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Different-distance sets in a graph

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In honor of Lutz Volkmann on the occasion of his seventy-fifth birthday.

Abstract: A set of vertices S in a connected graph G is a different-distance set if, for any vertex w outside S, no two vertices in S have the same distance to w. The lower and upper different-distance number of a graph are the order of a smallest, respectively largest, maximal different-distance set. We prove that a different-distance set induces either a special type of path or an independent set. We present properties of different-distance sets, and consider the different-distance numbers of paths, cycles, Cartesian products of bipartite graphs, and Cartesian products of complete graphs. We conclude with some open problems and questions.

Keywords: different-distance set, different-distance number

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1. Introduction

As early as 1975, Slater [5] introduced a location concept that has been the focus of much research since then. Let $S = \{s_1, s_2, \ldots, s_k\}$ be an *ordered* set of vertices in a connected graph G with distance function d. The S*-location* of a vertex v in G is the vector $(d(v, s_1), d(v, s_2), \ldots, d(v, s_k))$. Slater called the set S a *locating set* if, for any two distinct vertices of G, their S-locations are distinct. The *location number* of G is the order of a smallest locating set in G. In 1976 Harary and Melter independently introduced the same concept, but in their terminology, the S-location of v is the *metric representation* of v, a locating set is a *resolving set*, and the location number is the *metric dimension* of G. Although it is less imaginative, the Harary-Melter terminology has become the standard terminology. In this paper, we use Slater's terminology. To access the vast literature on the metric dimension of a graph, we just give references to two survey papers, [1], and [2].

In this paper we study the sets $S = \{s_1, s_2, \ldots, s_k\}$ having the property that, for any vertex w not in S, the S-location consists of k different numbers, that is, no two vertices in S have the same distance to w, for any $w \in V - S$. Equivalently, if $d(u, x) = d(v, x)$, for two distinct vertices u and v in S, then x must be in S as well. We call such a set a *different-distance set*. Note that a different-distance set need not be a locating set, and vice versa. For instance, take the graph consisting of a 6-cycle C and a chord between two opposite vertices x and y on C. The $S = \{x, y\}$ is a different-distance set, but not a locating set. On the other hand, a set of two vertices in a triangle forms a locating set, but is not a different-distance set. In this definition, we exclude the empty set and the whole vertex set. One could say that any vertex w outside a different-distance set can *distinguish* all vertices inside the set from each other, since their distances to w are all different.

In a complete graph, trivially, the singletons are the only different-distance sets. Intuitively, one might expect that the larger a set is, the less likely it is that its vertices will be distinguishable by all vertices outside the set. So it is natural to seek the maximal difference-distance sets in a graph. We introduce the lower and upper different-distance numbers, being the orders of a smallest and largest maximal different-distance set. These parameters may take values between 1 and $n - 1$, where n is the order of the graph. We prove that a different-distance set is either an independent set or induces a subgraph that is a pendant path, or a so-called bridging path. The first type of path extends the concept of pendant vertex (such as a leaf in a tree), whereas the latter type of path extends the concept of a bridge. We determine what kinds of sets are different-distance sets in a tree. Using these results, we determine the different-distance numbers of paths and cycles. We prove two crucial simple lemmata: (i) a different-distance set does not contain two vertices of a triangle, (ii) a different-distance set does not contain two vertices at distance 2 having at least two common neighbors. These two lemmata are our basic tool for determining the different-distance numbers of certain Cartesian products of graphs.

2. Different-Distance Sets

Let $G = (V, E)$ be a connected, simple graph without loops, with vertex set V and edge set E. The *order* of G is the number $|V|$. Let u and v be two vertices in V. The degree $deg_G(u)$ of u is the number of neighbors of u, that is, vertices adjacent to u. The *distance* $d_G(u, v)$ between u and v is the length of a shortest u, v-path, or

u, v-geodesic. The *interval* between u and v is the set

$$I_G(u, v) = \{w \mid d(u, w) + d(w, v) = d(u, v)\}.$$

See [4] for an extensive study of the interval function I_G of a graph G. When no confusion arises, we will delete the subscript G in deg_G, d_G, and I_G. For a subset S of V, we denote the subgraph induced by S by $G[S]$. Let $G(u, v)$ denote the subgraph of G induced by $I_G(u, v)$.

