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Total double Roman domination in graphs

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Abstract: Let G be a simple graph with vertex set V. A double Roman dominating function (DRDF) on G is a function $f : V \to \{0, 1, 2, 3\}$ satisfying that if $f(v) = 0$, then the vertex v must be adjacent to at least two vertices assigned 2 or one vertex assigned 3 under f, whereas if $f(v) = 1$, then the vertex v must be adjacent to at least one vertex assigned 2 or 3. The weight of a DRDF f is the sum $\sum_{v \in V} f(v)$. A total double Roman dominating function (TDRDF) on a graph G with no isolated vertex is a DRDF f on G with the additional property that the subgraph of G induced by the set $\{v \in V : f(v) \neq 0\}$ has no isolated vertices. The total double Roman domination number $\gamma_{tdR}(G)$ is the minimum weight of a TDRDF on G. In this paper, we give several relations between the total double Roman domination number of a graph and other domination parameters and we determine the total double Roman domination number of some classes of graphs.

Keywords: total double Roman domination, double Roman domination, total Roman domination, total domination, domination

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1. Introduction

Throughout this paper, G denotes a finite and simple graph with vertex set $V(G)$ and edge set $E(G)$. The *open neighborhood* of a vertex v in G is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$ and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. In general, for a set $X \subseteq V(G)$, its *open neighborhood* is the set $N(X) = \bigcup_{v \in X} N(v)$ and its *closed neighborhood* is the set $N[X] = N(X) \cup X$. The *degree* of a vertex v in G is $d(v) = d_G(v) = |N(v)|$. The minimum degree and maximum degree among all vertices of G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The graph G is k*-regular* if $d(v) = k$ for every vertex $v \in V(G)$. The *distance* $d(u, v)$ between two vertices u and v of a (connected) graph G is the length of a shortest (u, v)-path in G. The maximum distance among all pairs of vertices in G is the *diameter* of G, which is denoted by $\mathrm{diam}(G)$. For a subset S of vertices of G, we denote by $G[S]$ the subgraph induced by S.
We write P_n for the path of order n and C_n for the cycle of length n. A *star* S_n of order $n \geq 2$ is the complete bipartite graph $K_{1,\, n-1}$. We call the *center* of a star to be a vertex of maximum degree. A set S of vertices in G is an *independent set* if no two vertices of S are adjacent in G. A *leaf* of a graph G is a vertex of degree 1, while a *support vertex* of G is a vertex adjacent to a leaf. An edge incident with a leaf is a *pendant edge*. For a real-valued function $f : V(G) \to \mathbb{R}$ and $S \subseteq V(G)$, we define $f(S) = \sum_{x \in S} f(x)$.
A *packing* in G is a set of vertices that are pairwise at distance at least 3 apart; that is, if u and v are distinct vertices that belong to a packing S, then $d(u, v) \geq 3$. A *subdivision* of an edge uv is obtained by removing the edge uv, adding a new vertex w, and adding edges uw and wv.
For two vertex subsets X and Y of a graph G, if $Y \subseteq N[X]$, then we say that X dominates Y. A set $S \subseteq V(G)$ is a *dominating set* of G if S dominates $V(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set of G of cardinality $\gamma(G)$ is called a $\gamma(G)$*-set*. The domination in graphs has been studied extensively, for details, see, e.g., [6, 8, 11].
A *total dominating set*, abbreviated TD-set, of G is a set S of vertices of G such that every vertex of G is adjacent to a vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G. A TD-set of G of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$*-set*. The concept of total domination in graphs is now well studied [7, 9, 10].
A *Roman dominating function* (RDF) on a graph G is a function $f : V(G) \to \{0, 1, 2\}$ having the property that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The *weight* of a RDF f is the sum $\omega(f) = \sum_{v \in V(G)} f(v)$. The *total Roman dominating function* (TRDF) on a graph G with no isolated vertex is a RDF f on G with the additional property that the subgraph of G induced by the set $\{v \in V(G) : f(v) \neq 0\}$ has no isolated vertices. The *total Roman domination number* $\gamma_{tR}(G)$ is the minimum weight of a TRDF on G. A TRDF on G with weight $\gamma_{tR}(G)$ is called a $\gamma_{tR}(G)$*-function*. The literature on total Roman domination has been surveyed in [2–4].

