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On trees with equal Roman domination and outer-independent Roman domination number

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In honor of Lutz Volkmann on the occasion of his seventy-fifth birthday.

Abstract: A Roman dominating function (RDF) on a graph G is a function $f : V(G) \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. A Roman dominating function f is called an outer-independent Roman dominating function (OIRDF) on G if the set $\{v \in V \mid f(v) = 0\}$ is independent. The (outer-independent) Roman domination number $\gamma_R(G)$ ($\gamma_{oiR}(G)$) is the minimum weight of an RDF (OIRDF) on G. Clearly for any graph G, $\gamma_R(G) \leq \gamma_{oiR}(G)$. In this paper, we provide a constructive characterization of trees T with $\gamma_R(T) = \gamma_{oiR}(T)$.

Keywords: Roman domination, outer-independent Roman domination, tree

AMS Subject classification: 05C69

1. Introduction

Throughout this paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V, E). The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V(G)$, the *open neighborhood* of v is the set $N_G(v) = N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and its *closed neighborhood* is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg(v) = |N(v)|$. A *leaf* of G is a vertex with degree one in G, a *support vertex* is a vertex adjacent to a leaf, a *strong support vertex* is a support vertex adjacent to at least two leaves, an *end support vertex* is a support vertex whose all neighbors with exception at most one are leaves, and a *weak support vertex* is a support vertex with exactly one leaf neighbor. For every vertex $v \in V(G)$, the set of all leaves adjacent to

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v is denoted by L_v. A *double star* $DS_{p,q}$ is a tree containing exactly two non-pendant vertices which one is adjacent to p leaves and the other is adjacent to q leaves. We denote by P_n the path on n vertices. The *distance* $d_G(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest $u-v$ path in G. The *diameter* of a graph G, denoted by $\mathrm{diam}(G)$, is the greatest distance between two vertices of G. For a vertex v in a rooted tree T, let $C(v)$ denote the set of children of v, $D(v)$ denotes the set of descendants of v and $D[v] = D(v) \cup \{v\}$. Also, the *depth* of v, $\mathrm{depth}(v)$, is the largest distance from v to a vertex in $D(v)$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. A proper induced subgraph H of a graph G is called a pendant subgraph if there is exactly one edge between $V(H)$ and $V(G) - V(H)$.

A function $f : V(G) \to \{0,1,2\}$ is called a *Roman dominating function* (RDF) on G if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The *weight* of an RDF is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight of an RDF on G. Roman domination was introduced by Cockayne et al. in [8] and was inspired by the work of ReVelle and Rosing [10] and Stewart [12]. It is worth mentioning that since 2004, more than hundred papers have been published on this topic, where several new variations were introduced: weak Roman domination [9], Roman $\{2\}$-domination [7], maximal Roman domination [1], mixed Roman domination [3], double Roman domination [6], independent Roman domination [5], signed Roman domination [4, 11], signed total Roman domination [13, 14] and recently outer-independent Roman domination introduced by [2].

For a Roman dominating function f, let $V_i = \{v \in V \mid f(v) = i\}$ for $i = 0, 1, 2$. Since these three sets determine f, we can equivalently write $f = (V_0, V_1, V_2)$ (or $f = (V_0^f, V_1^f, V_2^f)$ to refer f). We note that $\omega(f) = |V_1| + 2|V_2|$.

A function $f : V(G) \to \{0,1,2\}$ is an *outer-independent Roman dominating function* (OIRDF) on G if f is an RDF and the set $\{v \in V \mid f(v) = 0\}$ is an independent set. The outer-independent Roman domination number $\gamma_{oiR}(G)$ is the minimum weight of an OIRDF on G. The concept of outer-independent Roman domination in graphs was introduced by Ahangar et al. in [2]. Since each outer-independent Roman dominating function is a Roman dominating function, we have the following observation.

Observation 1. *For every graph G, $\gamma_{oiR}(T) \geq \gamma_R(T)$.*

In this paper, we provide a constructive characterization of trees T with $\gamma_R(T) = \gamma_{oiR}(T)$.

We make use of the following observations in this paper.

Observation 2. *Let H be a subgraph of a graph G. If $\gamma_{oiR}(H) = \gamma_R(H)$, $\gamma_{oiR}(G) \leq \gamma_{oiR}(H) + s$ and $\gamma_R(G) \geq \gamma_R(H) + s$ for some non-negative integer s, then $\gamma_R(G) = \gamma_{oiR}(G)$, $\gamma_{oiR}(G) = \gamma_{oiR}(H) + s$ and $\gamma_R(G) = \gamma_R(H) + s$.*

Proof. We deduce from the assumptions and Observation 1 that

$$\gamma_{oiR}(G) \geq \gamma_R(G) \geq \gamma_R(H) + s = \gamma_{oiR}(H) + s \geq \gamma_{oiR}(G).$$

Hence, all inequalities occurring in above chain, become equalities and so $\gamma_R(G) = \gamma_{oiR}(G)$, $\gamma_{oiR}(G) = \gamma_{oiR}(H) + s$ and $\gamma_R(G) = \gamma_R(H) + s$. □

Observation 3. *Let H be a subgraph of a graph G. If $\gamma_R(G) = \gamma_{oiR}(G)$, $\gamma_R(G) \leq \gamma_R(H) + s$ and $\gamma_{oiR}(G) \geq \gamma_{oiR}(H) + s$ for some non-negative integer s, then $\gamma_R(H) = \gamma_{oiR}(H)$, $\gamma_{oiR}(G) = \gamma_{oiR}(H) + s$ and $\gamma_R(G) = \gamma_R(H) + s$.*

Proof. By Observation 1 and the assumptions, we have

$$\gamma_{oiR}(G) = \gamma_R(G) \leq \gamma_R(H) + s \leq \gamma_{oiR}(H) + s \leq \gamma_{oiR}(G).$$

Thus, all inequalities occurring in above chain, become equalities and so $\gamma_R(H) = \gamma_{oiR}(H)$, $\gamma_{oiR}(G) = \gamma_{oiR}(H) + s$ and $\gamma_R(G) = \gamma_R(H) + s$. □

2. A characterization of trees T with $\gamma_R(T) = \gamma_{oiR}(T)$

In this section we give a constructive characterization of all trees T satisfying $\gamma_R(T) = \gamma_{oiR}(T)$. We start with a definition.