Let $P = u_0 \to u_1 \to \ldots \to u_{2k}$ be a path of even length. Then we call the vertex u_k the *middle* of the path P. Let $d(u, v)$ be even, say $d(u, v) = 2k$, then the *middle* $M(u, v)$ of u and v is the subset of $I_G(u, v)$ consisting of the middles of the u, v-geodesics, that is, the vertices x with $d(u, x) = k = d(x, v)$. A subgraph H of G is an *isometric* subgraph, if $d_H(u, v) = d_G(u, v)$, for any two vertices of H, that is, H inherits its distance function from G. Let xy be an edge. By abuse of language, we will sometimes use the word *edge* also for the subgraph induced by xy, and also for the set $\{x, y\}$.

Definition 1. A set S of vertices in a connected graph G is a *different-distance set* if $d(u, w) \neq d(v, w)$, for any vertex w in $V - S$, and for any two vertices u and v in S.

Note that this is equivalent to

$$u, v \in S, d(x, u) = d(x, v) \Rightarrow x \in S.$$

The definition already gives an easy tool: if u and v are two vertices at even distance in a different-distance set S, then $M(u, v) \subset S$. We call this the *middle property* of a different-distance set.

The minimum and maximum orders of maximal different-distance sets S in G are called the *lower* and *upper difference-distance numbers* of G, and are denoted $dd(G)$ and $DD(G)$, respectively.

We assume that the entire vertex set $S = V$ is not a different-distance set, since there are no vertices in $V - S = \emptyset$ to demonstrate different distances to the vertices in S. Similarly, we exclude the empty set $\emptyset$, because in this case nothing is to demonstrate. On the other hand, we assume that, for any nontrivial connected graph G, any singleton set $S = \{u\}$ is a different-distance set, since no vertex in $V - S$ is equidistant to two vertices in S. Thus, it follows, by definition, that for any connected graph G of order $n \geq 2$, we have

$$1 \leq dd(G) \leq DD(G) \leq n - 1. \tag{1}$$

It is easily seen that these bounds are sharp. For the lower bound the next two examples serve the purpose.

Observation 1. *For any complete graph K_n, with $n > 1$, $dd(K_n) = DD(K_n) = 1$.*

Observation 2. *For the Petersen graph P, $dd(P) = DD(P) = 1$.*

We return to the upper bound later.
Obviously, in a connected bipartite graph of order at least 3, there is no vertex equidistant from two adjacent vertices. So we have also the following simple observation.

Observation 3. *Let G be a connected bipartite graph of order at least 3. Then $dd(G) \geq 2$.*

Nothing can be said as yet about the upper different-distance number of a bipartite graph.

Proposition 1. *Let $K_{m,n}$ be a complete bipartite graph of order at least 3. Then $dd(K_{m,n}) = DD(K_{m,n}) = 2$.*

Proof. By Observation 3, it suffices to show that a different-distance set S cannot contain more than two vertices. Assume the contrary. Then it will contain two distinct vertices u and v in the same part of the bipartition. Since u and v are equidistant from any vertex in $V - S$, the set S cannot be a different-distance set. □

First we prove a couple of lemmata that will give us the structure of the subgraph $G[S]$, for any different-distance set S in a connected graph G.

Lemma 1. *Let G be a connected graph, and let S be a different-distance set in G. Then the degree in $G[S]$ of any vertex in S is at most two.*

Proof. Assume that $G[S]$ contains a vertex u of degree at least 3, and let x, y, and z be neighbors of u in S. Let w be any vertex outside S. Consider a vertex in $\{u, x, y, z\}$ closest to w.
If it is u, then either w is equidistant from u and at least one of x, y, z, or w is equidistant from x, y, and z, which is impossible. So let, say, x be the vertex closest to w. Now u is one step farther away from w than x. To avoid that w is equidistant from x and one of y and z, it follows that y and z are at least one step farther away from w than x. Hence w is equidistant from u and one of y and z, or w is equidistant from y and z. This is also impossible. Hence, any vertex in $G[S]$ has degree at most 2 in $G[S]$. □

Lemma 2. *Let G be a connected graph, and let S be a different-distance set in G. Then $G[S]$ does not contain a cycle.*

Proof. Assume that $G[S]$ contains a cycle $C = c_1 \to c_2 \to \ldots \to c_k \to c_1$. Let w be any vertex outside S. Recall that w is not equidistant from any two vertices on C. Let x be the vertex on C closest to w, and let y and z be the neighbors of x on C. Then w must be equidistant from y and z. Since this is impossible, $G[S]$ cannot contain a cycle. □

With these two lemmata in hand, we can determine the structure of the subgraph $G[S]$, for any different-distance set S in G.

