On trees with equal Roman domination and outer-independent Roman domination number

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In honor of Lutz Volkmann on the occasion of his seventy-fifth birthday.

Abstract: A Roman dominating function (RDF) on a graph $G$ is a function $f : V(G) \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. A Roman dominating function $f$ is called an outer-independent Roman dominating function (OIRDF) on $G$ if the set $\{v \in V \mid f(v) = 0\}$ is independent. The (outer-independent) Roman domination number $\gamma_R(G)$ ($\gamma_{oiR}(G)$) is the minimum weight of an RDF (OIRDF) on $G$. Clearly for any graph $G$, $\gamma_R(G) \leq \gamma_{oiR}(G)$. In this paper, we provide a constructive characterization of trees $T$ with $\gamma_R(T) = \gamma_{oiR}(T)$.

Keywords: Roman domination, outer-independent Roman domination, tree

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1. Introduction

Throughout this paper, $G$ is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly $V,E$). The order $|V|$ of $G$ is denoted by $n = n(G)$. For every vertex $v \in V(G)$, the open neighborhood of $v$ is the set $N_G(v) = N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and its closed neighborhood is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The degree of a vertex $v \in V$ is $\deg(v) = |N(v)|$. A leaf of $G$ is a vertex with degree one in $G$, a support vertex is a vertex adjacent to a leaf, a strong support vertex is a support vertex adjacent to at least two leaves, an end support vertex is a support vertex whose all neighbors with exception at most one are leaves, and a weak support vertex is a support vertex with exactly one leaf neighbor. For every vertex $v \in V(G)$, the set of all leaves adjacent to

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Observation 1. For every graph $G$, the outer-independent Roman domination function $\gamma_{oiR}(G) \geq \gamma_R(T)$. 

In this paper, we provide a constructive characterization of trees $T$ with $\gamma_R(T) = \gamma_{oiR}(T)$. 

We make use of the following observations in this paper.

Observation 2. Let $H$ be a subgraph of a graph $G$. If $\gamma_{oiR}(H) = \gamma_R(H)$, $\gamma_{oiR}(G) \leq \gamma_{oiR}(H) + s$ and $\gamma_R(G) \geq \gamma_R(H) + s$ for some non-negative integer $s$, then $\gamma_R(G) = \gamma_{oiR}(G)$, $\gamma_{oiR}(G) = \gamma_{oiR}(H) + s$ and $\gamma_R(G) = \gamma_R(H) + s$. 

$v$ is denoted by $L_v$. A double star $DS_{p,q}$ is a tree containing exactly two non-pendant vertices which one is adjacent to $p$ leaves and the other is adjacent to $q$ leaves. We denote by $P_n$ the path on $n$ vertices. The distance $d_G(u,v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $u-v$ path in $G$. The diameter of a graph $G$, denoted by $\text{diam}(G)$, is the greatest distance between two vertices of $G$. For a vertex $v$ in a rooted tree $T$, let $C(v)$ denote the set of children of $v$, $D(v)$ denotes the set of descendants of $v$ and $D[v] = D(v) \cup \{v\}$. Also, the depth of $v$, $\text{depth}(v)$, is the largest distance from $v$ to a vertex in $D(v)$. The maximal subtree at $v$ is the subtree of $T$ induced by $D[v]$, and is denoted by $T_v$. A proper induced subgraph $H$ of a graph $G$ is called a pendant subgraph if there is exactly one edge between $V(H)$ and $V(G) - V(H)$. 

A function $f : V(G) \rightarrow \{0, 1, 2\}$ is called a Roman dominating function (RDF) on $G$ if every vertex $u \in V$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The weight of an RDF is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The Roman domination number $\gamma_R(G)$ is the minimum weight of an RDF on $G$. Roman domination was introduced by Cockayne et al. in [8] and was inspired by the work of ReVelle and Rosing [10] and Stewart [12]. It is worth mentioning that since 2004, more than hundred papers have been published on this topic, where several new variations were introduced: weak Roman domination [9], Roman $\{2\}$-domination [7], maximal Roman domination [1], mixed Roman domination [3], double Roman domination [6], independent Roman domination [5], signed Roman domination [4, 11], signed total Roman domination [13, 14] and recently outer-independent Roman domination introduced by [2].

For a Roman dominating function $f$, let $V_i = \{v \in V \mid f(v) = i\}$ for $i = 0, 1, 2$. Since these three sets determine $f$, we can equivalently write $f = (V_0, V_1, V_2)$ (or $f = (V_0^f, V_1^f, V_2^f)$ to refer $f$). We note that $\omega(f) = |V_1| + 2|V_2|$. 

A function $f : V(G) \rightarrow \{0, 1, 2\}$ is an outer-independent Roman dominating function (OIRDF) on $G$ if $f$ is an RDF and the set $\{v \in V \mid f(v) = 0\}$ is an independent set. The outer-independent Roman domination number $\gamma_{oiR}(G)$ is the minimum weight of an OIRDF on $G$. The concept of outer-independent Roman domination in graphs was introduced by Ahangar et al. in [2]. Since each outer-independent Roman dominating function is a Roman dominating function, we have the following observation.

Observation 1. For every graph $G$, $\gamma_{oiR}(T) \geq \gamma_R(T)$. 

In this paper, we provide a constructive characterization of trees $T$ with $\gamma_R(T) = \gamma_{oiR}(T)$. 

We make use of the following observations in this paper.

Observation 2. Let $H$ be a subgraph of a graph $G$. If $\gamma_{oiR}(H) = \gamma_R(H)$, $\gamma_{oiR}(G) \leq \gamma_{oiR}(H) + s$ and $\gamma_R(G) \geq \gamma_R(H) + s$ for some non-negative integer $s$, then $\gamma_R(G) = \gamma_{oiR}(G)$, $\gamma_{oiR}(G) = \gamma_{oiR}(H) + s$ and $\gamma_R(G) = \gamma_R(H) + s$. 