Definition 1. For a graph G and each vertex $v \in V(G)$, we say v has property P in G if there exists a $\gamma_{oiR}(G)$-function f such that $f(v) \neq 0$. Define

$$W_G = \{v \mid v \text{ has property } P \text{ in } G\}.$$

Proposition 1. Let G be a graph and v be a strong support vertex in G. Then there exists a $\gamma_{oiR}(G)$-function (resp. $\gamma_R(G)$-function) f such that $f(v) = 2$.

Proof. Suppose $w_1, w_2 \in L_v$ and let f be a $\gamma_{oiR}(G)$-function. If $f(v) = 2$, then we are done. Let $f(v) \leq 1$. If $f(v) = 1$, then we must have $f(w_1) = f(w_2) = 1$ and the function $g : V(G) \to \{0, 1, 2\}$ defined by $g(w_1) = g(w_2) = 0$, $g(v) = 2$ and $g(u) = f(u)$ otherwise, is an OIRDF of G of weight less than $\gamma_{oiR}(G)$ which is a contradiction. Hence, we assume $f(v) = 0$. Since f is an OIRDF of G, we have $f(x) \geq 1$ for each $x \in N(v)$. Now the function g defined above, is a $\gamma_{oiR}(G)$-function with $g(v) = 2$, as desired.
Using a similar argument, we can see that there exists a $\gamma_R(G)$-function f such that $f(v) = 2$. □

Corollary 1. Any strong support vertex of a graph G, has property P in G.

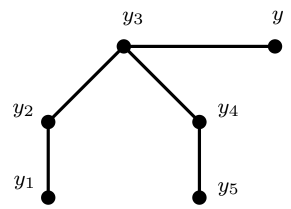

Figure 1. **The graph F_1 used in Operation $\mathcal{O}_7$**

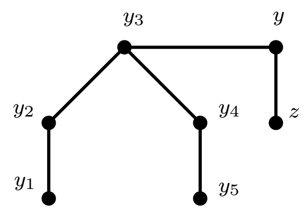

Figure 2. **The graph F_2 used in Operation $\mathcal{O}_8$**

In order to presenting our constructive characterization, we define a family of trees as follows. Let $\mathcal{T}$ be the family of trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ of trees for some $k \geq 1$, where $T_1 \in \{P_2, P_3, P_4\}$ and $T = T_k$. If $i \geq 2$, T_{i+1} can be obtained from T_i by one of the following operations.

Operation $\mathcal{O}_1$: If $x \in V(T_i)$ is a strong support vertex, then $\mathcal{O}_1$ adds a pendant edge xy to obtain T_{i+1}.

Operation $\mathcal{O}_2$: If $x \in V(T_i)$ is a strong support vertex or is adjacent to the center of a pendant star $K_{1,r}$ ($r \geq 1$), then $\mathcal{O}_2$ adds a star $K_{1,2}$ and joins x to the center of $K_{1,2}$ to obtain T_{i+1}.

Operation $\mathcal{O}_3$: If $x \in W_{T_i}$, then $\mathcal{O}_3$ adds a star $K_{1,r}$ ($r = 2, 3$) and joins x to a leaf of $K_{1,r}$ to obtain T_{i+1}.

Operation $\mathcal{O}_4$: If $x \in V(T_i)$ satisfies in one of the following statement:

1. x is a strong support vertex,
2. x is adjacent to the center of a pendant star $K_{1,r}$ ($r \geq 1$),
3. x is adjacent to a support vertex of a pendant path P_4,
4. x is adjacent to the center of a pendant path P_5,

then $\mathcal{O}_4$ adds a path $y_1y_2y_3y_4y_5$ or a path $y_1y_2y_3y_4$ and joins x to y_3 to obtain T_{i+1}.

Operation $\mathcal{O}_5$: If $x \in W_{T_i}$, then $\mathcal{O}_5$ adds a double star $DS_{2,1}$ and joins x to the support vertex of degree 2 in $DS_{2,1}$ to obtain T_{i+1}.

Operation $\mathcal{O}_6$: If $x \in V(T_i)$ satisfies in one of the following statement:

1. x is a strong support vertex,
2. x is a support vertex and there is a path xx_2x_1 such that $\deg(x_1) = 1$ and $\deg(x_2) = 2$,
3. there are two paths xx_2x_1 and xz_2z_1 such that $\deg(x_1) = \deg(z_1) = 1$ and $\deg(x_2) = \deg(z_2) = 2$,

then $\mathcal{O}_6$ adds a path y_1y_2 and joins x to y_1 to obtain T_{i+1}.

Operation $\mathcal{O}_7$: If $x \in W_{T_i}$, then $\mathcal{O}_7$ adds the graph F_1 illustrated in Figure 1 and joins x to y to obtain T_{i+1}.

Operation $\mathcal{O}_8$: If $x \in W_{T_i}$, then $\mathcal{O}_8$ adds the graph F_2 illustrated Figure 2 and joins x to y to obtain T_{i+1}.

Operation $\mathcal{O}_9$: If $x \in W_{T_i}$, then $\mathcal{O}_9$ adds the graph F_3 illustrated in Figure 3 and the edge xy to obtain T_{i+1}.

Operation $\mathcal{O}_{10}$: If $x \in W_{T_i}$, then $\mathcal{O}_{10}$ adds the graph F_4 illustrated Figure 4 and the edge xy to obtain T_{i+1}.

The proof of the first lemma is trivial by Proposition 1 and Observation 2 and therefore omitted.

Lemma 1. If T_i is a tree with $\gamma_R(T_i) = \gamma_{oiR}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_1$, then $\gamma_R(T_{i+1}) = \gamma_{oiR}(T_{i+1})$.

Lemma 2. If T_i is a tree with $\gamma_R(T_i) = \gamma_{oiR}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_2$, then $\gamma_R(T_{i+1}) = \gamma_{oiR}(T_{i+1})$.

