A note on the Roman domatic number of a digraph

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Abstract: A Roman dominating function on a digraph $D$ with vertex set $V(D)$ is a labeling $f: V(D) \rightarrow \{0, 1, 2\}$ such that every vertex with label 0 has an in-neighbor with label 2. A set $\{f_1, f_2, \ldots, f_d\}$ of Roman dominating functions on $D$ with the property that $\sum_{i=1}^{d} f_i(v) \leq 2$ for each $v \in V(D)$, is called a Roman dominating family (of functions) on $D$. The maximum number of functions in a Roman dominating family on $D$ is the Roman domatic number of $D$, denoted by $d_R(D)$. In this note, we study the Roman domatic number in digraphs, and we present some sharp bounds for $d_R(D)$. In addition, we determine the Roman domatic number of some digraphs. Some of our results are extensions of well-known properties of the Roman domatic number of undirected graphs.

Keywords: Digraphs, Roman dominating function, Roman domination number, Roman domatic number

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1. Introduction

In this paper, $D$ is a simple digraph with vertex set $V = V(D)$ and arc set $A = A(D)$. The order $|V|$ of $D$ is denoted by $n = n(D)$. We write $d^+_D(v) = d^+(v)$ for the out-degree of a vertex $v$ and $d^-_D(v) = d^-(v)$ for its in-degree. The minimum and maximum in-degree and minimum and maximum out-degree of $D$ are denoted by $\delta^- = \delta^-(D)$, $\Delta^- = \Delta^-(D)$, $\delta^+ = \delta^+(D)$ and $\Delta^+ = \Delta^+(D)$, respectively. If $uv$ is an arc of $D$, then we also write $u \rightarrow v$, and we say that $v$ is an out-neighbor of $u$ and $u$ is an in-neighbor of $v$. For a vertex $v$ of a digraph $D$, we denote the set of in-neighbors and out-neighbors of $v$ by $N^-_D(v) = N^-(v)$ and $N^+_D(v) = N^+(v)$, respectively. In addition, $N^-[v] = N^-(v) \cup \{v\}$ and $N^+[v] = N^+(v) \cup \{v\}$. If $X \subset V(D)$, then

$$N^+_D[X] = N^+[X] = \bigcup_{v \in X} N^+[v].$$
We write $K_n^*$ for the complete digraph of order $n$. An oriented cycle in a digraph is also called a cycle. For notation and graph theory terminology in general we follow [4].

A subset $S \subseteq V(D)$ is a dominating set if $N^+[S] = V(D)$. The domination number $\gamma(D)$ is the minimum cardinality of a dominating set of $D$. The domination number for digraphs was introduced by Lee [8]. A domatic partition is a partition of $V(D)$ into dominating sets, and the domatic number $d(D)$ is the largest number of sets in a domatic partition.

A Roman dominating function (RDF) on a digraph $D$ is defined in [7, 12] as a function $f: V(D) \to \{0, 1, 2\}$ satisfying the condition that every vertex $v$ with $f(v) = 0$ has an in-neighbor $u$ with $f(u) = 2$. The weight of an RDF $f$ is the value $\omega(f) = \sum_{v \in V(D)} f(v)$. The Roman domination number of a digraph $D$, denoted by $\gamma_R(D)$, is the minimum taken over the weights of all Roman dominating functions on $D$. A $\gamma_R(D)$-function is a Roman dominating function on $D$ with weight $\gamma_R(D)$. A Roman dominating function $f: V \to \{0, 1, 2\}$ can be represented by the ordered partition $(V_0, V_1, V_2)$ (or $(V_0^f, V_1^f, V_2^f)$ to refer to $f$) of $V(D)$, where $V_i = \{v \in V(D): f(v) = i\}$. In this representation, the weight of $f$ is $\omega(f) = |V_1| + 2|V_2|$. Since $V_1^f \cup V_2^f$ is a dominating set when $f$ is an RDF, and since placing weight 2 at the vertices of a dominating set yields an RDF, we observe that

$$\gamma(D) \leq \gamma_R(D) \leq 2\gamma(D).$$

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct Roman dominating functions on $D$ with $\sum_{i=1}^d f_i(v) \leq 2$ for each $v \in V(D)$, is called in [14] a Roman dominating family (of functions) on $D$. The maximum number of functions in a Roman dominating family (RD family) on $D$ is the Roman domatic number of $D$, denoted by $d_R(D)$. The Roman domatic number is well-defined and $d_R(D) \geq 1$ for all digraphs $D$ since the set consisting of any RDF forms an RD family on $D$ (see [14]).