A *double Roman dominating function* (DRDF) on a graph G is a function $f : V(G) \to \{0, 1, 2, 3\}$ having the property that if $f(v) = 0$, then the vertex v must be adjacent to at least two vertices assigned 2 or one vertex assigned 3 under f, whereas if $f(v) = 1$, then the vertex v must be adjacent to at least one vertex assigned 2 or 3. The *weight* of a DRDF f is the sum $\omega(f) = \sum_{v\in V(G)} f(v)$. The *double Roman domination number* of a graph G, denoted by $\gamma_{dR}(G)$, is the minimum weight of a DRDF on G. A $\gamma_{dR}(G)$-*function* is a DRDF on G with weight $\gamma_{dR}(G)$. For a sake of simplicity, a DRDF f on G will be represented by the ordered partition (V_0, V_1, V_2, V_3) of $V(G)$ induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i \in \{0, 1, 2, 3\}$. Double Roman domination was studied in [1, 5, 12, 14, 15] and elsewhere.
Recently, Shao et al. [13] initiated the study of total double Roman domination in graphs, which is a variant of double Roman domination. The *total double Roman dominating function* (TDRDF) on a graph G with no isolated vertex is a DRDF f on G with the additional property that the subgraph of G induced by the set $\{v \in V(G) : f(v) \neq 0\}$ has no isolated vertices. The *total double Roman domination number* $\gamma_{tdR}(G)$ is the minimum weight of a TDRDF on G. A TDRDF on G with weight $\gamma_{tdR}(G)$ is called a $\gamma_{tdR}(G)$-*function*.

2. Relations to other domination parameters

In this section, we shall relate the total double Roman domination number of graphs to other domination parameters such as domination number, total domination number, double Roman domination number and total Roman domination number.

Theorem 1. *For any graph G with no isolated vertex, $\gamma_{tdR}(G) \geq \gamma_t(G) + \gamma(G)$ and the lower bound is sharp for the disjoint union of copies of P_2.*

Proof. Let $f = (V_0, V_1, V_2, V_3)$ be an arbitrary $\gamma_{tdR}(G)$-function. Then $V_2 \cup V_3$ and $V_1 \cup V_2 \cup V_3$ are a dominating set and a TD-set of G, respectively, implying that $\gamma_{tdR}(G) = 3|V_3|+2|V_2|+|V_1| = (|V_3|+|V_2|+|V_1|)+(|V_3|+|V_2|)+|V_3| \geq \gamma_t(G)+\gamma(G)$, establishing the desired lower bound. □

Theorem 2. *For any graph G with no isolated vertex, $\gamma_{tdR}(G) \leq 3\gamma_t(G)$ with equality if and only if there exists a $\gamma_{tdR}(G)$-function f such that for each $v \in V(G)$, either $f(v) = 3$ or $f(v) = 0$.*

Proof. Let X be a $\gamma_t(G)$-set. Then the function f defined by $f(v) = 3$ for $v \in X$ and $f(v) = 0$ otherwise, is a TDRDF on G and so $\gamma_{tdR}(G) \leq \omega(f) = 3|X| = 3\gamma_t(G)$, establishing the desired upper bound.
Suppose that $\gamma_{tdR}(G) = 3\gamma_t(G)$. It is easy to see that the function f defined earlier is a TDRDF on G with weight $\omega(f) = 3|X| = 3\gamma_t(G)$, implying that f is a $\gamma_{tdR}(G)$-function such that for each $v \in V(G)$, either $f(v) = 3$ or $f(v) = 0$.

Conversely, suppose that there exists a $\gamma_{tdR}(G)$-function f such that for each $v \in V(G)$, either $f(v) = 3$ or $f(v) = 0$. Let $Y = \{v \in V(G) : f(v) = 3\}$. Clearly Y is a TD-set of G and hence $3\gamma_t(G) \leq 3|Y| = \gamma_{tdR}(G)$. On the other hand, as proven earlier, $\gamma_{tdR}(G) \leq 3\gamma_t(G)$. Thus we have $\gamma_{tdR}(G) = 3\gamma_t(G)$, completing the proof. □

Theorem 3. *For any graph G with no isolated vertex, $\gamma_{tdR}(G) \leq 4\gamma(G)$. Moreover, if $\gamma_{tdR}(G) = 4\gamma(G)$, then every $\gamma(G)$-set is a packing in G.*

Proof. Let S be an arbitrary $\gamma(G)$-set and let S' denote the set of vertices in S that are isolated in $G[S]$ (possibly, $S' = \emptyset$). For each vertex $v \in S'$, we select one neighbor of v and denote it by v'. Let $S'' = \bigcup_{v\in S'}\{v'\}$. Clearly, $S'' \subseteq V(G)\backslash S$ and $|S''| \leq |S'|$. Now define the function f by $f(x) = 3$ for each $x \in S$, $f(x) = 1$ for each $x \in S''$ and $f(x) = 0$ otherwise. Then f is a TDRDF on G and so

$$\gamma_{tdR}(G) \leq 3|S| + |S''| \leq 3|S| + |S'| \leq 4|S| = 4\gamma(G), \tag{1}$$

establishing the desired upper bound.
Suppose that $\gamma_{tdR}(G) = 4\gamma(G)$. Then we must have equality throughout the inequality chain (1). In particular, $|S| = |S'| = |S''|$, implying that $S' = S$ and so S is an independent set. Let u and v be two distinct vertices in S. Since S is an independent set, $d(u, v) \geq 2$. We now show that $d(u, v) \geq 3$. Suppose, to the contrary, that $d(u, v) = 2$. Let w be a common neighbor of u and v, and choose $u' = v' = w$ where, as before, u' and v' are the vertices chosen to be adjacent to u and v, respectively. With this choice of u' and v', we note that $|S''| < |S'|$, a contradiction. Thus $d(u, v) \geq 3$. This is true for every pair of distinct vertices in S, implying that S is a packing in G. Thus, every $\gamma(G)$-set is a packing in G. □