$\}$
Proof. We deduce from the assumptions and Observation 1 that
\[ \gamma_{oiR}(G) \geq \gamma_{R}(G) \geq \gamma_{R}(H) + s = \gamma_{oiR}(H) + s \geq \gamma_{oiR}(G). \]
Hence, all inequalities occurring in above chain, become equalities and so \( \gamma_{R}(G) = \gamma_{oiR}(G), \gamma_{oiR}(G) = \gamma_{oiR}(H) + s \) and \( \gamma_{R}(G) = \gamma_{R}(H) + s \).

\[ \square \]

Observation 3. Let \( H \) be a subgraph of a graph \( G \). If \( \gamma_{R}(G) = \gamma_{oiR}(G), \gamma_{R}(G) \leq \gamma_{R}(H) + s \) and \( \gamma_{oiR}(G) \geq \gamma_{oiR}(H) + s \) for some non-negative integer \( s \), then \( \gamma_{R}(H) = \gamma_{oiR}(H), \gamma_{oiR}(G) = \gamma_{oiR}(H) + s \) and \( \gamma_{R}(G) = \gamma_{R}(H) + s \).

\[ \square \]

2. A characterization of trees \( T \) with \( \gamma_{R}(T) = \gamma_{oiR}(T) \)

In this section we give a constructive characterization of all trees \( T \) satisfying \( \gamma_{R}(T) = \gamma_{oiR}(T) \). We start with a definition.

Definition 1. For a graph \( G \) and each vertex \( v \in V(G) \), we say \( v \) has property \( P \) in \( G \) if there exists a \( \gamma_{oiR}(G) \)-function \( f \) such that \( f(v) \neq 0 \). Define
\[ W_{G} = \{ v \mid v \text{ has property } P \text{ in } G \}. \]

Proposition 1. Let \( G \) be a graph and \( v \) be a strong support vertex in \( G \). Then there exists a \( \gamma_{oiR}(G) \)-function (resp. \( \gamma_{R}(G) \)-function) \( f \) such that \( f(v) = 2 \).

Proof. Suppose \( w_{1}, w_{2} \in L_{v} \) and let \( f \) be a \( \gamma_{oiR}(G) \)-function. If \( f(v) = 2 \), then we are done. Let \( f(v) \leq 1 \). If \( f(v) = 1 \), then we must have \( f(w_{1}) = f(w_{2}) = 1 \) and the function \( g : V(G) \to \{0, 1, 2\} \) defined by \( g(w_{1}) = g(w_{2}) = 0, g(v) = 2 \) and \( g(u) = f(u) \) otherwise, is an OIRDF of \( G \) of weight less than \( \gamma_{oiR}(G) \) which is a contradiction. Hence, we assume \( f(v) = 0 \). Since \( f \) is an OIRDF of \( G \), we have \( f(x) \geq 1 \) for each \( x \in N(v) \). Now the function \( g \) defined above, is a \( \gamma_{oiR}(G) \)-function with \( g(v) = 2 \), as desired.

Using a similar argument, we can see that there exists a \( \gamma_{R}(G) \)-function \( f \) such that \( f(v) = 2 \).

\[ \square \]

Corollary 1. Any strong support vertex of a graph \( G \), has property \( P \) in \( G \).
In order to presenting our constructive characterization, we define a family of trees as follows. Let $\mathcal{T}$ be the family of trees $T$ that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ of trees for some $k \geq 1$, where $T_1 \in \{P_2, P_3, P_4\}$ and $T = T_k$. If $i \geq 2$, $T_{i+1}$ can be obtained from $T_i$ by one of the following operations.

**Operation $O_1$:** If $x \in V(T_i)$ is a strong support vertex, then $O_1$ adds a pendant edge $xy$ to obtain $T_{i+1}$.

**Operation $O_2$:** If $x \in V(T_i)$ is a strong support vertex or is adjacent to the center of a pendant star $K_{1,r}$ ($r \geq 1$), then $O_2$ adds a star $K_{1,2}$ and joins $x$ to the center of $K_{1,2}$ to obtain $T_{i+1}$.

**Operation $O_3$:** If $x \in W_{T_i}$, then $O_3$ adds a star $K_{1,r}$ ($r = 2, 3$) and joins $x$ to a leaf of $K_{1,r}$ to obtain $T_{i+1}$.

**Operation $O_4$:** If $x \in V(T_i)$ satisfies in one of the following statement:

1. $x$ is a strong support vertex,
2. $x$ is adjacent to the center of a pendant star $K_{1,r}$ ($r \geq 1$),
3. $x$ is adjacent to a support vertex of a pendant path $P_4$,
4. $x$ is adjacent to the center of a pendant path $P_5$,

then $O_4$ adds a path $y_1y_2y_3y_4y_5$ or a path $y_1y_2y_3y_4$ and joins $x$ to $y_3$ to obtain $T_{i+1}$.

**Operation $O_5$:** If $x \in W_{T_i}$, then $O_5$ adds a double star $DS_{2,1}$ and joins $x$ to the support vertex of degree 2 in $DS_{2,1}$ to obtain $T_{i+1}$.
Operation $O_6$: If $x \in V(T_i)$ satisfies in one of the following statement:

1. $x$ is a strong support vertex,
2. $x$ is a support vertex and there is a path $xx_2x_1$ such that $\deg(x_1) = 1$ and $\deg(x_2) = 2$,
3. there are two paths $xx_2x_1$ and $xz_2z_1$ such that $\deg(x_1) = \deg(z_1) = 1$ and $\deg(x_2) = \deg(z_2) = 2$,

then $O_6$ adds a path $y_1y_2$ and joins $x$ to $y_1$ to obtain $T_{i+1}$.

Operation $O_7$: If $x \in W_{T_i}$, then $O_7$ adds the graph $F_1$ illustrated in Figure 1 and joins $x$ to $y$ to obtain $T_{i+1}$.

Operation $O_8$: If $x \in W_{T_i}$, then $O_8$ adds the graph $F_2$ illustrated Figure 2 and joins $x$ to $y$ to obtain $T_{i+1}$.

Operation $O_9$: If $x \in W_{T_i}$, then $O_9$ adds the graph $F_3$ illustrated in Figure 3 and the edge $xy$ to obtain $T_{i+1}$.

Operation $O_{10}$: If $x \in W_{T_i}$, then $O_{10}$ adds the graph $F_4$ illustrated Figure 4 and the edge $xy$ to obtain $T_{i+1}$.

The proof of the first lemma is trivial by Proposition 1 and Observation 2 and therefore omitted.

**Lemma 1.** If $T_i$ is a tree with $\gamma_R(T_i) = \gamma_{oiR}(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $O_1$, then $\gamma_R(T_{i+1}) = \gamma_{oiR}(T_{i+1})$.

**Lemma 2.** If $T_i$ is a tree with $\gamma_R(T_i) = \gamma_{oiR}(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $O_2$, then $\gamma_R(T_{i+1}) = \gamma_{oiR}(T_{i+1})$.