Our purpose in this paper is to improve some results given in [14], and to determine the Roman domatic number of some classes of digraphs. Some of our results are extensions of well-known properties of the Roman domatic number $d_R(G)$ of graphs $G$.

The following known results are interesting.

**Theorem 1 ([14]).** If $D$ is a digraph of order $n$, then $\gamma_R(D) \cdot d_R(D) \leq 2n$.

**Theorem 2 ([14]).** If $D$ is a digraph of order $n \geq 2$, then $\gamma_R(D) + d_R(D) \leq n + 2$.

### 2. Roman domatic number of digraphs

The following upper bound on the Roman domatic number can be found in [14].
Theorem 3. For every digraph $D$,
\[
d_R(D) \leq \delta^-(D) + 2
\]
and this bound is sharp.

We will prove the following extension of Theorem 3.

Theorem 4. For every digraph $D$,
\[
d_R(D) \leq \delta^-(D) + 2.
\]
Moreover, if $d_R(D) = \delta^-(D) + 2$, then the set of vertices of minimum in-degree is an independent set.

Proof. If $d_R(D) \leq 2$, then the bound is immediate. Let now $d_R(D) \geq 3$ and let \{${f_1, f_2, \ldots, f_d}$\} be an RD family on $D$ such that $d = d_R(D)$. Assume that $v$ is a vertex of minimum in-degree $\delta^-(D)$. Since the equality $\sum_{x \in N^-[v]} f_i(x) = 1$ holds for at most two indices $i \in \{1, 2, \ldots, d\}$, we have
\[
2d - 2 \leq \sum_{i=1}^{d} \sum_{x \in N^-[v]} f_i(x) = \sum_{x \in N^-[v]} \sum_{i=1}^{d} f_i(x) \leq \sum_{x \in N^-[v]} 2 = 2(\delta^-(D) + 1).
\]
This implies the desired bound $d_R(D) \leq \delta^-(D) + 2$. Moreover, if $d_R(D) = \delta^-(D) + 2$, then the following holds for every vertex $v$ of minimum in-degree.

(i) There exist precisely two indices $j, k \in \{1, 2, \ldots, d\}$ such that
\[
\sum_{x \in N^-[v]} f_j(x) = \sum_{x \in N^-[v]} f_k(x) = 1;
\]
(ii) For every index $i \in \{1, 2, \ldots, d\} \setminus \{j, k\}$: $\sum_{x \in N^-[v]} f_i(x) = 2$;
(iii) For every vertex $x \in N^-[v]$ and every index $i \in \{1, 2, \ldots, d\}$: $\sum_{i=1}^{d} f_i(x) = 2$.

In particular, (i) implies that $f_j(v) = f_k(v) = 1$ and $f_j(x) = f_k(x) = 0$ for every $x \in N^-[v)$, (ii) implies that for every index $i \in \{1, 2, \ldots, d\} \setminus \{j, k\}$ there exists a unique vertex $x_i \in N^-[v)$ such that $f_i(x_i) = 2$, and (iii) implies that $f_i(y) = 0$ for every vertex $x_i \neq y \in N^-[v]$. This is impossible if $D$ contains two adjacent vertices of minimum in-degree.

The complement $\overline{D}$ of a digraph $D$ is the digraph with vertex set $V(D)$ such that for any two distinct vertices $u, v$ the arc $uv$ belongs to $\overline{D}$ if and only if $uv$ does not belong to $D$. A digraph $D$ is in-regular when $\delta^-(D) = \Delta^-(D)$ and out-regular when $\delta^+(D) = \Delta^+(D)$. Xie, Hao and Wei [14] proved the following Nordhaus-Gaddum type inequality.
Theorem 5. If \( D \) is a digraph of order \( n \geq 2 \), then

\[
d_r(D) + d_r(D) \leq n + \epsilon
\]

with \( \epsilon = 1 \) when \( D \) is out-regular, \( \epsilon = 2 \) when \( D \) is not in-regular and \( \epsilon = 3 \) otherwise.