Proof. Let the Operation $\mathcal{O}_2$ add a star $K_{1,2}$ centered at y and join x to y. Clearly, any outer-independent Roman dominating function of T_i, can be extended to an outer-independent Roman dominating function of T_{i+1} by assigning weight 2 to y and 0 to the vertices in L_y yielding $\gamma_{oiR}(T_{i+1}) \leq \gamma_{oiR}(T_i) + 2$.
Now let f be a $\gamma_R(T_{i+1})$-function such that $f(y) + f(x)$ is as large as possible. Obviously $f(y) = 2$. If $f(x) \geq 1$, then the function f, restricted to T_i is an RDF of T_i and so $\gamma_R(T_{i+1}) \geq 2 + \gamma_R(T_i)$. Let $f(x) = 0$. Then x is not a strong support vertex and so x is adjacent to the center, say w, of a pendant star $K_{1,r}$ $(r \geq 1)$. We may assume without loss of generality that $f(w) = 2$. As above, the function f, restricted to T_i is an RDF of T_i and so $\gamma_R(T_{i+1}) \geq 2 + \gamma_R(T_i)$. Now the result follows by Observation 2. □

Lemma 3. If T_i is a tree with $\gamma_R(T_i) = \gamma_{oiR}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_3$, then $\gamma_R(T_{i+1}) = \gamma_{oiR}(T_{i+1})$.

Proof. Let $\mathcal{O}_3$ add a star $K_{1,r}$ ($r = 2, 3$) centered at z and an edge xy where y is a leaf of $K_{1,r}$. It is easy to see that $\gamma_R(T_{i+1}) \geq \gamma_R(T_i) + 2$. On the other hand, since $x \in W(T_i)$, there exists a $\gamma_{oiR}(T_i)$-function f with $f(x) \geq 1$. Then f can be extended to an outer-independent Roman dominating function of T_{i+1} by assigning weight 2 to z and 0 to the neighbors of z implying that $\gamma_{oiR}(T_{i+1}) \leq \gamma_{oiR}(T_i) + 2$. Now the result follows by Observation 2. $\square$

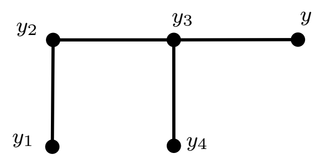

Figure 3. **The graph F_3 used in Operation $\mathcal{O}_9$**

Lemma 4. If T_i is a tree with $\gamma_R(T_i) = \gamma_{oiR}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_4$, then $\gamma_R(T_{i+1}) = \gamma_{oiR}(T_{i+1})$.

Proof. Let $\mathcal{O}_4$ add a path $y_1y_2y_3y_4y_5$ (resp. $y_1y_2y_3y_4$) and join x to y_3. Clearly, any outer-independent Roman domination function of T_i can be extended to an outer-independent Roman dominating function of T_{i+1} by assigning the value 2 to y_3, 1 to y_1, y_5 and 0 to y_2, y_4 (resp. the value 2 to y_3, 1 to y_1 and 0 to y_2, y_4) and so $\gamma_{oiR}(T_{i+1}) \leq \gamma_{oiR}(T_i) + 4$ (resp. $\gamma_{oiR}(T_{i+1}) \leq \gamma_{oiR}(T_i) + 3$).
Assume now that f is a $\gamma_R(T_{i+1})$-function such that $f(N[x])$ is as large as possible. Obviously, $f(y_3) = 2$. Since, x is a strong support vertex or is adjacent to the center of a pendant star $K_{1,r}$ ($r \geq 1$) or is adjacent to a support vertex of a pendant path P_4 or is adjacent to the center of a pendant path P_5, by the choice of f we have $f(y) = 2$ for some $y \in N[x] - \{y_3\}$. Hence, the function f, restricted to T_i is a Roman dominating function of T_i and we have $\gamma_R(T_{i+1}) \geq 4 + \gamma_R(T_i)$ (resp. $\gamma_R(T_{i+1}) \geq \gamma_R(T_i) + 3$). Now the result follows by Observation 2. $\square$

Lemma 5. If T_i is a tree with $\gamma_R(T_i) = \gamma_{oiR}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_5$, then $\gamma_R(T_{i+1}) = \gamma_{oiR}(T_{i+1})$.

Proof. Let $\mathcal{O}_5$ add a double star $DS_{2,1}$ with the support vertices a, b and join x to a where $\deg(a) = 2$. Since, $x \in W_{T_i}$, there exists a $\gamma_{oiR}(T_i)$-function f such that $f(x) \geq 1$. Then f can be extended to an outer-independent Roman dominating function of T_{i+1} by assigning weight 2 to b, 1 to the leaf adjacent to a and 0 to the vertices in $L_b \cup \{a\}$ yielding $\gamma_{oiR}(T_{i+1}) \leq \gamma_{oiR}(T_i) + 3$.

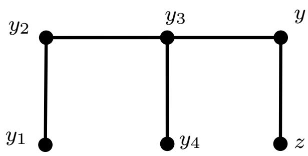

Figure 4. **The graph F_4 used in Operation $\mathcal{O}_{10}$**

Assume that w is the leaf adjacent to a and g is a $\gamma_R(T_{i+1})$-function such that $g(b)+g(w)$ is as large as possible. Obviously, we have $g(b) = 2$ and $g(w) = 1$. Then the function g, restricted to T_i is a Roman dominating function of T_i and we have $\gamma_R(T_{i+1}) \geq 3 + \gamma_R(T_i)$. Now the result follows by Observation 2. $\square$

Lemma 6. If T_i is a tree with $\gamma_R(T_i) = \gamma_{oiR}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_6$, then $\gamma_R(T_{i+1}) = \gamma_{oiR}(T_{i+1})$.

Proof. Let $\mathcal{O}_6$ add a path y_1y_2 and the edge xy_1. Suppose that f is a $\gamma_{oiR}(T_i)$-function such that $f(x)$ is as large as possible. Then $f(x) = 2$ and f can be extended to an outer-independent Roman dominating function of T_{i+1} by assigning a 1 to y_2 and a 0 to y_1 and this implies that $\gamma_{oiR}(T_{i+1}) \leq \gamma_{oiR}(T_i) + 1$.
On the other hand, if g is a $\gamma_R(T_{i+1})$-function, then $g(x) = 2$ and $g(y_2) = 1$, and the function g, restricted to T_i is a Roman dominating function of T_i yielding $\gamma_R(T_{i+1}) \geq 1 + \gamma_R(T_i)$. As in the above lemmas, we obtain $\gamma_{oiR}(T_{i+1}) = \gamma_R(T_{i+1})$. $\square$

Lemma 7. If T_i is a tree with $\gamma_R(T_i) = \gamma_{oiR}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_7$, then $\gamma_R(T_{i+1}) = \gamma_{oiR}(T_{i+1})$.