**Proof.** Let the Operation $O_2$ add a star $K_{1,2}$ centered at $y$ and join $x$ to $y$. Clearly, any outer-independent Roman dominating function of $T_i$ can be extended to an outer-independent Roman dominating function of $T_{i+1}$ by assigning weight 2 to $y$ and 0 to the vertices in $L_y$ yielding $\gamma_{oiR}(T_{i+1}) \leq \gamma_{oiR}(T_i) + 2$.

Now let $f$ be a $\gamma_R(T_{i+1})$-function such that $f(y) + f(x)$ is as large as possible. Obviously $f(y) = 2$. If $f(x) \geq 1$, then the function $f$, restricted to $T_i$ is an RDF of $T_i$ and so $\gamma_R(T_{i+1}) \geq 2 + \gamma_R(T_i)$. Let $f(x) = 0$. Then $x$ is not a strong support vertex and so $x$ is adjacent to the center, say $w$, of a pendant star $K_{1,r} \ (r \geq 1)$. We may assume without loss of generality that $f(w) = 2$. As above, the function $f$, restricted to $T_i$ is an RDF of $T_i$ and so $\gamma_R(T_{i+1}) \geq 2 + \gamma_R(T_i)$. Now the result follows by Observation 2.\square

**Lemma 3.** If $T_i$ is a tree with $\gamma_R(T_i) = \gamma_{oiR}(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $O_3$, then $\gamma_R(T_{i+1}) = \gamma_{oiR}(T_{i+1})$.
Proof. Let $O_3$ add a star $K_{1,r}$ ($r = 2, 3$) centered at $z$ and an edge $xy$ where $y$ is a leaf of $K_{1,r}$. It is easy to see that $\gamma_R(T_{i+1}) \geq \gamma_R(T_i) + 2$. On the other hand, since $x \in W(T_i)$, there exists a $\gamma_{oiR}(T_i)$-function $f$ with $f(x) \geq 1$. Then $f$ can be extended to an outer-independent Roman dominating function of $T_{i+1}$ by assigning weight $2$ to $z$ and $0$ to the neighbors of $z$ implying that $\gamma_{oiR}(T_{i+1}) \leq \gamma_{oiR}(T_i) + 2$. Now the result follows by Observation 2.

![Figure 3. The graph $F_3$ used in Operation $O_9$](image)

**Lemma 4.** If $T_i$ is a tree with $\gamma_R(T_i) = \gamma_{oiR}(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $O_4$, then $\gamma_R(T_{i+1}) = \gamma_{oiR}(T_{i+1})$.

*Proof.* Let $O_4$ add a path $y_1y_2y_3y_4y_5$ (resp. $y_1y_2y_3y_4$) and join $x$ to $y_3$. Clearly, any outer-independent Roman dominating function of $T_i$ can be extended to an outer-independent Roman dominating function of $T_{i+1}$ by assigning the value $2$ to $y_3$, $1$ to $y_1, y_5$ and $0$ to $y_2, y_4$ (resp. the value $2$ to $y_3$, $1$ to $y_1$ and $0$ to $y_2, y_4$) and so $\gamma_{oiR}(T_{i+1}) \leq \gamma_{oiR}(T_i) + 4$ (resp. $\gamma_{oiR}(T_{i+1}) \leq \gamma_{oiR}(T_i) + 3$).

Assume now that $f$ is a $\gamma_R(T_{i+1})$-function such that $f(N[x])$ is as large as possible. Obviously, $f(y_3) = 2$. Since, $x$ is a strong support vertex or is adjacent to the center of a pendant star $K_{1,r}$ ($r \geq 1$) or is adjacent to a support vertex of a pendant path $P_4$ or is adjacent to the center of a pendant path $P_5$, by the choice of $f$ we have $f(y) = 2$ for some $y \in N[x] \setminus \{y_3\}$. Hence, the function $f$, restricted to $T_i$ is a Roman dominating function of $T_i$ and we have $\gamma_R(T_{i+1}) \geq 4 + \gamma_R(T_i)$ (resp. $\gamma_R(T_{i+1}) \geq \gamma_R(T_i) + 3$). Now the result follows by Observation 2.

**Lemma 5.** If $T_i$ is a tree with $\gamma_R(T_i) = \gamma_{oiR}(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $O_5$, then $\gamma_R(T_{i+1}) = \gamma_{oiR}(T_{i+1})$.

*Proof.* Let $O_5$ add a double star $DS_{2,1}$ with the support vertices $a, b$ and join $x$ to $a$ where $\deg(a) = 2$. Since, $x \in W(T_i)$, there exists a $\gamma_{oiR}(T_i)$-function $f$ such that $f(x) \geq 1$. Then $f$ can be extended to an outer-independent Roman dominating function of $T_{i+1}$ by assigning weight $2$ to $b$, $1$ to the leaf adjacent to $a$ and $0$ to the vertices in $L_b \cup \{a\}$ yielding $\gamma_{oiR}(T_{i+1}) \leq \gamma_{oiR}(T_i) + 3$. 


Assume that $w$ is the leaf adjacent to $a$ and $g$ is a $\gamma_R(T_{i+1})$-function such that $g(b) + g(w)$ is as large as possible. Obviously, we have $g(b) = 2$ and $g(w) = 1$. Then the function $g$, restricted to $T_i$ is a Roman dominating function of $T_i$ and we have $\gamma_R(T_{i+1}) \geq 3 + \gamma_R(T_i)$. Now the result follows by Observation 2.

**Lemma 6.** If $T_i$ is a tree with $\gamma_R(T_i) = \gamma_{oiR}(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $O_6$, then $\gamma_R(T_{i+1}) = \gamma_{oiR}(T_{i+1})$.

**Proof.** Let $O_6$ add a path $y_1y_2$ and the edge $xy_1$. Suppose that $f$ is a $\gamma_{oiR}(T_i)$-function such that $f(x)$ is as large as possible. Then $f(x) = 2$ and $f$ can be extended to an outer-independent Roman dominating function of $T_{i+1}$ by assigning a 1 to $y_2$ and a 0 to $y_1$ and this implies that $\gamma_{oiR}(T_{i+1}) \leq \gamma_{oiR}(T_i) + 1$.

On the other hand, if $g$ is a $\gamma_R(T_{i+1})$-function, then $g(x) = 2$ and $g(y_2) = 1$, and the function $g$, restricted to $T_i$ is a Roman dominating function of $T_i$ yielding $\gamma_R(T_{i+1}) \geq 1 + \gamma_R(T_i)$. As in the above lemmas, we obtain $\gamma_{oiR}(T_{i+1}) = \gamma_R(T_{i+1})$.