As an application of Theorem 4, we shall prove the following improvement of Theorem 5.

Theorem 6. If \( D \) is a digraph of order \( n \), then

\[
d_r(D) + d_r(D) \leq n + 1.
\]

Proof. If, without loss of generality, \( D = K^*_n \), then \( d_r(D) = n \) and \( d_r(D) = 1 \) and the inequality holds. So assume that \( D \neq K^*_n \) and \( D \neq K^*_n \).

If \( D \) and \( \bar{D} \) contain adjacent vertices of minimum in-degree, then let \( x \) be a vertex such that \( d_\bar{D}(x) = \delta^-(D) \). By Theorem 4, it follows that

\[
d_r(D) + d_r(D) \leq (\delta^-(D) + 1) + (\delta^-(\bar{D}) + 1) \leq d_\bar{D}(x) + d_\bar{D}(x) + 2 = n + 1.
\]

If, without loss of generality, \( \bar{D} \) contains adjacent vertices of minimum in-degree and \( D \) does not contain adjacent vertices of minimum in-degree, then \( D \) is not in-regular and thus \( \delta^-(D) < \Delta^-(D) \). Let \( x \) and \( y \) be two vertices such that \( d_\bar{D}(x) = \delta^-(D) \) and \( d_\bar{D}(y) = \Delta^-(D) \). Using Theorem 3, we obtain

\[
d_r(D) + d_r(D) \leq (\delta^-(D) + 2) + (\delta^-(\bar{D}) + 1) \leq d_\bar{D}(x) + d_\bar{D}(y) + 3 < d_\bar{D}(y) + d_\bar{D}(y) + 3 = n + 2
\]

and thus \( d_r(D) + d_r(D) \leq n + 1 \).

It remains to check the case that neither \( D \) nor \( \bar{D} \) contain adjacent vertices of minimum in-degree. Then both digraphs are not in-regular. First we assume that \( \delta^-(D) \leq \Delta^-(D) - 2 \) or \( \delta^-(\bar{D}) \leq \Delta^-(\bar{D}) - 2 \), say \( \delta^-(D) \leq \Delta^-(D) - 2 \). Then let \( x \) and \( y \) be two vertices such that \( d_\bar{D}(x) = \delta^-(D) \) and \( d_\bar{D}(y) = \Delta^-(D) \). Again by Theorem 4, it follows that

\[
d_r(D) + d_r(D) \leq (\delta^-(D) + 2) + (\delta^-(\bar{D}) + 2) \leq d_\bar{D}(x) + d_\bar{D}(y) + 4 < d_\bar{D}(y) + d_\bar{D}(y) + 2 = n + 1.
\]

So assume that \( \delta^-(D) = \Delta^-(D) - 1 \) and \( \delta^-(\bar{D}) = \Delta^-(\bar{D}) - 1 \). If \( d_r(D) = \delta^-(D) + 2 \) and \( d_r(D) = \delta^-(\bar{D}) + 2 \), then let \( A = \{v: d_\bar{D}(v) = \delta^-(D)\} \) and \( B = \{v: d_\bar{D}(v) = \delta^-(\bar{D})\} \). Note that \( A \cup B = V(D) \). Since \( A \) is independent in \( D \), it follows that \( \bar{D}[A] \)
is complete. Likewise, $B$ is independent in $\overline{D}$ and $D[B]$ is complete. Let $d = d_R(D)$ and $\{f_1, f_2, \ldots, f_d\}$ be an RD family on $D$. Note that $f_i(a) \leq 1$ for every vertex $a \in A$ and thus, $f_i(b) = 2$ for at least one vertex $b \in B$ for every index $i$. It follows that $d \leq |B| \leq \delta^-(D) + 1$, a contradiction. Hence, without loss of generality, $d_R(D) \leq \delta^-(D) + 1$, and thus

$$d_R(D) + d_R(\overline{D}) \leq (\delta^-(D) + 1) + (\delta^-(\overline{D}) + 2) \leq d_D(x) + d_{\overline{D}}(y) + 3$$

$$\leq d_D(y) + d_{\overline{D}}(y) + 2 = n + 1.$$

This completes the proof.