Theorem 1. *Let G be a connected graph, and let S be a different-distance set in G. Then $G[S]$ is either a path or an independent set in G of order at least 2.*

Proof. It follows from Lemmata 1 and 2 that the components of $G[S]$ are paths. Assume that $G[S]$ is disconnected and there exists a component that is not a single vertex. We choose an edge xy in $G[S]$, and a vertex z in another component such that the distance between z and x is as small as possible. Let P be a geodesic between z and x. Then P has length at least 2, and no internal vertex of P is in S. So P must have odd length $2k+1$, because, otherwise, the middle vertex of P would be in S. Let w be the vertex of P with $d(z,w) = k+1 = d(w,x)+1$. Since w cannot be equidistant from x and y, we have $d(w,y) = k+1$, so that w is equidistant from z and y. This is impossible. Hence, if $G[S]$ is disconnected, then all components of $G[S]$ are single vertices. □

Next we determine what kinds of paths can constitute a different-distance set. A path of length 0 is just a single vertex, and as such it is always a different-distance set. A path of length 1 is just an edge xy. The vertices x and y form a different-distance set if and only if there is no vertex equidistant from x and y. This happens for instance in bipartite graphs, for each edge. It becomes more interesting when the path has length at least 2.
First we recall two well-known notions. A *cut-vertex* in a graph G is a vertex, the deletion of which increases the number of components. A *bridge* is an edge, the deletion of which increases the number of components. If G is connected to start with, the result of the deletion is a disconnected graph. It is easy to see that the two ends of a bridge form a different-distance set (unless $G = K_2$). The following definition can be viewed as a an extension of these notions of cut-vertex and bridge. Loosely speaking, a bridging path is a path P that connects two different parts of the graph in such a way that one can get from the one part to the other part only by traversing P. First we give the definition, and then we explain why we have chosen to formulate it in this way.

Definition 2. Let G be a connected graph. Let P be a path such that each of its internal vertices has degree 2 in G. The path P is a *bridging path* if the deletion of P results in a disconnected graph, and each edge of P is a bridge.

Note that a bridging path of order 1 is just a cut-vertex. Note that a bridge in itself need not be part of a bridging path: in K_2 the edge is a bridge, but if we delete the path containing this edge, then nothing is left, so this path is not 'bridging' anything. Let P be an x, y-path of order at least 2 such that all its internal vertices have degree 2. The deletion of P might disconnect the graph without P being a bridging path. For instance, take a 4-cycle C, let a, b be a pair of opposite vertices of C, and x, y the other pair of opposite vertices. Now we insert an x, y-path P of positive length. Then C minus the path P consists of two isolated vertices a and b, so is disconnected. But we do not need to traverse P to get from a to b. To make the x, y-path P (with all internal vertices of degree 2) into a bridging path, we need to add the property that each edge on P is a bridge.

There is yet another type of path that can be involved in a different-distance set. It is an extension of the notion of a pendant vertex or a leaf (a vertex of degree 1).

Definition 3. Let G be a connected graph. A *pendant path* in G is a path P such that exactly one of its ends has degree 1, and all internal vertices have degree 2.

By definition, the end of P, that has degree at least 2, is attached to a part of the graph distinct from P. Note that a pendant path is of order at least 2. Moreover, if G contains a pendant path of order k, then it also contains a bridging path of order $k-1$, viz. the pendant path minus its vertex of degree 1. Clearly, a 2-connected graph does not contain a pendant path or a bridging path. On the other hand, such paths can be abundant in trees.

Lemma 3. *Let G be a connected graph, and let P be a pendant path or bridging path of order k. Then the vertices of P form a different-distance set of order k.*

Proof. First let P be a pendant path. Let x be the end of P that is not of degree 1. Then any vertex not in P can reach each vertex of P only via x. It is easy to see that therefore there is no vertex outside P equidistant to any two vertices of P.

Next let P be a bridging path. When we delete the vertices of P, then the remaining graph $G-P$ consists of two or more components. Take any vertex w not in P. Then it lies in one of these components. This component together with P forms an isometric subgraph of G with P as a pendant path. Clearly, no two vertices of P are equidistant from w. □

Now we determine what kinds of paths of length at least 2 can constitute different-distance sets.