**Lemma 7.** If $T_i$ is a tree with $\gamma_R(T_i) = \gamma_{oiR}(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $O_7$, then $\gamma_R(T_{i+1}) = \gamma_{oiR}(T_{i+1})$.

**Proof.** Let $O_7$ add the graph $F_1$ and the edge $xy$. Since, $x \in W_{T_i}$, there exists a $\gamma_{oiR}(T_i)$-function $f$ such that $f(x) \geq 1$. Then $f$ can be extended to an outer-independent Roman dominating function of $T_{i+1}$ by assigning a 2 to $y_3$, a 1 to $y_1, y_5$ and a 0 to the vertices $N(y_3)$, and so $\gamma_{oiR}(T_{i+1}) \leq \gamma_{oiR}(T_i) + 4$.

On the other hand, let $g$ be a $\gamma_R(T_{i+1})$-function such that $g(y_3)$ is as large as possible. Clearly, $g(y_3) = 2$, $g(y_1) = g(y_5) = 1$ and $g(y) = 0$. Hence, $g$ restricted to $T_i$ is an RDF of $T_i$ implying that $\gamma_R(T_{i+1}) \geq 4 + \gamma_R(T_i)$. Now the result follows by Observation 2.

**Lemma 8.** If $T_i$ is a tree with $\gamma_R(T_i) = \gamma_{oiR}(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $O_8$, then $\gamma_R(T_{i+1}) = \gamma_{oiR}(T_{i+1})$.
Proof. Let $O_8$ add the graph $F_2$ and joins $x$ to $y$. Since, $x \in W_{T_1}$, there exists a $\gamma_{oiR}(T_i)$-function $f$ such that $f(x) \geq 1$ and $f$ can be extended to an outer-independent Roman dominating function of $T_{i+1}$ by assigning a 2 to $y_3$, a 1 to $y_1, y_5, z$ and a 0 to $y, y_2, y_4$. Thus $\gamma_{oiR}(T_{i+1}) \leq \gamma_{oiR}(T_i) + 5$.

Now let $g$ be a $\gamma_R(T_{i+1})$-function such that $g(y_3) + g(z)$ is as large as possible. Then we must have $g(y_3) = 2, g(z) = g(y_1) = g(y_5) = 1$ and $g(y) = 0$. Hence, the function $g$, restricted to $T_i$ is an Roman dominating function of $T_i$ and so $\gamma_R(T_{i+1}) \geq 5 + \gamma_R(T_i)$. Now the result follows by Observation 2. 

\[ \square \]

Lemma 9. If $T_i$ is a tree with $\gamma_R(T_i) = \gamma_{oiR}(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $O_9$, then $\gamma_R(T_{i+1}) = \gamma_{oiR}(T_{i+1})$.

Proof. Let $O_9$ add a graph $F_3$ and the edge $xy$. Since $x \in W_{T_1}$, there exists a $\gamma_{oiR}(T_i)$-function $f$ such that $f(x) \geq 1$ and $f$ can be extended to an outer-independent Roman dominating function of $T_{i+1}$ by assigning a 2 to $y_3$, 1 to $y_1$ and 0 to the vertices in $N(y_3)$. Hence, $\gamma_{oiR}(T_{i+1}) \leq \gamma_{oiR}(T_i) + 3$.

Now let $g$ be a $\gamma_R(T_{i+1})$-function such that $g(y_3)$ is as large as possible. Clearly, $g(y_3) = 2, g(y_1) = 1$ and $g(y) = 0$. Then the function $g$, restricted to $T_i$ is an RDF of $T_i$ yielding $\gamma_R(T_{i+1}) \geq 3 + \gamma_R(T_i)$. Now the result follows by Observation 2. \[ \square \]

The proof of the next lemma is similar to the proof of Lemma 9 and therefore it is omitted.

Lemma 10. If $T_i$ is a tree with $\gamma_R(T_i) = \gamma_{oiR}(T_i)$ and $T_{i+1}$ is a tree obtained from $T_i$ by Operation $O_{10}$, then $\gamma_R(T_{i+1}) = \gamma_{oiR}(T_{i+1})$.

Theorem 4. If $T \in \mathcal{T}$, then $\gamma_R(T) = \gamma_{oiR}(T)$.

Proof. If $T \in \{P_2, P_3, P_4\}$, then obviously $\gamma_R(T) = \gamma_{oiR}(T)$. Suppose now that $T \in \mathcal{T}$. Then there exists a sequence of trees $T_1, T_2, \ldots, T_k (k \geq 1)$ such that $T_1 \in \{P_2, P_3, P_4\}, T = T_k$ and if $k \geq 2$, then $T_{i+1}$ can be obtained from $T_i$ by one of the Operations $O_1, O_2, \ldots, O_{10}$ for $i = 1, 2, \ldots, k - 1$. We apply induction on the number of operations used to construct $T$. If $k = 1$, the result is trivial. Assume the result holds for each tree $T \in \mathcal{T}$ which can be obtained from a sequence of operations of length $k - 1$ and let $T' = T_{k-1}$. By the induction hypothesis, we have $\gamma_R(T') = \gamma_{oiR}(T')$. Since $T = T_k$ is obtained by one of the Operations $O_1, O_2, \ldots, O_{10}$ from $T'$, we conclude from the above Lemmas that $\gamma_R(T) = \gamma_{oiR}(T)$.

Now we are ready to prove our main result.

Theorem 5. Let $T$ be a non-trivial tree. Then $\gamma_R(T) = \gamma_{oiR}(T)$ if and only if $T \in \mathcal{T}$. 

Proof. According to Theorem 4, we need only to prove necessity. Let $T$ be a tree of order $n \geq 3$ with $\gamma_R(T) = \gamma_{oiR}(T)$. The proof is by induction on $n$. If $n \leq 3$, then clearly $T \in \mathcal{T}$. Let $n \geq 4$ and let the statement hold for all trees of order less than $n$. Assume that $T$ is a tree of order $n$ with $\gamma_R(T) = \gamma_{oiR}(T)$. If $\text{diam}(T) \leq 3$, then $T = P_4$ or $T$ is a star or double star. If $T = P_4$, then obviously $T \in \mathcal{T}$, if $T$ is a star, then $T$ can be obtained from $P_3$ by repeated application of operation $\mathcal{O}_1$, and if $T$ is a double star different from $P_4$, then $T$ can be obtained from $P_3$ by applying operation $\mathcal{O}_2$ once and operation $\mathcal{O}_1$ repeatedly, and this implies that $T \in \mathcal{T}$. Hence let $\text{diam}(T) \geq 4$.