We remark that the upper bound of Theorem 3 is sharp. For example, for $k \geq 2$, let G be obtained from the disjoint union of k stars $S_{i_1}, S_{i_2}, \ldots, S_{i_k}$, where $i_j \geq 4$ for $1 \leq j \leq k$, by selecting one leaf from each star and adding some edges joining these k selected leaves so that the resulting graph is connected. It is easy to see that $\gamma(G) = k$ and the set of k support vertices of G forms the unique $\gamma(G)$-set, which is also a packing in G.
Let f be an arbitrary $\gamma_{tdR}(G)$-function and let v be an arbitrary support vertex of G. One can check that $f(N[v]) \geq 4$ and so $\gamma_{tdR}(G) \geq 4k = 4\gamma(G)$. Moreover, it follows from Theorem 3 that $\gamma_{tdR}(G) \leq 4\gamma(G)$. As a result, we obtain $\gamma_{tdR}(G) = 4\gamma(G)$.
It remains an open problem to find a necessary and sufficient condition for equality to hold in Theorem 3. We remark that if G is a graph with no isolated vertex such that every $\gamma(G)$-set is a packing in G, then it is not necessarily true that $\gamma_{tdR}(G) = 4\gamma(G)$. For example, for $t \geq 4$, suppose that G_t is obtained from a star S_t with center v by subdividing $t - 2$ edges twice and subdividing the remaining edge exactly once. One can check that $\gamma(G_t) = t - 1$ and the set of $t - 1$ support vertices of G_t forms the unique $\gamma(G_t)$-set, which is also a packing in G_t.

However, the function $f : V(G_t) \to \{0,1,2,3\}$ that assigns the value 1 to every leaf, the value 2 to v and to every support vertex, and the value 0 to the remaining vertices of G_t, is a TDRDF on G_t and hence $\gamma_{tdR}(G_t) \leq \omega(f) = 3t - 1 < 4\gamma(G_t)$.

Proposition 1 ([5]). *For any graph G, there exists a $\gamma_{dR}(G)$-function such that no vertex needs to be assigned the value 1.*

Theorem 4. *For any graph G of order n with no isolated vertex,*

$$\gamma_{tdR}(G) \leq 3\gamma_{dR}(G)/2 - 1/2$$

with equality if and only if $\Delta = n - 1$ $(n \geq 3)$.

Proof. If G has two vertices, then the assertion is trivial. So in the following we may assume that G has at least three vertices. By Proposition 1, we can assume that $f = (V_0, V_1, V_2, V_3)$ is a $\gamma_{dR}(G)$-function, where $V_1 = \emptyset$. If there exists some vertex, say v, in $V_2 \cup V_3$ such that $N(v) \cap V_0 = \emptyset$, then $N(v) \subseteq V_2 \cup V_3$ and so the function g defined by $g(v) = 1$ and $g(x) = f(x)$ for each $x \in V(G)\backslash\{v\}$, is a DRDF on G with weight at most $\omega(f) - 1$, a contradiction. Therefore, we have that for $x \in V_2 \cup V_3$, $N(x) \cap V_0 \neq \emptyset$.

Suppose first that V_3 dominates V_0 and $V_2 = \emptyset$. For each vertex $x \in V_3$, we select an arbitrary neighbor, say x', of x in V_0 and let $X = \bigcup_{x \in V_3}\{x'\}$. If $|V_3| \geq 2$, that is, if $\gamma_{dR}(G) \geq 6$, then the function h_1 defined by $h_1(x) = 1$ for each $x \in X$ and $h_1(x) = f(x)$ otherwise, is a TDRDF on G and so

$$\gamma_{tdR}(G) \leq \omega(h_1) = 3|V_3| + |X| \leq 4|V_3| = \frac{4}{3}\gamma_{dR}(G) \leq \frac{3}{2}\gamma_{dR}(G) - 1.$$

If $|V_3| = 1$, then clearly $\Delta = n - 1$ and hence one can verify that $\gamma_{tdR}(G) = 4$ and $\gamma_{dR}(G) = 3$, implying that $\gamma_{tdR}(G) = \frac{3}{2}\gamma_{dR}(G) - \frac{1}{2}$.