Proof. Let $\mathcal{O}_7$ add the graph F_1 and the edge xy. Since, $x \in W_{T_i}$, there exists a $\gamma_{oiR}(T_i)$-function f such that $f(x) \geq 1$. Then f can be extended to an outer-independent Roman dominating function of T_{i+1} by assigning a 2 to y_3, a 1 to y_1, y_5 and a 0 to the vertices $N(y_3)$, and so $\gamma_{oiR}(T_{i+1}) \leq \gamma_{oiR}(T_i) + 4$.
On the other hand, let g be a $\gamma_R(T_{i+1})$-function such that $g(y_3)$ is as large as possible. Clearly, $g(y_3) = 2$, $g(y_1) = g(y_5) = 1$ and $g(y) = 0$. Hence, g restricted to T_i is an RDF of T_i implying that $\gamma_R(T_{i+1}) \geq 4+\gamma_R(T_i)$. Now the result follows by Observation 2. $\square$

Lemma 8. If T_i is a tree with $\gamma_R(T_i) = \gamma_{oiR}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_8$, then $\gamma_R(T_{i+1}) = \gamma_{oiR}(T_{i+1})$.

Proof. Let $\mathcal{O}_8$ add the graph F_2 and joins x to y. Since, $x \in W_{T_i}$, there exists a $\gamma_{oiR}(T_i)$-function f such that $f(x) \geq 1$ and f can be extended to an outer-independent Roman dominating function of T_{i+1} by assigning a 2 to y_3, a 1 to y_1, y_5, z and a 0 to y, y_2, y_4. Thus $\gamma_{oiR}(T_{i+1}) \leq \gamma_{oiR}(T_i) + 5$.
Now let g be a $\gamma_R(T_{i+1})$-function such that $g(y_3)+g(z)$ is as large as possible. Then we must have $g(y_3) = 2, g(z) = g(y_1) = g(y_5) = 1$ and $g(y) = 0$. Hence, the function g, restricted to T_i is an Roman dominating function of T_i and so $\gamma_R(T_{i+1}) \geq 5+\gamma_R(T_i)$. Now the result follows by Observation 2. □

Lemma 9. If T_i is a tree with $\gamma_R(T_i) = \gamma_{oiR}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_9$, then $\gamma_R(T_{i+1}) = \gamma_{oiR}(T_{i+1})$.

Proof. Let $\mathcal{O}_9$ add a graph F_3 and the edge xy. Since $x \in W_{T_i}$, there exists a $\gamma_{oiR}(T_i)$-function f such that $f(x) \geq 1$ and f can be extended to an outer-independent Roman dominating function of T_{i+1} by assigning a 2 to y_3, 1 to y_1 and 0 to the vertices in $N(y_3)$. Hence, $\gamma_{oiR}(T_{i+1}) \leq \gamma_{oiR}(T_i) + 3$.
Now let g be a $\gamma_R(T_{i+1})$-function such that $g(y_3)$ is as large as possible. Clearly, $g(y_3) = 2$, $g(y_1) = 1$ and $g(y) = 0$. Then the function g, restricted to T_i is an RDF of T_i yielding $\gamma_R(T_{i+1}) \geq 3 + \gamma_R(T_i)$. Now the result follows by Observation 2. □

The proof of the next lemma is similar to the proof of Lemma 9 and therefore it is omitted.

Lemma 10. If T_i is a tree with $\gamma_R(T_i) = \gamma_{oiR}(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_{10}$, then $\gamma_R(T_{i+1}) = \gamma_{oiR}(T_{i+1})$.

Theorem 4. If $T \in \mathcal{T}$, then $\gamma_R(T) = \gamma_{oiR}(T)$.

Proof. If $T \in \{P_2, P_3, P_4\}$, then obviously $\gamma_R(T) = \gamma_{oiR}(T)$. Suppose now that $T \in \mathcal{T}$. Then there exists a sequence of trees $T_1, T_2, \ldots, T_k$ $(k \geq 1)$ such that $T_1 \in \{P_2, P_3, P_4\}$, $T = T_k$ and if $k \geq 2$, then T_{i+1} can be obtained from T_i by one of the Operations $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_{10}$ for $i = 1, 2, \ldots, k-1$. We apply induction on the number of operations used to construct T. If $k = 1$, the result is trivial. Assume the result holds for each tree $T \in \mathcal{T}$ which can be obtained from a sequence of operations of length $k-1$ and let $T' = T_{k-1}$. By the induction hypothesis, we have $\gamma_R(T') = \gamma_{oiR}(T')$. Since $T = T_k$ is obtained by one of the Operations $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_{10}$ from T', we conclude from the above Lemmas that $\gamma_R(T) = \gamma_{oiR}(T)$. □

Now we are ready to prove our main result.

Theorem 5. Let T be a non-trivial tree. Then $\gamma_R(T) = \gamma_{oiR}(T)$ if and only if $T \in \mathcal{T}$.

Proof. According to Theorem 4, we need only to prove necessity. Let T be a tree of order $n \geq 3$ with $\gamma_R(T) = \gamma_{oiR}(T)$. The proof is by induction on n. If $n \leq 3$, then clearly $T \in \mathcal{T}$. Let $n \geq 4$ and let the statement hold for all trees of order less than n. Assume that T is a tree of order n with $\gamma_R(T) = \gamma_{oiR}(T)$. If $\mathrm{diam}(T) \leq 3$, then $T = P_4$ or T is a star or double star. If $T = P_4$, then obviously $T \in \mathcal{T}$, if T is a star, then T can be obtained from P_3 by repeated application of operation $\mathcal{O}_1$, and if T is a double star different from P_4, then T can be obtained from P_3 by applying operation $\mathcal{O}_2$ once and operation $\mathcal{O}_1$ repeatedly, and this implies that $T \in \mathcal{T}$. Hence let $\mathrm{diam}(T) \geq 4$.