Let $v_1v_2 \ldots v_k$ ($k \geq 5$) be a diametral path in $T$ such that $\text{deg}_T(v_2)$ is as large as possible and root $T$ at $v_k$. If $\text{deg}_T(v_2) \geq 4$, then clearly $\gamma_R(T-v_1) = \gamma_{oiR}(T-v_1)$. It follows from the induction hypothesis that $T - v_1 \in \mathcal{T}$ and hence $T$ can be obtained from $T - v_1$ by Operation $\mathcal{O}_1$ implying that $T \in \mathcal{T}$. Assume that $\text{deg}_T(v_2) \leq 3$.

First let $\text{deg}_T(v_3) = 2$. Suppose that $T' = T - v_3$. Clearly, any Roman dominating function of $T'$ can be extended to a Roman dominating function of $T$ by assigning a 2 to $v_2$ and a 0 to the vertices in $N_T(v_2)$ yielding

$$\gamma_R(T) \leq \gamma_R(T') + 2. \quad (1)$$

Similarly, any outer-independent Roman dominating function of $T'$ can be extended to an outer-independent Roman dominating function of $T$ by assigning a 1 to $v_3$, a 2 to $v_2$ and a 0 to the vertices in $L_{v_2}$ yielding $\gamma_{oiR}(T) \leq \gamma_{oiR}(T') + 3$. On the other hand, assume that $f$ is a $\gamma_{oiR}(T)$-function. Clearly $f(N[v_2]) \geq 2$. If $f(v_3) \leq 1$, then the function $f$ restricted to $T'$ is an OIRDF of $T'$ that implies $\gamma_{oiR}(T) \geq \gamma_{oiR}(T') + 2$. If $f(v_3) = 2$, then $f(L_{v_2}) \geq 1$, and the function $g : V(T') \to \{0, 1, 2\}$ defined by $g(v_4) = \min\{f(v_4) + 1, 2\}$ and $g(x) = f(x)$ otherwise is an OIRDF of $T'$ of weight at most $\gamma_{oiR}(T) - 2$ yielding $\gamma_{oiR}(T) \geq \gamma_{oiR}(T') + 2$. Thus $\gamma_{oiR}(T) - 3 \leq \gamma_{oiR}(T') \leq \gamma_{oiR}(T) - 2$. If $\gamma_{oiR}(T') = \gamma_{oiR}(T) - 3$, then

$$\gamma_{oiR}(T') = \gamma_{oiR}(T) - 3 = \gamma_R(T) - 3 \leq \gamma_R(T') - 1$$

which is a contradiction by Observation 1. Hence,

$$\gamma_{oiR}(T') = \gamma_{oiR}(T) - 2. \quad (2)$$

By (1), (2) and Observation 3, we obtain $\gamma_R(T') = \gamma_{oiR}(T')$. By the induction hypothesis we have $T' \in \mathcal{T}$. Now we show that $v_4 \in W_{T'}$. Let $h$ be a $\gamma_{oiR}(T)$-function such that $h(v_4)$ is as large as possible. Clearly $h(N[v_2]) \geq 2$. Since $\gamma_R(T) = \gamma_{oiR}(T)$, $h$ is also a $\gamma_R(T)$-function. If $h(v_4) \geq 1$, then $h$ restricted to $T'$ is a $\gamma_{oiR}(T')$-function and we are done. Assume that $h(v_4) = 0$. Since $h$ is an OIRDF of $T$, we must have $h(v_3) \geq 1$. If $h(v_3) = 1$, then to Roman dominate the vertices of $v_1, v_2$, we must have $h(N[v_2] - \{v_3\}) \geq 2$. But then the function $h_1 : V(T) \to \{0, 1, 2\}$ defined by $h_1(v_2) = 2, h_1(x) = 0$ for $x \in N(v_2)$ and $h_1(x) = h(x)$ otherwise, is an RDF of $T$ of
On the other hand, let $f$ dominate $\omega$ weight less than 194. On trees with equal Roman domination and... yield a Roman dominating function of $T$ defined by $h_T$.

Let $T'$ be a $\gamma_{oiR}(T)$-function such that $f(v_2) + f(v_3)$ is as large as possible. Clearly, $f(v_2) = 2$. If $f(v_3) \geq 1$, then the function $f$, restricted to $T'$ is an outer-independent Roman dominating function of $T'$ which implies that $\gamma_{oiR}(T') \leq \gamma_{oiR}(T) - 2$. Assume that $f(v_3) = 0$. We deduce from the assumption that $v_3$ is not a strong support vertex, and thus $v_3$ has one child with depth 1, say $u$, different from $v_2$. Since $f$ is an OIRDF of $T$, we have $f(u) \geq 1$. To Roman dominate the leaves adjacent to $u$, we must have $f(u) + f(L(u)) \geq 2$. Without loss of generality, we may assume that $f(u) = 2$. Now the function $f$, restricted to $T'$ is an outer-independent Roman dominating function of $T'$ which implies that $\gamma_{oiR}(T') \leq \gamma_{oiR}(T) - 2$. Thus

\[
\gamma_{oiR}(T') \leq \gamma_{oiR}(T) - 2
\]  

We conclude from (3), (4), Observation 3 and the induction hypothesis that $T' \in \mathcal{T}$. Now $T$ can be obtained from $T'$ by Operation $O_3$ and so $T \in \mathcal{T}$.

Subcase 1.2 $\deg_T(v_3) = 3$ and $v_3$ has one child with depth 0.

Assume that $T' = T - T_{v_3}$ and $L(v_3) = \{u\}$. Clearly, any Roman dominating function of $T'$ can be extended to a Roman dominating function of $T$ by assigning weight 2 to $v_2$, 1 to $u$ and 0 to the vertices in $N_T(v_2)$ which implies that

\[
\gamma_R(T) \leq \gamma_R(T') + 3.
\]  

Consider now a $\gamma_{oiR}(T)$-function $f$ such that $f(v_2) + f(u)$ is as large as possible. Then we must have $f(v_2) = 2$, $f(u) = 1$ and $f(x) = 0$ for $x \in N(v_2)$. Then the function $f$, restricted to $T'$ is an outer-independent Roman dominating function of $T'$ which implies that

\[
\gamma_{oiR}(T') \leq \gamma_{oiR}(T) - 3.
\]
It follows from (3), (4) and Observation 3 that \( \gamma_{oiR}(T') = \gamma_R(T') \) and by the induction hypothesis we have \( T' \in \mathcal{T} \). Since \( \gamma_{oiR}(T') = \gamma_R(T') \), the function \( f \) restricted to \( T' \) is a \( \gamma_{oiR}(T') \)-function. Since \( f \) is an OIRDF of \( T \) and \( f(v_3) = 0 \), we deduce that \( f(v_4) \geq 1 \) and \( v_4 \in W_{T'} \). Now \( T \) can be obtained from \( T' \) by Operation \( \mathcal{O}_5 \) yielding \( T \in \mathcal{T} \).