Suppose now that V_3 dominates V_0 and $V_2 \neq \emptyset$. As shown earlier, $N(x) \cap V_0 \neq \emptyset$ for $x \in V_2 \cup V_3$. Hence we may assume that there exist $u \in V_3$ and $v \in V_2$ such that u and v have a common neighbor w in V_0. For each vertex $x \in V_2\backslash\{v\}$, we select an arbitrary neighbor, say x', of x in V_0 and let $A = \bigcup_{x \in V_2\backslash\{v\}}\{x'\}$. Similarly, for each vertex $x \in V_3\backslash\{u\}$, we select an arbitrary neighbor, say x', of x in V_0 and let $B = \bigcup_{x \in V_3\backslash\{u\}}\{x'\}$. Observe that $|A| \leq |V_2\backslash\{v\}| = |V_2| - 1$ and $|B| \leq |V_3\backslash\{u\}| = |V_3| - 1$. Then the function h_2 defined by $h_2(x) = 1$ for each $x \in A \cup B \cup \{w\}$ and $h_2(x) = f(x)$ otherwise, is a TDRDF on G and so

$$\begin{aligned}
\gamma_{tdR}(G) &\leq 3|V_3| + 2|V_2| + |A \cup B \cup \{w\}| \\
&\leq 3|V_3| + 2|V_2| + (|V_3| + |V_2| - 1) \\
&\leq 3(3|V_3| + 2|V_2|)/2 - 1 \\
&= 3\gamma_{dR}(G)/2 - 1.
\end{aligned}$$

Suppose next that V_3 does not dominate V_0. This implies that there exist two distinct vertices $u, v \in V_2$ and $w \in V_0$ such that w is adjacent to u and v. As shown earlier, $N(x) \cap V_0 \neq \emptyset$ for each $x \in V_2 \cup V_3$. For each vertex $x \in V_2 \backslash \{u, v\}$, we select an arbitrary neighbor, say x', of x in V_0 and let $C = \bigcup_{x \in V_2 \backslash \{u,v\}} \{x'\}$. Similarly, for each vertex $x \in V_3$, we select an arbitrary neighbor, say x', of x in V_0 and let $D = \bigcup_{x \in V_3} \{x'\}$. Observe that $|C| \leq |V_2 \backslash \{u, v\}| = |V_2| - 2$ and $|D| \leq |V_3|$. Then the function h_3 defined by $h_3(x) = 1$ for each $x \in C \cup D \cup \{w\}$ and $h_3(x) = f(x)$ otherwise, is a TDRDF on G and so

$$\begin{aligned}
\gamma_{tdR}(G) &\leq 3|V_3| + 2|V_2| + |C \cup D \cup \{w\}| \\
&\leq 3|V_3| + 2|V_2| + (|V_3| + |V_2| - 1) \\
&\leq 3(3|V_3| + 2|V_2|)/2 - 1 \\
&= 3\gamma_{dR}(G)/2 - 1,
\end{aligned}$$

which completes our proof. □

Theorem 5. *For any connected graph G of order $n \geq 2$,*

$$\gamma_{tR}(G) + 1 \leq \gamma_{tdR}(G) \leq 2\gamma_{tR}(G) - 1.$$

Further, the following holds.

(a) *The lower bound is sharp for any graph with $\Delta = n - 1$.*

(b) *$\gamma_{tdR}(G) = 2\gamma_{tR}(G) - 1$ if and only if $G = P_2$.*

Proof. To show the lower bound, let $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{tdR}(G)$-function. If $V_3 \neq \emptyset$, then every vertex in V_3 can be reassigned the value 2 and the resulting function is a TRDF on G and hence

$$\gamma_{tdR}(G) = \big(2(|V_2| + |V_3|) + |V_1|\big) + |V_3| \geq \gamma_{tR}(G) + |V_3| \geq \gamma_{tR}(G) + 1.$$

Assume now that $V_3 = \emptyset$. We conclude from the definition of a $\gamma_{tdR}(G)$-function that $V_2 \neq \emptyset$. Assume first that $V_0 \neq \emptyset$. Note that all vertices in V_0 must have at least two neighbors in V_2. In this case one vertex in V_2 can be reassigned the value 1 and the resulting function is a TRDF on G and hence

$$\gamma_{tdR}(G) = 2|V_2| + |V_1| = \big(2(|V_2| - 1) + (|V_1| + 1)\big) + 1 \geq \gamma_{tR}(G) + 1.$$

Assume second that $V_0 = \emptyset$. In this case all vertices in V_2 can be reassigned the value 1 and the resulting function is a TRDF on G and so

$$\gamma_{tdR}(G) = 2|V_2| + |V_1| = (|V_2| + |V_1|) + |V_2| \geq \gamma_{tR}(G) + 1.$$

It remains for us to show that $\gamma_{tdR}(G) \leq 2\gamma_{tR}(G) - 1$ with equality if and only if $G = P_2$. If G has two vertices, that is, if $G = P_2$, then the assertion is trivial. So in the following we may assume that G has at least three vertices. Let g be a $\gamma_{tR}(G)$-function such that the number of vertices assigned 0 under g is maximum and let $V_i' = \{v \in V(G) : g(v) = i\}$ for $i \in \{0, 1, 2\}$.