Let $v_1 v_2 \dots v_k$ $(k \geq 5)$ be a diametral path in T such that $\deg_T(v_2)$ is as large as possible and root T at v_k. If $\deg_T(v_2) \geq 4$, then clearly $\gamma_R(T - v_1) = \gamma_{oiR}(T - v_1)$. It follows from the induction hypothesis that $T - v_1 \in \mathcal{T}$ and hence T can be obtained from $T - v_1$ by Operation $\mathcal{O}_1$ implying that $T \in \mathcal{T}$. Assume that $\deg_T(v_2) \leq 3$.

First let $\deg_T(v_3) = 2$. Suppose that $T' = T - T_{v_3}$. Clearly, any Roman dominating function of T' can be extended to a Roman dominating function of T by assigning a 2 to v_2 and a 0 to the vertices in $N_T(v_2)$ yielding

$$\gamma_R(T) \leq \gamma_R(T') + 2. \tag{1}$$

Similarly, any outer-independent Roman dominating function of T' can be extended to an outer-independent Roman dominating function of T by assigning a 1 to v_3, a 2 to v_2 and a 0 to the vertices in L_{v_2} yielding $\gamma_{oiR}(T) \leq \gamma_{oiR}(T') + 3$. On the other hand, assume that f is a $\gamma_{oiR}(T)$-function. Clearly $f(N[v_2]) \geq 2$. If $f(v_3) \leq 1$, then the function f restricted to T' is an OIRDF of T' that implies $\gamma_{oiR}(T) \geq \gamma_{oiR}(T') + 2$. If $f(v_3) = 2$, then $f(L_{v_2}) \geq 1$, and the function $g : V(T') \to \{0, 1, 2\}$ defined by $g(v_4) = \min\{f(v_4) + 1, 2\}$ and $g(x) = f(x)$ otherwise is an OIRDF of T' of weight at most $\gamma_{oiR}(T) - 2$ yielding $\gamma_{oiR}(T) \geq \gamma_{oiR}(T') + 2$. Thus $\gamma_{oiR}(T) - 3 \leq \gamma_{oiR}(T') \leq \gamma_{oiR}(T) - 2$. If $\gamma_{oiR}(T') = \gamma_{oiR}(T) - 3$, then

$$\gamma_{oiR}(T') = \gamma_{oiR}(T) - 3 = \gamma_R(T) - 3 \leq \gamma_R(T') - 1$$

which is a contradiction by Observation 1. Hence,

$$\gamma_{oiR}(T') = \gamma_{oiR}(T) - 2. \tag{2}$$

By (1), (2) and Observation 3, we obtain $\gamma_R(T') = \gamma_{oiR}(T')$. By the induction hypothesis we have $T' \in \mathcal{T}$. Now we show that $v_4 \in W_{T'}$. Let h be a $\gamma_{oiR}(T)$-function such that $h(v_4)$ is as large as possible. Clearly $h(N[v_2]) \geq 2$. Since $\gamma_R(T) = \gamma_{oiR}(T)$, h is also a $\gamma_R(T)$-function. If $h(v_4) \geq 1$, then h restricted to T' is a $\gamma_{oiR}(T')$-function and we are done. Assume that $h(v_4) = 0$. Since h is an OIRDF of T, we must have $h(v_3) \geq 1$. If $h(v_3) = 1$, then to Roman dominate the vertices of v_1, v_2, we must have $h(N[v_2] - \{v_3\}) \geq 2$. But then the function $h_1 : V(T) \to \{0, 1, 2\}$ defined by $h_1(v_2) = 2, h_1(x) = 0$ for $x \in N(v_2)$ and $h_1(x) = h(x)$ otherwise, is an RDF of T of

weight less than $\omega(h)$ which is a contradiction. Assume that $h(v_3) = 2$. To Roman dominate v_1, we must have $f(v_1)+f(v_2) \geq 1$. Now the function $h_2 : V(T) \to \{0, 1, 2\}$ defined by $h_2(v_4) = \min\{2, h(v_4) + 1\}$, $h_2(v_2) = 2, h_2(x) = 0$ for $x \in N(v_2)$ and $h_2(x) = h(x)$ otherwise, is an OIRDF of T of weight $\omega(h)$ contradicting the choice of h. Thus $v_4 \in W_{T'}$. Now T can be obtained from T' by Operation $\mathcal{O}_3$ and so $T \in \mathcal{T}$.
Now, let $\deg_T(v_3) \geq 3$. We consider the following cases.

Case 1. $\deg_T(v_2) = 3$.
We distinguish the following subcases.

Subcase 1.1 v_3 has two children with depth 0 or v_3 has one child with depth 1 other than v_2.
Let $T' = T - T_{v_2}$. Clearly, any Roman dominating function of T' can be extended to a Roman dominating function of T by assigning a 2 to v_2 and 0 to the vertices in L_{v_2} yielding

$$\gamma_R(T) \leq \gamma_R(T') + 2. \tag{3}$$

On the other hand, let f be a $\gamma_{oiR}(T)$-function such that $f(v_2) + f(v_3)$ is as large as possible. Clearly, $f(v_2) = 2$. If $f(v_3) \geq 1$, then the function f, restricted to T' is an outer-independent Roman dominating function of T' which implies that $\gamma_{oiR}(T') \leq \gamma_{oiR}(T) - 2$. Assume that $f(v_3) = 0$. We deduce from the assumption that v_3 is not a strong support vertex, and thus v_3 has one child with depth 1, say u, different from v_2. Since f is an OIRDF of T, we have $f(u) \geq 1$. To Roman dominate the leaves adjacent to u, we must have $f(u) + f(L(u)) \geq 2$. Without loss of generality, we may assume that $f(u) = 2$. Now the function f, restricted to T' is an outer-independent Roman dominating function of T' which implies that $\gamma_{oiR}(T') \leq \gamma_{oiR}(T) - 2$. Thus

$$\gamma_{oiR}(T') \leq \gamma_{oiR}(T) - 2 \tag{4}$$

We conclude from (3), (4), Observation 3 and the induction hypothesis that $T' \in \mathcal{T}$. Now T can be obtained from T' by Operation $\mathcal{O}_2$ and so $T \in \mathcal{T}$.