**Case 2.** \( \deg_T(v_2) = 2 \).

First let \( \deg_T(v_2) \geq 4 \). Assume that \( T' = T - \{v_1, v_2\} \). By the choice of diametrical path, \( v_3 \) is a strong support vertex or is a weak support vertex and there is a path \( x x_2 x_1 \) such that \( \deg(x_1) = 1 \) and \( \deg(x_2) = 2 \) or there are two paths \( x x_2 x_1 \) and \( x_2 z_1 \) such that \( \deg(x_1) = \deg(z_1) = 1 \) and \( \deg(x_2) = \deg(z_2) = 2 \). If \( f \) is a \( \gamma_{oiR}(T) \)-function, then \( f \) is a \( \gamma_R(T) \)-function (since \( \gamma_{oiR}(T) = \gamma_R(T) \)) and we must have \( f(v_3) = 2 \) and \( f(v_1) = 1 \). Thus the function \( f \) restricted to \( T' \) is an OIRDF of \( T' \) and so \( \gamma_{oiR}(T) \geq \gamma_{oiR}(T') + 1 \). On the other hand, if \( g \) is a \( \gamma_R(T') \)-function such that \( g(v_3) \) is as large as possible, then clearly \( g(v_3) = 2 \) and \( g \) can be extended to an RDF of \( T \) by assigning a 1 to \( v_1 \) and a 0 to \( v_2 \) yielding \( \gamma_R(T) \leq \gamma_R(T') + 1 \). We conclude from Observation 3 that \( \gamma_{oiR}(T') = \gamma_R(T') \) and by the induction hypothesis we have \( T' \in \mathcal{T} \). Now \( T \) can be obtained from \( T' \) by Operation \( \mathcal{O}_6 \) and so \( T \in \mathcal{T} \).

Now, let \( \deg_T(v_3) = 3 \). Then \( T_{v_3} \) is a pendant path \( P_4 \) or a pendant path \( P_5 \) in \( T \). If \( \text{diam}(T) = 4 \), then \( T \) can be obtained from \( P_4 \) by Operation \( \mathcal{O}_6 \) or from \( P_2 \) by Operations \( \mathcal{O}_3, \mathcal{O}_6 \) and so \( T \in \mathcal{T} \). Suppose that \( \text{diam}(T) \geq 5 \). We distinguish the following subcases.

**Subcase 2.1.** \( v_3 \) is a support vertex and \( \deg_T(v_4) = 2 \).

Let \( u \) be the leaf adjacent to \( v_3 \) and let \( T' = T - T_{v_4} \). Clearly, any Roman dominating function of \( T' \) can be extended to a Roman dominating function of \( T \) by assigning a 2 to \( v_3 \), 1 to \( v_1 \) and 0 to \( v_2, v_4 \) which implies that \( \gamma_R(T) \leq \gamma_R(T') + 3 \). On the other hand, let \( f \) be a \( \gamma_{oiR}(T) \)-function such that \( f(v_3) \) is as large as possible. Then we must have \( f(v_3) = 2 \), \( f(v_1) = 1 \) and \( f(v_2) = f(u) = 0 \). Now the function \( g : V(T') \to \{0, 1, 2\} \) defined by \( g(v_3) = \min\{2, f(v_3) + f(v_4)\} \) and \( g(x) = f(x) \) otherwise, is an OIRDF of \( T' \) of weight \( \gamma_{oiR}(T) - 3 \) and this implies that \( \gamma_{oiR}(T') \leq \gamma_{oiR}(T) - 3 \). It follows that Observation 3 that \( \gamma_{oiR}(T') = \gamma_R(T') \) and so \( g \) is a \( \gamma_R(T') \)-function yielding \( v_5 \in W_{T'} \). By the induction hypothesis, we obtain \( T' \in \mathcal{T} \). Now \( T \) can be obtained from \( T' \) by Operation \( \mathcal{O}_9 \) and so \( T \in \mathcal{T} \).

**Subcase 2.2.** \( v_3 \) is a support vertex and \( v_4 \) has a child with depth 2.

Let \( u \) be the leaf adjacent to \( v_3 \) and let \( v_4 y_3 y_2 y_1 \) be a path in \( T \) such that \( y_3 \notin \{v_3, v_5\} \). By the choice of diametrical path we have \( \deg(y_2) = 1 \). Considering above arguments, we may assume that \( T_{y_3} = P_4 \) or \( T_{y_3} = P_5 \) since otherwise we can rename \( y_i \) as \( v_i \) and are in the case that \( \deg_T(v_3) = 2 \) which we have considered already. Let \( T' = T - T_{v_4} \). As in the Subcase 2.1, we have \( \gamma_R(T) \leq \gamma_R(T') + 3 \). Now let \( f \) be a \( \gamma_{oiR}(T) \)-function such that \( f(y_3) \) is as large as possible. Clearly, \( f(y_3) = 2 \) and \( f(u) + f(v_1) + f(v_2) + f(v_3) \geq 3 \). Hence the function \( f \) restricted to \( T' \) is an OIRDF of \( T' \) of weight at most \( \gamma_{oiR}(T) - 3 \) implying that \( \gamma_{oiR}(T) \geq \gamma_{oiR}(T') + 3 \). It follows that Observation 3 and the induction hypothesis that \( T' \in \mathcal{T} \). Since \( T \) can be obtained from \( T' \) by Operation \( \mathcal{O}_4 \), we have \( T \in \mathcal{T} \).
Subcase 2.3. \( v_3 \) is a support vertex and \( v_4 \) has a child with depth 1.
Let \( u \) be the leaf adjacent to \( v_3 \) and let \( v_4 y_2 y_1 \) be a path in \( T \) such that \( y_2 \not= v_5 \). Let \( T' = T - T_{v_3} \). As in the Subcase 2.1, we have \( \gamma_R(T) \leq \gamma_R(T') + 3 \). Now let \( f \) be a \( \gamma_{oiR}(T) \)-function such that \( f(v_3) + f(y_2) \) is as large as possible. Then clearly, \( f(v_3) = 2 \), \( f(u) + f(v_1) + f(v_2) + f(v_3) \geq 3 \) and either \( f(v_4) \geq 1 \) or \( f(v_4) = 0 \) and \( f(y_2) = 2 \). Hence, the function \( f \) restricted to \( T' \) is an OIRDF of \( T' \) of weight at most \( \gamma_{oiR}(T) - 3 \) yielding \( \gamma_{oiR}(T') \geq \gamma_{oiR}(T') + 3 \). By Observation 3 and the induction hypothesis, we have \( T' \in \mathcal{T} \). Since \( T \) can be obtained from \( T' \) by Operation \( O_4 \), we have \( T \in \mathcal{T} \).