We now claim that $V_2' \neq \emptyset$. Suppose, to the contrary, that $V_2' = \emptyset$. This implies that $V_0' = \emptyset$ and $V_1' = V(G)$. Thus we have $\gamma_{tR}(G) = |V_1'| = n$. Let $P = v_1v_2\cdots v_d$ be a longest path of G. One can check that the function h_1 defined by $h_1(v_1) = 0$, $h_1(v_2) = 2$ and $h_1(x) = 1$ otherwise, is also a $\gamma_{tR}(G)$-function, a contradiction to the choice of g. Consequently, we conclude that $V_2' \neq \emptyset$.

Assume that $|V_2'| = 1$. Let $V_2' = \{v\}$. Since g is a $\gamma_{tR}(G)$-function, there exists some vertex $u \in N(v) \cap V_1'$. One can verify that the function h_2 defined by $h_2(v) = 3$, $h_2(x) = 2$ for $x \in V_1' \backslash \{u\}$ and $h_2(x) = f(x)$ otherwise, is a TDRDF on G and hence $\gamma_{tdR}(G) \leq 2(|V_1'| - 1) + 4 = 2(|V_1'| + 2|V_2'|) - 2 = 2\gamma_{tR}(G) - 2$.

Assume now that $|V_2'| \geq 2$. Then every vertex in V_2' is reassigned the value 3 and every vertex in V_1' is reassigned the value 2 and the resulting function is a TDRDF on G and hence $\gamma_{tdR}(G) \leq 3|V_2'| + 2|V_1'| = 2(|V_1'| + 2|V_2'|) - |V_2'| \leq 2\gamma_{tR}(G) - 2$, which completes our proof. □

3. Paths, cycles and complete r-partite graphs

In this section, we determine the total double Roman domination numbers of paths, cycles and complete r-partite graphs.

Lemma 1. *Let f be a $\gamma_{tdR}(C_n)$-function such that the number of vertices assigned 0 under f is minimum. Then $f(x) \neq 3$ for each $x \in V(C_n)$.*

Proof. Let $C_n = v_0v_1\cdots v_{n-1}v_0$. Suppose, to the contrary, that there exists some vertex, say v_0, of C_n such that $f(v_0) = 3$. If $f(v_{n-1}) \geq 1$ and $f(v_1) \geq 1$, then clearly the function g defined by $g(v_0) = 2$ and $g(v_i) = f(v_i)$ otherwise, is a TDRDF on C_n with weight $\omega(f) - 1$, a contradiction. Thus exactly one of v_{n-1} and v_1 is assigned 0 under f since f is a $\gamma_{tdR}(C_n)$-function. Without loss of generality, assume that $f(v_{n-1}) \geq 1$ and $f(v_1) = 0$. Then the function h defined by $h(v_0) = 2$, $h(v_1) = 1$ and $h(v_i) = f(v_i)$ otherwise, is a TDRDF on C_n with weight $\omega(f)$ and so h is also a $\gamma_{tdR}(C_n)$-function, a contradiction to our assumption that the number of vertices assigned 0 under f is minimum. Consequently, we have that for $i \in \{0, 1, \ldots, n-1\}$, $f(v_i) \neq 3$. □

Lemma 2. *Let f be a $\gamma_{tdR}(C_n)$-function such that the number of vertices assigned 0 under f is minimum and let u, v and w be three consecutive vertices of C_n. If $f(u) = 0$ and $f(v) = 2$, then $f(w) = 1$.*

Proof. Let u' be the unique neighbor of u different from v. It follows from Lemma 1 that $f(x) \in \{0,1,2\}$ for each $x \in V(C_n)$. Moreover, since $f(u) = 0$, we have $f(u') = 2$ and $f(w) \in \{1,2\}$. If $f(w) = 2$, then the function g defined by $g(u) = g(v) = 1$ and $g(x) = f(x)$ otherwise, is also a $\gamma_{tdR}(C_n)$-function, a contradiction to the choice of f. Therefore, we have $f(w) = 1$. $\square$

Proposition 2. *For $n \geq 3$,*

$$\gamma_{tdR}(C_n) = \left\lceil \frac{6n}{5} \right\rceil.$$

Proof. If $n = 3$, then the result is trivial. Hence we may assume that $n \geq 4$. Let $C_n = v_0 v_1 v_2 \cdots v_{n-1} v_0$ and let f be a $\gamma_{tdR}(C_n)$-function such that the number of vertices assigned 0 under f is minimum. It follows from Lemma 1 that $f(v_i) \in \{0,1,2\}$ for each $i \in \{0,1,\ldots,n-1\}$.
We now claim that $\gamma_{tdR}(C_n) \geq \lceil \frac{6n}{5} \rceil$. Assume firstly that $f(v_i) \in \{1,2\}$ for each $i \in \{0,1,\ldots,n-1\}$. Note that there do not exist three consecutive vertices of C_n assigned 1 under f. This implies that $f(v_i) + f(v_{i+1}) + f(v_{i+2}) \geq 4$ for each $i \in \{0,1,\ldots,n-1\}$, where the indices are taken modulo n. Let $n = 3s + t \geq 4$, where s is a positive integer and $t \in \{0,1,2\}$. Without loss of generality, we may assume that $f(v_{n-1}) = 2$. Therefore, if $t = 0$, then