Subcase 1.2 $\deg_T(v_3) = 3$ and v_3 has one child with depth 0 .
Assume that $T' = T - T_{v_3}$ and $L(v_3) = \{u\}$. Clearly, any Roman dominating function of T' can be extended to a Roman dominating function of T by assigning weight 2 to v_2 , 1 to u and 0 to the vertices in $N_T(v_2)$ which implies that

$$\gamma_R(T) \leq \gamma_R(T') + 3. \tag{5}$$

Consider now a $\gamma_{oiR}(T)$-function f such that $f(v_2)+f(u)$ is as large as possible. Then we must have $f(v_2) = 2$, $f(u) = 1$ and $f(x) = 0$ for $x \in N(v_2)$. Then the function f, restricted to T' is an outer-independent Roman dominating function of T' which implies that

$$\gamma_{oiR}(T') \leq \gamma_{oiR}(T) - 3. \tag{6}$$

It follows from (3), (4) and Observation 3 that $\gamma_{oiR}(T') = \gamma_R(T')$ and by the induction hypothesis we have $T' \in \mathcal{T}$. Since $\gamma_{oiR}(T') = \gamma_R(T')$, the function f restricted to T' is a $\gamma_{oiR}(T')$-function. Since f is an OIRDF of T and $f(v_3) = 0$, we deduce that $f(v_4) \geq 1$ and so $v_4 \in W_{T'}$. Now T can be obtained from T' by Operation $\mathcal{O}_5$ yielding $T \in \mathcal{T}$.

Case 2. $\deg_T(v_2) = 2$.
First let $\deg_T(v_3) \geq 4$. Assume that $T' = T - \{v_1, v_2\}$. By the choice of diametrical path, v_3 is a strong support vertex or is a weak support vertex and there is a path xx_2x_1 such that $\deg(x_1) = 1$ and $\deg(x_2) = 2$ or there are two paths xx_2x_1 and xz_2z_1 such that $\deg(x_1) = \deg(z_1) = 1$ and $\deg(x_2) = \deg(z_2) = 2$. If f is a $\gamma_{oiR}(T)$-function, then f is a $\gamma_R(T)$-function (since $\gamma_{oiR}(T) = \gamma_R(T)$) and we must have $f(v_3) = 2$ and $f(v_1) = 1$. Thus the function f restricted to T' is an OIRDF of T' and so $\gamma_{oiR}(T) \geq \gamma_{oiR}(T') + 1$. On the other hand, if g is a $\gamma_R(T')$-function such that $g(v_3)$ is as large as possible, then clearly $g(v_3) = 2$ and g can be extended to an RDF of T by assigning a 1 to v_1 and a 0 to v_2 yielding $\gamma_R(T) \leq \gamma_R(T') + 1$. We conclude from Observation 3 that $\gamma_{oiR}(T') = \gamma_R(T')$ and by the induction hypothesis we have $T' \in \mathcal{T}$. Now T can be obtained from T' by Operation $\mathcal{O}_6$ and so $T \in \mathcal{T}$.
Now, let $\deg_T(v_3) = 3$. Then T_{v_3} is a pendant path P_4 or a pendant path P_5 in T. If $\mathrm{diam}(T) = 4$, then T can be obtained from P_4 by Operation $\mathcal{O}_6$ or from P_2 by Operations $\mathcal{O}_3, \mathcal{O}_6$ and so $T \in \mathcal{T}$. Suppose that $\mathrm{diam}(T) \geq 5$. We distinguish the following subcases.

Subcase 2.1. v_3 is a support vertex and $\deg_T(v_4) = 2$.
Let u be the leaf adjacent to v_3 and let $T' = T - T_{v_4}$. Clearly, any Roman dominating function of T' can be extended to a Roman dominating function of T by assigning a 2 to v_3 , 1 to v_1 and 0 to v_2, v_4, u which implies that $\gamma_R(T) \leq \gamma_R(T') + 3$. On the other hand, let f be a $\gamma_{oiR}(T)$-function such that $f(v_3)$ is as large as possible. Then we must have $f(v_3) = 2$, $f(v_1) = 1$ and $f(v_2) = f(u) = 0$. Now the function $g : V(T') \to \{0, 1, 2\}$ defined by $g(v_5) = \min\{2, f(v_5) + f(v_4)\}$ and $g(x) = f(x)$ otherwise, is an OIRDF of T' of weight $\gamma_{oiR}(T) - 3$ and this implies that $\gamma_{oiR}(T') \leq \gamma_{oiR}(T) - 3$. It follows that Observation 3 that $\gamma_{oiR}(T') = \gamma_R(T')$ and so g is a $\gamma_R(T')$-function yielding $v_5 \in W_{T'}$. By the induction hypothesis, we obtain $T' \in \mathcal{T}$. Now T can be obtained from T' by Operation $\mathcal{O}_9$ and so $T \in \mathcal{T}$.

Subcase 2.2. v_3 is a support vertex and v_4 has a child with depth 2.
Let u be the leaf adjacent to v_3 and let $v_4y_3y_2y_1$ be a path in T such that $y_3 \notin \{v_3, v_5\}$. By the choice of diametrical path we have $\deg(y_2) = 1$. Considering above arguments, we may assume that $T_{y_3} = P_4$ or $T_{y_3} = P_5$ since otherwise we can rename y_i as v_i and are in the case that $\deg_T(v_3) = 2$ which we have considered already. Let $T' = T - T_{v_3}$. As in the Subcase 2.1, we have $\gamma_R(T) \leq \gamma_R(T') + 3$. Now let f be a $\gamma_{oiR}(T)$-function such that $f(y_3)$ is as large as possible. Clearly, $f(y_3) = 2$ and $f(u) + f(v_1) + f(v_2) + f(v_3) \geq 3$. Hence the function f restricted to T' is an OIRDF of T' of weight at most $\gamma_{oiR}(T) - 3$ implying that $\gamma_{oiR}(T) \geq \gamma_{oiR}(T') + 3$. It follows that Observation 3 and the induction hypothesis that $T' \in \mathcal{T}$. Since T can be obtained from T' by Operation $\mathcal{O}_4$, we have $T \in \mathcal{T}$.