Theorem 2. *Let G be a connected graph, and let S be a subset of the vertices of G with $|S| \geq 3$. Then S is a different-distance set such that $G[S]$ is a path if and only if S is the set of vertices of a bridging path or a pendant path.*

Proof. Let S be a different-distance set such that $G[S]$ is a path P of length at least 2, with ends, say, x and y. Note that x and y cannot be adjacent. First we prove that any internal vertex of P has degree 2 in G. Suppose to the contrary that u is an internal vertex of P with $deg_G(u) \geq 3$. Let v and w be the neighbors of u on P, and let z be a neighbor of u not on P. Then z is equidistant from at least two of u, v, w. Thus z must be in S, so that $G[S]$ is not a path. Hence $deg_G(u) = 2$.

If one of the ends of P has degree 1, then, G being connected, there are neighbors of the other end of P that are not in P. So P is a pendant path.

So let both ends x and y of P be vertices of degree at least 2. Hence P is not a pendant path. Suppose that P is not a bridging path. If we delete any internal vertex u from P, then x and y remain in the same component. So there is a path between x and y in $G - u$. Since all internal vertices of P are of degree 2 in G, this path cannot contain any internal vertex of P. Let Q be a shortest x, y-path in $G - u$. Then P and Q have only x and y in common. Since x and y are not adjacent, Q has length at least 2. Let z be an internal vertex of Q. Due to the minimality of the length of Q, it follows that the subpath of Q between x and z is a shortest x, z-path. Similarly, the subpath of Q between y and z is a shortest y, z-path. Let k be the length of Q. If k is even, then the middle of Q is equidistant from x and y, so it must be in S, contradicting the fact that $G[S] = P$. So $k = 2\ell + 1$, for some positive integer ℓ. Let z be the vertex on Q at distance $\ell + 1$ from x and at distance ℓ from y. Then z is equidistant from x and the neighbor y' of y on P. Again this is an impossibility. Thus we have shown that P is a bridging path.

The converse follows from Lemma 3. □

Let G be a connected graph. The *bridging-path number* $Bp(G)$ of G is the number of vertices in a longest path in G that is either a bridging path or a pendant path. Note that $Bp(G) = 0$ if and only if G is 2-connected. The previous theorem has the following simple corollary. Recall that any single vertex is a different-distance set, and that the two ends of a bridge also form a different-distance set.

Corollary 1. *Let G be a connected graph. Then $Bp(G) \leq DD(G)$.*

This corollary is especially interesting in the case of trees. Here, in general, cut-vertices, bridges, bridging paths, and pendant paths are abundant. Of course the question arises whether there are independent different-distance sets in a tree, and if so, what kind of independent sets in a tree can be different-distance sets. Let W be a set of vertices on a path P. We call W *equally spaced* on P, if we can write $W = \{s_1, s_2, \ldots, s_k\}$ such that s_1 and s_k are the ends of P, and if we go from s_1 towards s_k along P, then we encounter the vertices of W in the order according to their labels, that is, first s_1, then s_2, then s_3, and so forth, until we reach s_k. Moreover $d(s_i, s_{i+1}) = d(s_{i+1}, s_{i+2})$, for $i = 1, 2, \ldots, k-2$. We call this distance the distance at which the vertices of W are spaced along the path P.

As preparation for the next theorem, we exhibit in Figure 1 a tree T with an independent different-distance set of order 4, viz. the set of grey vertices. It is easy to verify that it is the only maximum different-distance set in T, so $DD(T) = 4$. Since any edge not incident with one of the grey vertices of degree 2 forms a maximal different-distance set, we have $dd(T) = 2$. The other two maximal different-distance sets are the two bridging paths of length 2 (containing a grey vertex as its internal vertex). Note that we can add any number of leaves at any vertex of T without changing the maximum different-distance set of the four grey vertices.

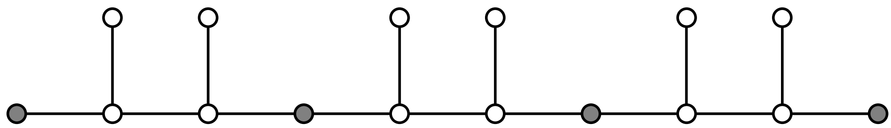

Figure 1. **A maximum independent different-distance set in a tree.**

Theorem 3. *Let T be a tree, and let S be an independent different-distance set of order k in T. Then the vertices of S are equally spaced along a path P in T at an odd distance. Moreover all vertices of S that are internal vertices of P have degree 2 in T.*