$$\begin{aligned}
\gamma_{tdR}(C_n) &= \sum_{k=0}^{s-1} (f(v_{3k}) + f(v_{3k+1}) + f(v_{3k+2})) \\
&\geq 4s \geq \left\lceil \frac{6n}{5} \right\rceil,
\end{aligned}$$

if $t = 1$, then

$$\begin{aligned}
\gamma_{tdR}(C_n) &= f(v_{n-1}) + \sum_{k=0}^{s-1} (f(v_{3k}) + f(v_{3k+1}) + f(v_{3k+2})) \\
&\geq 4s + 2 > \left\lceil \frac{6n}{5} \right\rceil,
\end{aligned}$$

and if $t = 2$, then

$$\begin{aligned}
\gamma_{tdR}(C_n) &= (f(v_{n-2}) + f(v_{n-1})) + \sum_{k=0}^{s-1} (f(v_{3k}) + f(v_{3k+1}) + f(v_{3k+2})) \\
&\geq 4s + 3 > \left\lceil \frac{6n}{5} \right\rceil.
\end{aligned}$$

So in the following we may assume that there exists some vertex, say v_0, of C_n such that $f(v_0) = 0$. Clearly $f(v_1) = f(v_{n-1}) = 2$. It follows from Lemma 2 that $f(v_2) =$

$f(v_{n-2}) = 1$. This implies that if $n = 4$, then $\gamma_{tdR}(C_4) = \sum_{i=0}^{3} f(v_i) = 5 = \lceil \frac{6n}{5} \rceil$ and if $n = 5$, then $\gamma_{tdR}(C_5) = \sum_{i=0}^{4} f(v_i) = 6 = \lceil \frac{6n}{5} \rceil$. Since $f(v_2) = f(v_{n-2}) = 1$, we have $f(v_3) \in \{1, 2\}$ and $f(v_{n-3}) \in \{1, 2\}$. Moreover, since there do not exist three consecutive vertices of C_n assigned 1 under f, we conclude that if $n = 6$, then $f(v_3) = 2$ and so $\gamma_{tdR}(C_6) = \sum_{i=0}^{5} f(v_i) = 8 = \lceil \frac{6n}{5} \rceil$; and if $n = 7$, then at least one of $f(v_3)$ and $f(v_4)$ is equal to 2, implying that $f(v_3) + f(v_4) \geq 3$, and so $\gamma_{tdR}(C_7) = \sum_{i=0}^{6} f(v_i) \geq 9 = \lceil \frac{6n}{5} \rceil$. Hence we may assume that $n \geq 8$. Recall that $f(v_3) \in \{1, 2\}$.

Suppose that $f(v_3) = 1$. Clearly $f(v_4) = 2$. Let C_{n-5} be the cycle of length $n - 5$ obtained from $C_n = v_0v_1v_2 \cdots v_{n-1}v_0$ by deleting the five vertices v_0, v_1, v_2, v_3 and v_{n-1} and adding a new edge v_4v_{n-2}. Thus the function f restricted to $V(C_n)\backslash\{v_0, v_1, v_2, v_3, v_{n-1}\}$ is obviously a TDRDF on C_{n-5}, implying that $\gamma_{tdR}(C_n) = \omega(f) \geq \gamma_{tdR}(C_{n-5}) + 6$.

Suppose next that $f(v_3) = 2$. Since $f(v_{n-2}) = 1$, we obtain $f(v_{n-3}) \in \{1, 2\}$. Let C_{n-5} be the cycle of length $n - 5$ obtained from $C_n = v_0v_1v_2 \cdots v_{n-1}v_0$ by deleting the five vertices v_0, v_1, v_2, v_{n-2} and v_{n-1} and adding a new edge v_3v_{n-3}. Then the function f restricted to $V(C_n)\backslash\{v_0, v_1, v_2, v_{n-2}, v_{n-1}\}$ is obviously a TDRDF on C_{n-5}, implying that $\gamma_{tdR}(C_n) = \omega(f) \geq \gamma_{tdR}(C_{n-5}) + 6$.