Subcase 2.4. \( v_3 \) is a support vertex and \( v_4 \) is a strong support vertex.
Let \( u \) be the leaf adjacent to \( v_3 \) and let \( T' = T - T_{v_3} \). As in the Subcase 2.1, we have \( \gamma_R(T) \leq \gamma_R(T') + 3 \). Consider now a \( \gamma_{oiR}(T) \)-function \( f \) such that \( f(v_4) = 2 \) to according Proposition 1. Clearly, \( f(u) + f(v_1) + f(v_2) + f(v_3) \geq 3 \) and the function \( f \) restricted to \( T' \) is an OIRDF of \( T' \) of weight at most \( \gamma_{oiR}(T) - 3 \) yielding \( \gamma_{oiR}(T') \geq \gamma_{oiR}(T') + 3 \). By Observation 3 and the induction hypothesis, we have \( T' \in \mathcal{T} \). Now, \( T \) can be obtained from \( T' \) by Operation \( O_4 \), we have \( T \in \mathcal{T} \).

Subcase 2.5. \( v_3 \) is a support vertex, \( \deg_T(v_4) = 3 \) and \( v_4 \) has a child with depth 0. Let \( z \) be the leaf adjacent to \( v_4 \). If \( \text{diam}(T) = 4 \), then \( T \) can be obtained from \( P_3 \) by Operation \( O_4 \) and so \( T \in \mathcal{T} \). Let \( \text{diam}(T) \geq 5 \) and let \( T' = T - T_{v_4} \). Clearly, any \( \gamma_R(T') \)-function can be extended to an RDF of \( T \) by assigning a 2 to \( v_3 \), a 1 to \( v_1, z \) and a 0 to \( u, v_2, v_4 \) and so

\[
\gamma_R(T) \leq \gamma_R(T') + 4. \tag{7}
\]

Let \( f \) be a \( \gamma_{oiR}(T) \)-function such that \( f(v_3) + f(z) \) is as large as possible. Clearly, \( f(v_3) = 2 \), \( f(V(T_{v_3})) \geq 3 \) and \( f(V(T_{v_4})) \geq 4 \). We claim that \( f(v_4) = 0 \). Suppose, to the contrary, that \( f(v_4) \geq 1 \). Since \( \gamma_R(T) = \gamma_{oiR}(T) \), \( f \) is also a \( \gamma_R(T) \)-function. This implies that \( f(v_4) = 2 \), otherwise we must have \( f(z) = 1 \) and the function \( h : V(T) \to \{0, 1, 2\} \) defined by \( h(v_4) = 0 \), and \( h(t) = f(t) \) otherwise, is an RDF of \( T \) of weight less that \( \omega(f) = \gamma_R(T) \) which is a contradiction. Now the function \( g : V(T) \to \{0, 1, 2\} \) defined by \( g(v_4) = 0, g(z) = 1, g(v_5) = \min\{2, f(v_5) + 1\} \) and \( g(x) = f(x) \) otherwise, is a \( \gamma_{oiR}(T) \)-function contradicting the choice of \( f \). Thus \( f(v_4) = 0 \) and so \( f(v_3) \geq 1 \) because \( f \) is an OIRDF of \( T \). Then the function \( f \) restricted to \( T' \) is an OIRDF of \( T' \) and so

\[
\gamma_{oiR}(T) \geq \gamma_{oiR}(T') + 4. \tag{8}
\]

We deduce from (7), (8) and Observation 3 that \( \gamma_{oiR}(T') = \gamma_R(T') \) and so \( f \) restricted to \( T' \) is a \( \gamma_{oiR}(T') \)-function implying that \( v_5 \in W_{T'} \). By the induction hypothesis, we have \( T' \in \mathcal{T} \) and since \( T \) can be obtained from \( T' \) by Operation \( O_{10} \), we have \( T \in \mathcal{T} \).

Subcase 2.6. There is a pendant path \( v_3 y_2 y_1 \) such that \( y_2 \not\in \{v_2, v_4\} \), and \( \deg_T(v_4) = 2 \).
Then \( T_{v_3} = P_5 \). If \( \text{diam}(T) = 4 \), then \( T \) can be obtained from \( P_5 \) by Operation \( O_6 \) and so \( T \in \mathcal{T} \). Suppose \( \text{diam}(T) \geq 5 \) and let \( T' = T - T_{v_4} \). Clearly, any Roman
dominating function of $T'$ can be extended to a Roman dominating function of $T$ by assigning a 2 to $v_3$, 1 to $v_1, y_1$ and 0 to $v_2, y_2, v_4$ which implies that

$$\gamma_R(T) \leq \gamma_R(T') + 4. \quad (9)$$

Consider now a $\gamma_{oiR}(T)$-function $f$ such that $f(v_3)$ is as large as possible. Then we must have $f(v_3) = 2$, $f(v_1) = f(y_1) = 1$ and $f(v_2) = f(y_2) = 0$. Now the function $g : V(T') \to \{0, 1, 2\}$ defined by $g(v_5) = \min\{2, f(v_5) + f(v_4)\}$ and $g(x) = f(x)$ otherwise, is an OIRDF of $T'$ of weight $\gamma_{oiR}(T) - 4$ yielding

$$\gamma_{oiR}(T') \leq \gamma_{oiR}(T) - 4. \quad (10)$$

By inequalities (9), (10) and Observation 3, we have $\gamma_{oiR}(T') = \gamma_R(T')$ and so $g$ is a $\gamma_R(T')$-function. Since $f$ is an OIRDF of $T$, we must have $f(v_4) + f(v_5) \geq 1$ and so $g(v_5) = \min\{2, f(v_5) + f(v_4)\} \geq 1$ yielding $v_5 \in W_{T'}$. By the induction hypothesis, we obtain $T' \in \mathcal{T}$. Since $T$ can be obtained from $T'$ by Operation $O_7$, we have $T \in \mathcal{T}$.