Proof. A single vertex satisfies the requirements of being equally spaced on a path at an odd distance. So, for $k = 1$, the conditions are trivially satisfied. If S consists of two vertices, say x and y, then $d(x, y)$ must be odd, for otherwise the middle of the x, y-path would be in S as well. Again the conditions are trivially satisfied.
So assume that $|S| \geq 3$. Let x and y be two vertices of S at minimum distance, and let P_{xy} be the path connecting x and y. Because of the minimality of $d(x, y)$, this distance must be odd. Moreover, no other vertex of P_{xy} can be in S. These vertices divide the tree into three parts: the subtree T_x consisting of the vertices that can be reached from x without using any edge of P_{xy}, the subtree T_y consisting of the vertices that can be reached from y without using any edge of P_{xy}, and the subtree T_{xy} consisting of the vertices that can be reached from x as well as y only by using some edges of P_{xy}. We can view T as consisting of the subtree T_x on the left, the subtree T_y on the right, and the subtree T_{xy} in the middle.
First suppose that S contains a vertex in T_{xy} distinct from x and y. Let z be such a vertex with $d(z, x) + d(z, y)$ as small as possible. Note that, due to the minimality of $d(x, y)$, we have $d(z, x) \geq d(x, y)$ as well as $d(z, y) \geq d(x, y)$. Moreover z cannot be on the path P_{xy}. Since a tree is bipartite one of $d(z, x)$ and $d(z, y)$ must be even, say $d(z, x)$. But then the middle w of the z, x-path must be in S. Because no internal vertex of P_{xy} is in S, it follows that w is not on P_{xy}. Hence, going from z to w in T, we get closer to x as well as y, which means that $d(w, x) < d(z, x)$ and $d(w, y) < d(z, y)$. This contradicts the minimality of $d(z, x) + d(z, y)$. Hence we conclude that S does not contain a third vertex in T_{xy}.

So all vertices of S lie in T_x or T_y. Without loss of generality, we may assume that S contains a vertex of T_x besides x. Let z be such a vertex at minimum distance from x. Since there is no vertex in S between z and x, the distance between z and x is odd. Also we have $d(z,x) \geq d(x,y)$, because of the minimality of $d(x,y)$. Now the distance between z and y is even, so the middle w of the z,y-path is in S. Then w lies on the z,x-path P_{zx}, and is closer to x than z. This is only possible if $w = x$. So $d(z,x) = d(x,y)$. Hence also z and x are vertices in S at minimum distance. Therefore, the same arguments as above apply also on z and x. To make this more precise, we divide T_x using z into two subtrees. Let T_{zx} be the subtree of T_x consisting of x and the vertices that can be reached from x using some edges of P_{zx}, and let T_z be the subtree of T_x that can be reached from z without using edges from P_{zx}. As above S cannot contain a vertex in T_{zx} other than z and x. So if S contains another vertex besides x, y, and z, then it lies in T_z or T_y.
Continuing this way, we conclude that indeed the vertices of S all lie on one path, in such a way that $S = \{s_1, s_2, \ldots, s_k\}$, and $d(s_i, s_{i+1}) = 2\ell + 1$, for some integer ℓ, for $i = 1, 2, \ldots, k-1$. Let P be the path between s_1 and s_k. If s_i with $1 < i < k$ would have another neighbor besides the two on P, then this neighbor would be equidistant from s_{i-1} and s_{i+1}. Since this is impossible, s_i must have degree 2 in T. This concludes the proof. □

We proceed with two basic lemmata that will turn out to be very useful in the next section.

Lemma 4 (Triangle Lemma). *Let G be a connected graph, and let S be a different-distance set in G. Then S does not contain two vertices of a triangle in G.*

Proof. Assume to the contrary that S contains two vertices u and v of a triangle on u, v, w. Since w is equidistant from u and v, it follows that w is in S as well. Take any other vertex x in G, and let u be a vertex in the triangle closest to x. Now either x is equidistant from u and v or from u and w, or x is equidistant from v and w. In each case x is in S. This implies that all vertices of G should be in S, a contradiction. □

The Triangle Lemma suggests that the more triangles there are in a connected graph, the lower its different-distance numbers will be. We will see instances of this intuition in Section 4.
We call two vertices u and v of G an *opposite pair* if they are opposite vertices in a 4-cycle, that is, there are two vertices x and y such that $u \to x \to v \to y \to u$ is a 4-cycle in G. Note that there are four possible situations: (i) the 4-cycle is induced, (ii), xy is an edge, but uv is not, (iii) uv is an edge, but xy is not; (iv) the 4-cycle induces a K_4.