Subcase 2.3. v_3 is a support vertex and v_4 has a child with depth 1.
Let u be the leaf adjacent to v_3 and let $v_4y_2y_1$ be a path in T such that $y_2 \neq v_5$. Let $T' = T - T_{v_3}$. As in the Subcase 2.1, we have $\gamma_R(T) \leq \gamma_R(T') + 3$. Now let f be a $\gamma_{oiR}(T)$-function such that $f(v_3) + f(y_2)$ is as large as possible. Then clearly, $f(v_3) = 2$, $f(u) + f(v_1) + f(v_2) + f(v_3) \geq 3$ and either $f(v_4) \geq 1$ or $f(v_4) = 0$ and $f(y_2) = 2$. Hence, the function f restricted to T' is an OIRDF of T' of weight at most $\gamma_{oiR}(T) - 3$ yielding $\gamma_{oiR}(T) \geq \gamma_{oiR}(T') + 3$. By Observation 3 and the induction hypothesis, we have $T' \in \mathcal{T}$. Since T can be obtained from T' by Operation $\mathcal{O}_4$, we have $T \in \mathcal{T}$.

Subcase 2.4. v_3 is a support vertex and v_4 is a strong support vertex.
Let u be the leaf adjacent to v_3 and let $T' = T - T_{v_3}$. As in the Subcase 2.1, we have $\gamma_R(T) \leq \gamma_R(T') + 3$. Consider now a $\gamma_{oiR}(T)$-function f such that $f(v_4) = 2$ to according Proposition 1. Clearly, $f(u) + f(v_1) + f(v_2) + f(v_3) \geq 3$ and the function f restricted to T' is an OIRDF of T' of weight at most $\gamma_{oiR}(T) - 3$ yielding $\gamma_{oiR}(T) \geq \gamma_{oiR}(T') + 3$. By Observation 3 and the induction hypothesis, we have $T' \in \mathcal{T}$. Now, T can be obtained from T' by Operation $\mathcal{O}_4$, we have $T \in \mathcal{T}$.

Subcase 2.5. v_3 is a support vertex, $\deg_T(v_4) = 3$ and v_4 has a child with depth 0.
Let z be the leaf adjacent to v_4. If $\mathrm{diam}(T) = 4$, then T can be obtained from P_3 by Operation $\mathcal{O}_4$ and so $T \in \mathcal{T}$. Let $\mathrm{diam}(T) \geq 5$ and let $T' = T - T_{v_4}$. Clearly, any $\gamma_R(T')$-function can be extended to an RDF of T by assigning a 2 to v_3, a 1 to v_1, z and a 0 to u, v_2, v_4 and so

$$\gamma_R(T) \leq \gamma_R(T') + 4. \tag{7}$$

Let f be a $\gamma_{oiR}(T)$-function such that $f(v_3) + f(z)$ is as large as possible. Clearly, $f(v_3) = 2$, $f(V(T_{v_3})) \geq 3$ and $f(V(T_{v_4})) \geq 4$. We claim that $f(v_4) = 0$. Suppose, to the contrary, that $f(v_4) \geq 1$. Since $\gamma_R(T) = \gamma_{oiR}(T)$, f is also a $\gamma_R(T)$-function. This implies that $f(v_4) = 2$, otherwise we must have $f(z) = 1$ and the function $h : V(T) \to \{0, 1, 2\}$ defined by $h(v_4) = 0$, and $h(t) = f(t)$ otherwise, is an RDF of T of weight less that $\omega(f) = \gamma_R(T)$ which is a contradiction. Now the function $g : V(T) \to \{0, 1, 2\}$ defined by $g(v_4) = 0, g(z) = 1, g(v_5) = \min\{2, f(v_5) + 1\}$ and $g(x) = f(x)$ otherwise, is a $\gamma_{oiR}(T)$-function contradicting the choice of f. Thus $f(v_4) = 0$ and so $f(v_5) \geq 1$ because f is an OIRDF of T. Then the function f restricted to T' is an OIRDF of T' and so

$$\gamma_{oiR}(T) \geq \gamma_{oiR}(T') + 4. \tag{8}$$

We deduce from (7), (8) and Observation 3 that $\gamma_{oiR}(T') = \gamma_R(T')$ and so f restricted to T' is a $\gamma_{oiR}(T')$-function implying that $v_5 \in W_{T'}$. By the induction hypothesis, we have $T' \in \mathcal{T}$ and since T can be obtained from T' by Operation $\mathcal{O}_{10}$, we have $T \in \mathcal{T}$.

Subcase 2.6. There is a pendant path $v_3y_2y_1$ such that $y_2 \notin \{v_2, v_4\}$, and $\deg_T(v_4) = 2$.
Then $T_{v_3} = P_5$. If $\mathrm{diam}(T) = 4$, then T can be obtained from P_5 by Operation $\mathcal{O}_6$ and so $T \in \mathcal{T}$. Suppose $\mathrm{diam}(T) \geq 5$ and let $T' = T - T_{v_4}$. Clearly, any Roman

dominating function of T' can be extended to a Roman dominating function of T by assigning a 2 to v_3 , 1 to v_1, y_1 and 0 to v_2, y_2, v_4 which implies that

$$\gamma_R(T) \leq \gamma_R(T') + 4. \tag{9}$$

Consider now a $\gamma_{oiR}(T)$-function f such that $f(v_3)$ is as large as possible. Then we must have $f(v_3) = 2$, $f(v_1) = f(y_1) = 1$ and $f(v_2) = f(y_2) = 0$. Now the function $g : V(T') \to \{0, 1, 2\}$ defined by $g(v_5) = \min\{2, f(v_5) + f(v_4)\}$ and $g(x) = f(x)$ otherwise, is an OIRDF of T' of weight $\gamma_{oiR}(T) - 4$ yielding

$$\gamma_{oiR}(T') \leq \gamma_{oiR}(T) - 4. \tag{10}$$

By inequalities (9), (10) and Observation 3, we have $\gamma_{oiR}(T') = \gamma_R(T')$ and so g is a $\gamma_R(T')$-function. Since f is an OIRDF of T, we must have $f(v_4) + f(v_5) \geq 1$ and so $g(v_5) = \min\{2, f(v_5) + f(v_4)\} \geq 1$ yielding $v_5 \in W_{T'}$. By the induction hypothesis, we obtain $T' \in \mathcal{T}$. Since T can be obtained from T' by Operation $\mathcal{O}_7$, we have $T \in \mathcal{T}$.