Consequently, we obtain that if $n = 5k + l \geq 8$, where k is a positive integer and $l \in \{0, 1, 2, 3, 4\}$, then $\gamma_{tdR}(C_n) \geq \gamma_{tdR}(C_{n-5}) + 6$ and hence

$$\gamma_{tdR}(C_n) \geq \gamma_{tdR}(C_{n-5}) + 6 \geq \gamma_{tdR}(C_{n-10}) + 12 \geq \cdots,$$

implying that if $l \in \{0, 1, 2\}$, then

$$\gamma_{tdR}(C_n) \geq \gamma_{tdR}(C_{5+l}) + 6(k - 1) \geq \left\lceil \frac{6(5 + l)}{5} \right\rceil + 6(k - 1) = \left\lceil \frac{6n}{5} \right\rceil,$$

and if $l \in \{3, 4\}$, then

$$\gamma_{tdR}(C_n) \geq \gamma_{tdR}(C_l) + 6k \geq \left\lceil \frac{6l}{5} \right\rceil + 6k = \left\lceil \frac{6n}{5} \right\rceil.$$

It remains for us to show that $\gamma_{tdR}(C_n) \leq \lceil \frac{6n}{5} \rceil$. Let $n = 5k + l \geq 4$, where k is a nonnegative integer and $l \in \{0, 1, 2, 3, 4\}$. If $l \in \{0, 1\}$, then the function f defined by

$$f(v_i) = \begin{cases} 1, & \text{if } i \in \{5j, 5j + 4\} \text{ for } 0 \leq j \leq k - 1, \\ 0, & \text{if } i = 5j + 2 \text{ for } 0 \leq j \leq k - 1, \\ 2, & \text{otherwise}, \end{cases}$$

is a TDRDF on C_n and hence

$$\gamma_{tdR}(C_n) \leq 2k + 2(n - 3k) = 6k + 2l = \left\lceil \frac{6n}{5} \right\rceil,$$

if $l \in \{2,3\}$, then the function f defined by

$$f(v_i) = \begin{cases} 2, & \text{if } i \in \{5j+1, 5j+3\} \text{ for } 0 \le j \le k, \\ 0, & \text{if } i = 5j+2 \text{ for } 0 \le j \le k-1, \\ 1, & \text{otherwise,} \end{cases}$$

is a TDRDF on C_n and hence

$$\gamma_{tdR}(C_n) \le 2(2k+1) + (n-3k-1) = 6k+l+1 = \left\lceil \frac{6n}{5} \right\rceil,$$

and if $l = 4$, then the function f defined by

$$f(v_i) = \begin{cases} 2, & \text{if } i \in \{5j+1, 5j+3\} \text{ for } 0 \le j \le k, \\ 0, & \text{if } i = 5j+2 \text{ for } 0 \le j \le k, \\ 1, & \text{otherwise,} \end{cases}$$

is a TDRDF on C_n and hence

$$\gamma_{tdR}(C_n) \le 2(2k+2) + (n-3k-3) = 6k+l+1 = \left\lceil \frac{6n}{5} \right\rceil,$$

which completes our proof. □

Proposition 3. *For $n \ge 2$,*

$$\gamma_{tdR}(P_n) = \begin{cases} 6, & \text{if } n = 4, \\ \lceil \frac{6n}{5} \rceil, & \text{otherwise.} \end{cases}$$

Proof. If $n \in \{2,3,4\}$, then the result is trivial. Hence we may assume that $n = 5k+l \ge 5$, where k is a positive integer and $l \in \{0,1,2,3,4\}$. We conclude from Proposition 2 that $\gamma_{tdR}(P_n) \ge \gamma_{tdR}(C_n) = \lceil \frac{6n}{5} \rceil$. It remains for us to show that $\gamma_{tdR}(P_n) \le \lceil \frac{6n}{5} \rceil$. Let $P_n = v_1 v_2 \cdots v_n$. If $l \in \{0,1\}$, then the function f defined by

$$f(v_i) = \begin{cases} 1, & \text{if } i \in \{5j+1, 5j+5\} \text{ for } 0 \le j \le k-1, \\ 0, & \text{if } i = 5j+3 \text{ for } 0 \le j \le k-1, \\ 2, & \text{otherwise,} \end{cases}$$

is a TDRDF on P_n and hence

$$\gamma_{tdR}(P_n) \le 2k + 2(n-3k) = 6k+2l = \left\lceil \frac{6n}{5} \right\rceil,$$

if $l \in \{2, 3\}$, then the function f defined by

$$f(v_i) = \begin{cases} 2, & \text{if } i = 5j + 2 \; when \; 0 < i \le k \; and \; i = 5j + 4 \; when \; 0 \le k \le j - 1, \\ 0, & \text{if } i = 5j + 3 \text{ for } 0 \le j \le k - 1, \\ 1, & \text{otherwise,} \end{cases}$$

is a TDRDF on P_n and hence

$$\gamma_{tdR}(P_n) \le 2(2k + 1) + (n - 3k - 1) = 6k + l + 1 = \left\lceil \frac{6n}{5} \right\rceil,$$

and if $l = 4$, then the function f defined by

$$f(v_i) = \begin{cases} 2, & \text{if } i \in \{5j + 2, 5j + 4\} \text{ for } 0 \le j \le k - 1, \\ & \text{or } i \in \{n - 3, n - 1\}, \\ 0, & \text{if } i = 5j + 3 \text{ for } 0 \le j \le k - 1, \text{or } i = n - 2, \\ 1, & \text{otherwise,} \end{cases}$$

is a TDRDF on P_n and hence

$$\gamma_{tdR}(P_n) \le 2(2k + 2) + (n - 3k - 3) = 6k + 5 = \left\lceil \frac{6n}{5} \right\rceil,$$

which completes our proof. □

In order to determine the total double Roman domination number of the complete r-partite graph, we first give a proposition as follows.