Lemma 5 (Opposite-Pair Lemma). *Let G be a connected graph, and let S be a different-distance set in G. Then S does not contain an opposite pair.*

Proof. Let S be a different-distance set, and let u and v be an opposite pair in a 4-cycle C with x and y being the other two vertices in C. In the situations (iii) and (iv) above, u and v are part of a triangle, so, by the Triangle Lemma, they cannot be both in S. In situation (ii), x and y are in the middle of u and v, so if u and v would both be in S, then x and y would be in S as well. This is impossible by the Triangle Lemma.
Finally, let C be an induced 4-cycle. Assume, to the contrary, that u and v are in S. Then, necessarily, S also contains x and y, these vertices being in the middle of u and v. Since, by Lemma 2, S does not contain a cycle, this is a contradiction. □

The Opposite-Pair Lemma suggests that the more 4-cycles there are in a connected graph, the lower its different-distance numbers will be. We will see instances of this intuition in Section 4. First we apply the two previous lemmata to a simple case.

Theorem 4. *Let $G = K_{a_1,a_2,\ldots,a_n}$ be an n-partite graph with $n \geq 3$. Then $dd(G) = DD(G) = 1$.*

Proof. Note that $n \geq 3$ is essential in the following arguments. Let S be a different-distance set. Any two vertices in the same part of the n-partition form an opposite pair. So S contains at most one vertex from each part. Any two vertices in different parts are vertices of a triangle. So S contains no vertices in more than one part. Hence S contains just one vertex. □

Observations 1 and 2 and Theorem 4 suggest the following question.

Question 1. Can we characterize the connected graphs G with $dd(G) = 1$, or with $DD(G) = 1$?

In Section 4.2 we will encounter another class of graphs G with $DD(G) = 1$. In the next section we deal with the graphs attaining the upper bound of $n - 1$ in (1).

3. The Different-Distance Numbers of Paths and Cycles

By definition, the longest pendant path of a path is a subpath obtained by deleting one of the two endpoints of the path. We denote the path on n vertices by P_n.

Theorem 5. *For any connected graph G of order n, $DD(G) = n - 1$ if and only if $G = P_n$.*

Proof. Deleting one end of P_n results in a pendant path of order $n - 1$. Hence, by Theorems 1 and 2, we have $DD(P_n) = n - 1$.
Assume therefore that G is a graph of order n for which $DD(G) = n - 1$. Let S be a different-distance set of order $n - 1$, and let v be the vertex not in S. Then the

distances of v to the vertices in S must all be different. Since G is connected, and the maximum possible distance in a graph of order n is $n-1$, the distances of v to the vertices in S must take the values $1, 2, \ldots, n-1$. Hence there is a vertex w at distance $n-1$ from v. So G is a path of order n. □

The paths also provide an example that the gap between $dd(G)$ and $DD(G)$ can be arbitrarily large.

Theorem 6. *For a path P_n of order n,*

$$dd(P_n) = \begin{cases} n-1 & \text{if } n-1 \textit{ is a power of 2} \\ \frac{n-1}{r}+1 & \text{otherwise, where } r \textit{ is the largest odd prime divisor of } n-1. \end{cases}$$

Proof. First let $n-1 = 2^t$, for some positive integer t. Let S be a different-distance set. If S would contain both ends of P_n, then, by the middle property, all vertices of P_n would necessarily be in S. So S does not contain both ends. Now, for S to be maximal, it follows that $G[S]$ has to be one of the pendant paths of order $n-1$. So we have $dd(P_n) = DD(P_n) = n-1$.
Next let $n-1$ not be a power of 2, and let r be the largest prime divisor of $n-1$. Again let S be a different-distance set. If S does not contain both ends of P_n, then, for S to be maximal, it must be one of the pendant paths of order $n-1$. Assume that S contains both ends of P_n. Then $G[S]$ cannot be connected, for otherwise it would be P_n itself. By Theorem 3, S consists of vertices equally spaced along P_n at an odd distance. If this distance is t, then $|S| = \frac{n-1}{t}+1$. If t is not prime, say $t = a \times b$ with both a and b odd, then we can add vertices to S such that it becomes an equally spaced set at distance a. So S is not maximal. Hence S is maximal if and only if it is equally spaced at distance t with t a prime number. The smallest order for such a maximal different-distance set is realized with the distance being the largest odd prime divisor r of $n-1$. □