**Subcase 2.7.** There is a pendant path $v_3y_2y_1$ such that $y_2 \notin \{v_2, v_4\}$, and $v_4$ has a child with depth 2.

Let $v_4z_3z_2z_1$ be a path in $T$ such that $z_3 \notin \{v_3, v_5\}$. By the choice of diametrical path we have $\deg(z_2) = 1$. If $\deg(z_3) = 2$, then $T$ can be obtained from $T - T_{z_3}$ by Operation $O_3$ (see the third paragraph of the proof), if $\deg(z_3) \geq 4$, then $T$ can be obtained from $T - \{z_1, z_2\}$ by Operation $O_6$ (see the first paragraph of Case 2) and if $\deg(v_3) = 3$ and $z_3$ is a support vertex, then $T$ can be obtained from $T - T_{z_3}$ by Operation $O_4$ (see Subcase 2.2). Henceforth, we may assume that $T_{y_3} = P_5$.

Let $T' = T - T_{v_3}$. Clearly, any Roman dominating function of $T'$ can be extended to a Roman dominating function of $T$ by assigning a 2 to $v_3$, 1 to $v_1, y_1$ and 0 to $v_2, y_2$ and so

$$\gamma_R(T) \leq \gamma_R(T') + 4. \quad (11)$$

Consider now a $\gamma_{oiR}(T)$-function $f$ such that $f(v_3) + f(z_3)$ is as large as possible. Then we must have $f(v_3) = f(z_3) = 2$, $f(v_1) = f(y_1) = 1$ and $f(v_2) = f(y_2) = 0$, and $f$ restricted to $T'$ is an OIRDF of $T'$ of weight $\gamma_{oiR}(T) - 4$ and so

$$\gamma_{oiR}(T) \geq \gamma_{oiR}(T') + 4. \quad (12)$$

By inequalities (11), (12), Observation 3 and the induction hypothesis $T' \in \mathcal{T}$. Since $T$ can be obtained from $T'$ by Operation $O_4$, we have $T \in \mathcal{T}$.

**Subcase 2.8.** There is a pendant path $v_3y_2y_1$ such that $y_2 \notin \{v_2, v_4\}$, and $v_4$ has a child $z_2$ with depth 1.

Let $v_4z_2z_1$ be a path in $T$. Assume that $T' = T - T_{v_3}$. As in the Subcase 2.7, we can see that $\gamma_R(T) \leq \gamma_R(T') + 4$. Now let $f$ be a $\gamma_{oiR}(T)$-function such that $f(v_3) + f(z_2)$ is as large as possible. Then clearly, $f(v_3) = 2$, $f(y_2) + f(y_1) + f(v_1) + f(v_2) + f(v_3) \geq 4$ and either $f(v_4) \geq 1$ or $f(v_4) = 0$ and $f(z_2) = 2$. Hence, the function $f$ restricted to
$T'$ is an OIRDF of $T'$ of weight at most $\gamma_{oiR}(T) - 4$ yielding $\gamma_{oiR}(T) \geq \gamma_{oiR}(T') + 4$. By Observation 3 and the induction hypothesis, we obtain $T' \in \mathcal{T}$ and since $T$ can be obtained from $T'$ by Operation $O_4$, we have $T \in \mathcal{T}$.

Subcase 2.9. There is a pendant path $v_3y_2y_1$ such that $y_2 \notin \{v_2, v_4\}$, and $v_4$ is a strong support vertex. Let $T' = T - T_{v_4}$. As in the subcase 2.4, we can see that $T' \in \mathcal{T}$, and since $T$ can be obtained from $T'$ by Operation $O_4$, we have $T \in \mathcal{T}$.

Subcase 2.10. There is a pendant path $v_3y_2y_1$ such that $y_2 \notin \{v_2, v_4\}$, $\deg_T(v_4) = 3$ and $v_4$ has a child with depth one. Let $z$ be the child of $v_4$ with depth one and let $T' = T - T_{v_4}$. Clearly, any $\gamma_R(T')$-function can be extended to an RDF of $T$ by assigning a 2 to $v_3$, a 1 to $v_1, y_1, z$ and a 0 to $y_2, v_2, v_4$ and so $\gamma_R(T) \leq \gamma_R(T') + 5$. Consider now a $\gamma_{oiR}(T)$-function $f$ such that $f(v_3) + f(z)$ is as large as possible. Clearly, $f(v_3) = 2$, $f(V(T_{v_4})) \geq 4$ and $f(V(T_{v_4})) \geq 5$. We claim that $f(v_4) = 0$. Suppose, to the contrary, that $f(v_4) \geq 1$. Since $\gamma_R(T) = \gamma_{oiR}(T)$, $f$ is also a $\gamma_R(T)$-function. This implies that $f(v_4) = 2$, otherwise we must have $f(z) = 1$ and the function $h : V(T) \to \{0, 1, 2\}$ defined by $h(v_4) = 0$, and $h(t) = f(t)$ otherwise, is an RDF of $T$ of weight less that $\omega(f) = \gamma_R(T)$ which is a contradiction. Define $g : V(T) \to \{0, 1, 2\}$ by $g(v_4) = 0, g(z) = 1, g(v_5) = \min\{2, f(v_5) + 1\}$ and $g(x) = f(x)$ otherwise. Clearly, $g$ is a $\gamma_{oiR}(T)$-function contradicting the choice of $f$. Thus $f(v_4) = 0$ and so $f(v_5) \geq 1$ because $f$ is an OIRDF of $T$. Now the function $f$ restricted to $T'$ is an OIRDF of $T'$ and so $\gamma_{oiR}(T') \geq \gamma_{oiR}(T') + 5$. We deduce from Observation 3 that $\gamma_{oiR}(T') = \gamma_R(T')$ and hence $f$ restricted to $T'$ is a $\gamma_{oiR}(T')$-function with $f(v_5) \geq 1$ implying that $v_5 \in W_{T'}$. On the other hand, by the induction hypothesis, we have $T' \in \mathcal{T}$ and since $T$ can be obtained from $T'$ by Operation $O_8$, we have $T \in \mathcal{T}$. This completes the proof. 

References