If $D$ is isomorphic to the complete graph $K_n^*$, then $d_R(D) = n$ and $d_R(\overline{D}) = 1$ and therefore $d_R(D) + d_R(\overline{D}) = n + 1$. This example demonstrates that the bound (1) is sharp.

For out-regular graphs, the following upper bound on the Roman domatic number is valid (see [14]).

**Theorem 7.** If $D$ is a $\delta$-out-regular digraph of order $n = p(\delta + 1) + r$ with integers $p \geq 1$ and $0 \leq r \leq \delta$, then

$$d_R(D) \leq \delta + \varepsilon$$

with $\varepsilon = 1$ when $\delta = 0$ or $r = 0$ or $2r = \delta + 1$ and $\varepsilon = 0$ otherwise.

Let $q \geq 4$ be an integer, and let $K_{q,q}^*$ be the complete bipartite digraph with the partite sets $\{u_1, u_2, \ldots, u_q\}$ and $\{v_1, v_2, \ldots, v_q\}$. Define the RDFs $f_1, f_2, \ldots, f_q$ by $f_i(u_i) = f_i(v_i) = 2$ and $f_i(u_j) = f_i(v_j) = 0$ for $j \neq i$ and $1 \leq i, j \leq q$. Because of $\sum_{i=1}^{q} f_i(x) = 2$ for each vertex $x$ in $K_{q,q}^*$, the set $\{f_1, f_2, \ldots, f_q\}$ is an RD family on $K_{q,q}^*$. Thus $d_R(K_{q,q}^*) \geq q$ and therefore Theorem 7 implies that $d_R(K_{q,q}^*) = q$. This example demonstrates that Theorem 7 is sharp.

If $D$ is a digraph, then we denote by $D^{-1}$ the digraph obtained by reversing all arcs of $D$. A digraph without cycles of length 2 is called an oriented graph. An oriented graph $D$ is a tournament when either $xy \in A(D)$ or $yx \in A(D)$ for each pair of distinct vertices $x, y \in V(D)$.

For regular tournaments the following upper bound is valid.

**Theorem 8.** If $D$ is a $\delta$-regular tournament of order $n \geq 3$, then

$$d_R(D) + d_R(D^{-1}) \leq n - 1.$$

**Proof.** Since $D$ is a $\delta$-regular tournament, $D^{-1}$ is also a $\delta$-regular tournament of order $n = 2\delta + 1$. If $\delta \geq 2$, then Theorem 7 leads to

$$d_R(D) + d_R(D^{-1}) \leq 2\delta = n - 1.$$

Since this remains valid for $\delta = 1$, the proof is complete. \qed
The class of round regular tournaments shows that the bound in Theorem 8 is sharp. A tournament $D$ on $V = \{v_0, v_1, \ldots, v_{2p}\}$ is called round if for every vertex $v_i$ the out- and in-neighborhood are given by $N^+(v_i) = \{v_{i+1}, v_{i+2}, \ldots, v_{i+p}\}$ and $N^-(v_i) = \{v_{i-1}, v_{i-2}, \ldots, v_{i-p}\}$, where all indices are taken modulo $n = 2p + 1$. For $i = 1, 2, \ldots, p$, define $f_i: V \to \{0, 1, 2\}$ by $f_i(v_i) = f_i(v_{i+p+1}) = 2$ and $f_i(v_j) = 0$ for $j \notin \{i, i + p + 1\}$. Then $\{f_1, f_2, \ldots, f_p\}$ is an RD family on $D$. Since $D$ is isomorphic to $D^{-1}$, it follows that $d_R(D) \geq p$ and $d_R(D^{-1}) \geq p$ and thus $d_R(D) + d_R(D^{-1}) = 2p = n - 1$ by Theorem 8.