It follows from Theorem 5 that, for any positive integer k, there exists a graph G, for which $DD(G) = k$. However, one may ask whether this also holds for 2-connected graphs. To answer this question we determine the different-distance numbers of cycles. First we observe that, since there are no pendant paths or bridging paths, the only different-distance sets inducing a connected subgraph are those that induce a single vertex or a P_2. So any different-distance set of order at least 3 must be independent. We use this observation below frequently.
Let C be a cycle, and let x and y be distinct vertices of C. Then there are two x, y-paths on C that share only their endpoints x and y. We denote these paths as P_{xy} and Q_{xy}, such that the length of P_{xy} is at most that of Q_{xy}. Then P_{xy} is a shortest x, y-path. If Q_{xy} has the same length, then it is also a shortest x, y-path. Otherwise it is a longer path. If either of these two paths is of even length, then the middle of that path is equidistant from x and y. We will use this observation in the proofs of the following lemma and propositions.

Lemma 6. *Let S be an independent different-distance set in a cycle with $|S| > 3$. Then the vertices of S are equally spaced along the cycle at an odd distance.*

Proof. Let x and y be two vertices in S at minimum distance. Since S is independent, x and y are non-adjacent. The minimality implies that $d(x, y)$ is odd. Let z be another vertex in S closest to x or y, say closest to x. Then $d(z, x)$ is also odd. By the minimality of $d(x, y)$, we have $d(z, x) \geq d(x, y)$. Now the z, y-path containing x is an even path. So its middle must lie in S. Since this path shares only z, x, and y with S, its middle must be x, so that $d(z, x) = d(x, y)$. If S contains yet another vertex, then let w be a vertex that is closest to z or y, say to z. Applying the same argument as above to the vertices w, z, and x instead of z, x, and y, it follows that $d(w, z) = d(z, x) = d(x, y)$. Continuing this way, we conclude that the vertices of S are equally spaced along a path P at odd distance $r = d(x, y)$. This path begins in a vertex u of S and ends in a vertex v of S. Consider the other u, v-path Q on C. Note that Q shares only u and v with S. Hence Q must have odd length, for otherwise, the middle of Q should also be in S. If the length of Q would be larger than r, then let u' be the first vertex of S on P after u. Then Q together with u, u'-path of length r would be an even path of length greater than $2r$, and its middle would not be in S. So also $d(u, v) = r$, and the vertices of S are equally spaced along C at odd distance r. □

Note that in the situation of Lemma 6, the odd distance r must be a divisor of the order of the cycle, and the number of vertices in S is then $\frac{n}{r}$. We denote the cycle on n vertices by C_n. For convenience, we split the results on $dd(C)$ and $DD(C)$ into a couple of propositions, depending on the value of n

Proposition 2. *If n is a power of 2, then $dd(C_n) = DD(C_n) = 2$.*

Proof. Any different-distance set of order at least 3 is independent. By Lemma 6, there are no such sets. Any two vertices at odd distance form a different-distance set, so these are the maximal ones. □

Proposition 3. *If n is an odd prime, then $dd(C_n) = DD(C_n) = 1$.*

Proof. By Lemma 6, there can be no different-distance set of order at least 3. For any two vertices x and y, one of the connecting paths in the cycle is even. So its middle is equidistant from x and y. Hence there are no different-distance sets of order 2. Therefore any single vertex constitutes a maximal different-distance set. □

Proposition 4. *If n is even with an odd divisor, then $dd(C_n) = 2$ and $DD(C_n) = \frac{n}{r}$, with r being the smallest odd divisor of n.*

Proof. In an even cycle, every two adjacent vertices form a different-distance set. Since such a set induces a connected subgraph, it must be maximal. So $dd(C_n) = 2$. Let us determine $DD(C_n)$. First consider the case where $n = 2p$, with p an odd prime number. Since n is even, any two vertices at odd distance form a different-distance set. The only way to space vertices equally at odd distance is to take two vertices at distance p in the cycle. So $DD(C_{2p}) = 2 = \frac{n}{p}$.
Next let n not be 2 times an odd prime. Observe that, for any odd divisor t of n, we can find $\frac{n}{t}$ vertices that are equally spaced along the cycle. These form a different-distance set S. By Lemma 6, there are no other independent different-distance sets. So, if we take r to be a smallest odd divisor of n, then we maximize the order of S, giving us the value $DD(C_n) = \frac{n}{r}$. □

