z}_n - \{0\}$, is the graph on the vertex set $\{x_i, y_i \mid i \in \mathbb{Z}_n\}$ with adjacencies $x_i x_{i+1}, x_i y_i$, and $y_i y_{i+k}$ for all i.

Example 5. If $n \equiv_4 0$ and k is odd then $\gamma(P(n,k)) \equiv \gamma g(P(n,k)) \equiv \gamma g_e(P(n,k))$ and $d(P(n,k)) = gd(P(n,k)) = g_e d(P(n,k))$.

Proof. A graph $P(n, k)$ is bipartite if and only if n is even and k is odd ([1]), and it is an EDP-graph if and only if $n \equiv_4 0$ and k is odd ([15]). Now the required immediately follows by Theorem 12. □

Let S be a subset of $\mathbb{Z}_n$ such that $0 \notin S$ and $x \in S$ implies $-x \in S$. The *circulant graph* with distance set S is the graph $C(n; S)$ with vertex set $\mathbb{Z}_n$ and vertex x is adjacent to vertex y if and only if $x - y \in S$. It is clear from the definition that $C(n; S)$ is vertex-transitive and regular of degree $|S|$.

Example 6. Let $G = C(n = (2k+1)t; \{1, .., k\} \cup \{n-1, ..., n-k\})$, where $k, t \geq 1$. Then G is an EDP-graph, $\gamma(G) = t$ and G has only one ED-partition ([15]). By Theorem 12, $\gamma(G) \equiv \gamma g(G)$ and $d(G) = gd(G)$.

Example 7. Let $G = C(n; \{\pm 1, \pm s\})$ where $2 \leq s \leq n-2$, $s \neq n/2$, $s \equiv_5 \pm 2$ and $5|n$. Then G is an EDP-graph, $\gamma(G) = n/5$ and G has only one ED-partition ([15]). By Theorem 12, $\gamma(G) \equiv \gamma g(G)$ and $d(G) = gd(G)$.

Example 8. Let $G = C_5 \Box C_{5k}$, $k \geq 1$. Then G is an EDP-graph, $\gamma(G) \equiv \gamma g(G) \equiv \gamma g_e(G)$ and $d(G) = gd(G) = g_e d(G)$.

Proof. Define the sets D_i as the union of $\{x_{i,j} \mid j \equiv_5 0\}$, $\{x_{1+i,j} \mid j \equiv_5 3\}$, $\{x_{2+i,j} \mid j \equiv_5 1\}$, $\{x_{3+i,j} \mid j \equiv_5 4\}$, and $\{x_{4+i,j} \mid j \equiv_5 2\}$, $i = 0, 1..., 4$. Clearly, all these sets are efficient dominating and they form an ED-partition of G. Thus G is an EDP-graph with $\gamma(G) = 5k$ (the fact that $\gamma(G) = 5k$ is known, see [11]). Since G has no triangles, by Theorem 12 we immediately have $\gamma(G) \equiv \gamma g(G) \equiv \gamma g_e(G)$ and $d(G) = gd(G) = g_e d(G)$. □

5. Problems

We conclude the paper by two problems.

Problem 1. Let G be a graph and $\mu \in \{gd, g_e d, gp, g_e p\}$. Find a nontrivial characterization of graphs with $\mu(G) \leq 2$.

Problem 2. Let G be an n-order graph and $\nu \in \{g, g_e, \gamma g, \gamma g_e\}$. Find results on ν-excellent graphs.

Acknowledgements

The author express his sincere thanks to the anonymous referees for their meticulous and thorough reading of the paper that greatly improved its exposition.

References

[1] D.W. Bange, A.E Barkauskas, and P.J. Slater, *Efficient dominating sets in graphs*, Applications of Discrete Mathematics, SIAM - Philadelphia, 1988, pp. 189–199.

[2] J. Cao, B. Wu, and M. Shi, *The geodetic number of* $c_m \times c_n$, International Conference on Management and Service Science, Wuhan, 9 2009, pp. 1–3.

[3] E.J. Cockayne and S.T. Hedetniemi, *Towards a theory of domination in graphs*, Networks **7** (1977), 247–261.

[4] M.C. Dourado, F. Protti, D. Rautenbach, and J.L. Szwarcfiter, *Some remarks on the geodetic number of a graph*, Discrete Math. **310** (2010), 832–837.

[5] H. Escuardo, R. Gera, A. Hansberg, N. Jafari Rad, and L. Volkmann, *Geodetic domination in graphs*, J. Combin. Math. Combin. Comput. **77** (2011), 89–101.

[6] J.F. Fink, M.S. Jacobson, L.F. Kinch, and J. Roberts, *On graphs having domination number half their order*, Period. Math. Hungar. **16** (1985), 287–293.

[7] T. Gallai, *Uber extreme punkt- und kantenmengen*, Ann. Univ. Sci. Budapest, Eotvos Sect. Math. **2** (1959), 133–138.

[8] A. Hansberg and L. Volkmann, *On the geodetic and geodetic domination numbers of a graph*, Discrete Math. **310** (2010), 2140–2146.

[9] F. Harary, E. Loukakis, and C. Tsouros, *The geodetic number of a graph*, Mathl. Comput. Modelling **17** (1993), no. 11, 89–95.

[10] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker - New York, 1998.

[11] S. Klaviar and N. Seifter, *Dominating cartesian products of cycles*, Discrete Appl. Math. **59** (1995), 129–136.

[12] O. Ore, *Theory of graphs*, American Mathematical Society - Providence, 1967.

[13] C. Payan and N.H. Xuong, *Domination-balanced graphs*, J. Graph Theory **6** (1982), 23–32.

[14] B. Randerath and L. Volkmann, *Characterization of graphs with equal domination and covering number*, Discrete Math. **191** (1998), 159–169.

[15] V. Samodivkin, *Common extremal graphs for three inequalities involving domination parameters*, Trans. Comb. **6** (2017), no. 3, 1–9.

[16] A.P. Santhakumaran and J. John, *Edge geodetic number of a graph*, J. Discrete Math. Sci. Cryptogr. **10** (2007), no. 3, 415–432.

[17] ———, *The upper edge geodetic number and the forcing edge geodetic number of a graph*, Opuscula Math. **29** (2009), no. 4, 427–441.

[18] A.P. Santhakumaran and S.V. Ullas Chandran, *Comment on "edge geodetic covers in graphs"*, Proyecciones Journal of Mathematics **34** (2015), no. 4, 343–350.

[19] P. Arul Paul Sudhahar, A. Ajitha, and A. Subramanian, *Edge geodetic domination number of a graph*, Int. J. Math. Appl. **4** (2016), no. 3-B, 45–50.