Proposition 4. *For any graph G of order $n \ge 3$ with no isolated vertex, $\gamma_{tdR}(G) = 4$ if and only if $\Delta = n - 1$.*

Proof. The sufficiency is trivial. Now we show the necessity. Let $\gamma_{tdR}(G) = 4$ and let $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{tdR}(G)$-function. Then $\gamma_{tdR}(G) = |V_1| + 2|V_2| + 3|V_3| = 4$. Thus we have that one of the following holds:
(1) $|V_1| = |V_3| = 1$ and $|V_2| = 0$,
(2) $|V_1| = 2$, $|V_2| = 1$ and $|V_3| = 0$,
(3) $|V_1| = |V_3| = 0$ and $|V_2| = 2$.
Suppose that (1) holds and let $V_3 = \{v\}$. Then we have that each vertex of $V(G) \setminus \{v\}$ is adjacent to v. This implies that $\Delta = n - 1$.
Suppose that (2) holds. Obviously $|V_0| = 0$ and so $n = |V_0| + |V_1| + |V_2| + |V_3| = 3$, implying that $\Delta = n - 1$.
Suppose that (3) holds. Let $V_2 = \{u, v\}$. We observe that u is adjacent to v and each vertex of $V(G) \setminus \{u, v\}$ is adjacent to both of u and v. Thus $\Delta = n - 1$. □

Proposition 5. *Let $G = K_{n_1,n_2,\ldots,n_r}$ be the complete r-partite graph with $r \geq 2$ and $1 \leq n_1 \leq n_2 \leq \ldots \leq n_r$ of order $n = n_1 + n_2 + \ldots + n_r \geq 3$.*

(a) *If $n_1 = 1$, then $\gamma_{tdR}(G) = 4$.*

(b) *If $n_1 = 2$, then $\gamma_{tdR}(G) = 5$.*

(c) *If $n_1 \geq 3$, then $\gamma_{tdR}(G) = 6$.*

Proof. Let $X_1, X_2, \ldots, X_r$ be the partite sets of G. Case (a) is clear by Proposition 4. Since $\Delta(G) \leq n-2$ in the Cases (b) and (c), Proposition 4 implies $\gamma_{tdR}(G) \geq 5$ in these cases.

Let $n_1 = 2$, and let $X_1 = \{u, v\}$. Define the function f by $f(u) = f(v) = 2$, $f(z) = 1$ for an arbitrary vertex different from u and v and $f(x) = 0$ otherwise. Then f is a TDRDF on G and hence $\gamma_{tdR}(G) \leq 5$. Thus $\gamma_{tdR}(G) = 5$ in Case (b).

Let now $n_1 \geq 3$. Suppose to the contrary that $\gamma_{tdR}(G) = 5$ and let $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{tdR}(G)$-function. Then $\gamma_{tdR}(G) = |V_1| + 2|V_2| + 3|V_3| = 5$. Thus we have that one of the following holds:

(1) $|V_3| = |V_2| = 1$ and $|V_1| = 0$,

(2) $|V_3| = 1$, $|V_1| = 2$ and $|V_2| = 0$,

(3) $|V_2| = 2$, $|V_1| = 1$ and $|V_3| = 0$,

(4) $|V_2| = 1$, $|V_1| = 3$ and $|V_3| = 0$.

Suppose that (1) or (2) holds and let $V_3 = \{v\}$. Then each vertex of $V(G)\backslash\{v\}$ is adjacent to v, a contradiction.

Suppose that (3) holds and let $V_2 = \{u, v\}$ and $V_1 = \{z\}$. Then each vertex of $V(G)\backslash\{u, v, z\}$ is adjacent to u and v and z is adjacent to u or v. This yields the contradiction $\Delta(G) \geq n-2$. If (4) holds, then we obtain the contradiction $n = 4$.

Therefore $\gamma_{tdR}(G) \geq 6$. On the other hand let $u \in X_1$ and $v \in X_2$, and define the function g by $g(u) = g(v) = 3$ and $g(x) = 0$ otherwise. Then g is a TDRDF on G and hence $\gamma_{tdR}(G) \leq 6$. Thus $\gamma_{tdR}(G) = 6$ in case (c). □

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