For arbitrary oriented digraphs we shall prove a slightly weaker upper bound.

**Theorem 9.** If $D$ is an oriented graph of order $n$, then

$$d_R(D) + d_R(D^{-1}) \leq n + 1.$$ 

**Proof.** For regular tournaments the bound is true by Theorem 8.

If $D$ is an almost-regular tournament, then $D^{-1}$ is also an almost-regular tournament and $\delta^-(D) = \delta^-(D^{-1}) = (n - 2)/2$. Furthermore, $D$ and $D^{-1}$ contain adjacent vertices of minimum in-degree. By Theorem 3, it follows that

$$d_R(D) + d_R(D^{-1}) \leq \delta^-(D) + 1 + \delta^-(D^{-1}) + 1 = n.$$ 

If $D$ is a tournament that is neither regular nor almost-regular, then $\delta^-(D) + \delta^-(D^{-1}) \leq n - 3$. Again Theorem 3 implies that

$$d_R(D) + d_R(D^{-1}) \leq \delta^-(D) + 2 + \delta^-(D^{-1}) + 2 \leq n + 1.$$ 

If $D$ is not a tournament, then it is a subdigraph of a tournament. Since every RD family on $D$ is also an RD family on every superdigraph $D'$ of $D$, it follows that $d_R(D) \leq d_R(D')$. Using this observation, it is immediate that $d_R(D) + d_R(D^{-1}) \leq n + 1$ for every oriented digraph $D$. This completes the proof. 

3. The Roman domatic number of a graph

A Roman dominating function on a graph $G = (V(G), E(G))$ is defined in [10, 13] as a function $f: V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex $v$ with $f(v) = 0$ is adjacent to at least one vertex $u$ with $f(u) = 2$. The weight of an RDF $f$ is the value $\omega(f) = \sum_{v \in V(G)} f(v)$. The Roman domination number of a graph $G$, denoted by $\gamma_R(G)$, is the minimum weight of an RDF on $G$. In [1–3, 5, 6, 9] the reader can find a lot of results on Roman domination.

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct RDFs on $G$ with $\sum_{i=1}^{d} f_i(v) \leq 2$ for each $v \in V(G)$, is called a Roman dominating family (of functions) on $G$. The maximum number of functions in an RD family on $G$ is the Roman domatic number of $G$, denoted by $d_R(G)$. 
The associated digraph $D(G)$ of a graph $G$ is the digraph obtained from $G$ by replacing each edge of $G$ by a 2-cycle. Since $N_{D(G)}[v] = N_G[v]$ for each vertex $v \in V(G) = V(D(G))$, the following useful observation is valid.

**Observation 10.** If $D(G)$ is the associated digraph of a graph $G$, then $\gamma_R(D(G)) = \gamma_R(G)$ and $d_R(D(G)) = d_R(G)$.

There are a lot of interesting applications of Observation 10, as for example the following results. Using Theorem 1, we obtain the first one.

**Corollary 1 ([11]).** If $G$ is a graph of order $n$, then $\gamma_R(G) \cdot d_R(G) \leq 2n$.

Theorem 2 and Observation 10 imply the next corollary immediately.

**Corollary 2 ([11]).** If $G$ is a graph of order $n \geq 2$, then $\gamma_R(G) + d_R(G) \leq n + 2$.

Since $\delta^{-}(D(G)) = \delta(G)$ and a set of vertices is independent in $G$ if and only if it is independent in $D(G)$, Theorem 4 and Observation 10 lead to the following result.

**Corollary 3.** If $G$ is a graph, then $d_R(G) \leq \delta(G) + 2$. Moreover, if $d_R(G) = \delta(G) + 2$, then the set of vertices of minimum degree is an independent set.

The bound $d_R(G) \leq \delta(G) + 2$ can be found in [11]. Finally, we obtain the following Nordhaus-Gaddum bound in view of Theorem 6 and Observation 10.

**Corollary 4.** If $G$ is a graph of order $n$, then
\[ d_R(G) + d_R(G) \leq n + 1. \]

In [11] Corollary 4 was only proved for regular graphs.

**References**