Subcase 2.7. There is a pendant path $v_3y_2y_1$ such that $y_2 \notin \{v_2, v_4\}$, and v_4 has a child with depth 2.

Let $v_4z_3z_2z_1$ be a path in T such that $z_3 \notin \{v_3, v_5\}$. By the choice of diametrical path we have $\deg(z_2) = 1$. If $\deg(z_3) = 2$, then T can be obtained from $T - T_{z_3}$ by Operation $\mathcal{O}_3$ (see the third paragraph of the proof), if $\deg(z_3) \geq 4$, then T can be obtained from $T - \{z_1, z_2\}$ by Operation $\mathcal{O}_6$ (see the first paragraph of Case 2) and if $\deg(v_3) = 3$ and z_3 is a support vertex, then T can be obtained from $T - T_{z_3}$ by Operation $\mathcal{O}_4$ (see Subcase 2.2). Henceforth, we may assume that $T_{y_3} = P_5$. Let $T' = T - T_{v_3}$. Clearly, any Roman dominating function of T' can be extended to a Roman dominating function of T by assigning a 2 to v_3 , 1 to v_1, y_1 and 0 to v_2, y_2 and so

$$\gamma_R(T) \leq \gamma_R(T') + 4. \tag{11}$$

Consider now a $\gamma_{oiR}(T)$-function f such that $f(v_3) + f(z_3)$ is as large as possible. Then we must have $f(v_3) = f(z_3) = 2$, $f(v_1) = f(y_1) = 1$ and $f(v_2) = f(y_2) = 0$, and f restricted to T' is an OIRDF of T' of weight $\gamma_{oiR}(T) - 4$ and so

$$\gamma_{oiR}(T) \geq \gamma_{oiR}(T') + 4. \tag{12}$$

By inequalities (11), (12), Observation 3 and the induction hypothesis $T' \in \mathcal{T}$. Since T can be obtained from T' by Operation $\mathcal{O}_4$, we have $T \in \mathcal{T}$.

Subcase 2.8. There is a pendant path $v_3y_2y_1$ such that $y_2 \notin \{v_2, v_4\}$, and v_4 has a child z_2 with depth 1.

Let $v_4z_2z_1$ be a path in T. Assume that $T' = T - T_{v_3}$. As in the Subcase 2.7, we can see that $\gamma_R(T) \leq \gamma_R(T') + 4$. Now let f be a $\gamma_{oiR}(T)$-function such that $f(v_3) + f(z_2)$ is as large as possible. Then clearly, $f(v_3) = 2$, $f(y_2) + f(y_1) + f(v_1) + f(v_2) + f(v_3) \geq 4$ and either $f(v_4) \geq 1$ or $f(v_4) = 0$ and $f(z_2) = 2$. Hence, the function f restricted to

T' is an OIRDF of T' of weight at most $\gamma_{oiR}(T) - 4$ yielding $\gamma_{oiR}(T) \geq \gamma_{oiR}(T') + 4$. By Observation 3 and the induction hypothesis, we obtain $T' \in \mathcal{T}$ and since T can be obtained from T' by Operation $\mathcal{O}_4$, we have $T \in \mathcal{T}$.

Subcase 2.9. There is a pendant path $v_3y_2y_1$ such that $y_2 \notin \{v_2, v_4\}$, and v_4 is a strong support vertex.
Let $T' = T - T_{v_3}$. As in the subcase 2.4, we can see that $T' \in \mathcal{T}$, and since T can be obtained from T' by Operation $\mathcal{O}_4$, we have $T \in \mathcal{T}$.

Subcase 2.10. There is a pendant path $v_3y_2y_1$ such that $y_2 \notin \{v_2, v_4\}$, $\deg_T(v_4) = 3$ and v_4 has a child with depth one.
Let z be the child of v_4 with depth one and let $T' = T - T_{v_4}$. Clearly, any $\gamma_R(T')$-function can be extended to an RDF of T by assigning a 2 to v_3, a 1 to v_1, y_1, z and a 0 to y_2, v_2, v_4 and so $\gamma_R(T) \leq \gamma_R(T') + 5$. Consider now a $\gamma_{oiR}(T)$-function f such that $f(v_3) + f(z)$ is as large as possible. Clearly, $f(v_3) = 2$, $f(V(T_{v_3})) \geq 4$ and $f(V(T_{v_4})) \geq 5$. We claim that $f(v_4) = 0$. Suppose, to the contrary, that $f(v_4) \geq 1$. Since $\gamma_R(T) = \gamma_{oiR}(T)$, f is also a $\gamma_R(T)$-function. This implies that $f(v_4) = 2$, otherwise we must have $f(z) = 1$ and the function $h : V(T) \to \{0, 1, 2\}$ defined by $h(v_4) = 0$, and $h(t) = f(t)$ otherwise, is an RDF of T of weight less that $\omega(f) = \gamma_R(T)$ which is a contradiction. Define $g : V(T) \to \{0, 1, 2\}$ by $g(v_4) = 0, g(z) = 1, g(v_5) = \min\{2, f(v_5) + 1\}$ and $g(x) = f(x)$ otherwise. Clearly, g is a $\gamma_{oiR}(T)$-function contradicting the choice of f. Thus $f(v_4) = 0$ and so $f(v_5) \geq 1$ because f is an OIRDF of T. Now the function f restricted to T' is an OIRDF of T' and so $\gamma_{oiR}(T) \geq \gamma_{oiR}(T') + 5$. We deduce from Observation 3 that $\gamma_{oiR}(T') = \gamma_R(T')$ and hence f restricted to T' is a $\gamma_{oiR}(T')$-function with $f(v_5) \geq 1$ implying that $v_5 \in W_{T'}$. On the other hand, by the induction hypothesis, we have $T' \in \mathcal{T}$ and since T can be obtained from T' by Operation $\mathcal{O}_8$, we have $T \in \mathcal{T}$. This completes the proof. □

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