Proposition 5. *If n is odd but not prime, then $dd(C_n) = \frac{n}{t}$ with t being a largest odd divisor of n, and $DD(C_n) = \frac{n}{r}$ with r being a smallest odd divisor of n.*

Proof. In an odd cycle, for any two vertices u and v, there are two edge disjoint paths on the cycle from u to v, one of these two paths must be of even length, and the middle of this path is equidistant to both u and v. Therefore, no set of order 2 can be a different-distance set of an odd cycle.
Any equally spaced set of vertices at odd distance is of order at least 3, and is a different-distance set. So the maximal ones are those of order $\frac{n}{p}$, where p is a prime divisor of n. This gives us the values of the different-distance numbers. □

4. Cartesian Products

First we present the basic facts and notation for the Cartesian product of graphs. For an extensive discussion of this graph product, we refer the reader to [3].
Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two connected graphs. The *Cartesian product* of these graphs is the graph $G = G_1 \Box G_2$ with vertex set $V_1 \times V_2$, where $(u_1, u_2)(v_1, v_2)$ is an edge if and only if either u_1v_1 is an edge in G_1 and $u_2 = v_2$ or $u_1 = v_1$ and u_2v_2 is an edge in G_2. If G_2 consists of a single vertex, then $G_1 \Box G_2$ is isomorphic to G_1. Similarly, $G_1 \Box G_2$ is isomorphic to G_2 when G_1 consists of a single vertex. In the sequel, we will always assume that both G_1 and G_2 have order at least 2. An example of the Cartesian product of two paths is the $m \times n$ *grid*: it is the graph $P_m \Box P_n$, with $m, n \geq 2$, where P_k is the path on k vertices, see Figure 2 for the case $P_5 \Box P_3$. We call two vertices of degree 2 in the grid *opposite corners* if their distance is $m + n$. The vertices (u, r) and (v, t) are opposite corners in Figure 2. Any geodesic between (u, r) and (v, t) can be obtained by starting in (u, r), and then in each step either moving to the right or moving upwards until (v, t) is reached. A *boundary path* between two opposite corners of the grid consists of either a horizontal path followed by a vertical path, or a vertical path followed by a horizontal path. Clearly, if two opposite corners of the grid are at even distance, then their middle $M((u, r)(v, t))$ contains an opposite pair. The middle of (u, r) and (v, t) in Figure 2 consists of the three grey vertices.

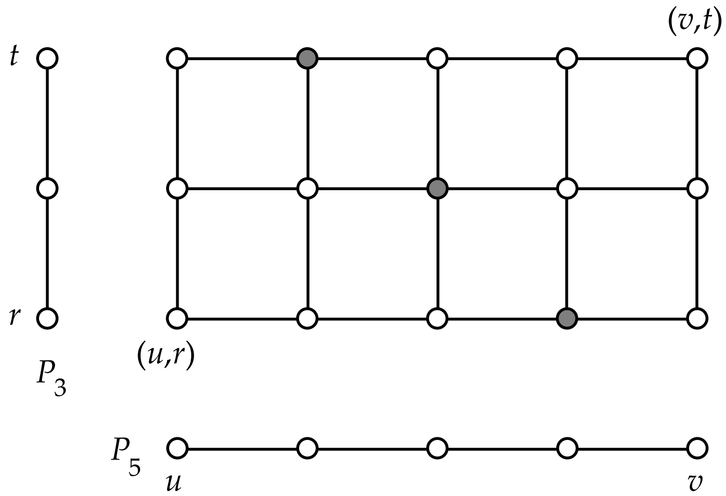

Figure 2. **The grid** $P_5 \Box P_3$

Let u_1 be a vertex in G_1, and u_2 be a vertex in G_2.We call the subgraph induced by the vertices in $V_1 \times \{u_2\}$ a *horizontal fiber* in G, and denote it by $G_1 \Box u_2$. We call the subgraph of G induced by the vertices $\{u_1\} \times V_2$ a *vertical fiber* in G, and we denote it by $u_1 \Box G_2$. Note that $G_1 \Box u_2$ is the unique horizontal fiber containing (u_1, u_2), and $u_1 \Box G_2$ is the unique vertical fiber containing (u_1, u_2). Moreover, the horizontal fiber $G_1 \Box u_2$ and the vertical fiber $u_1 \Box G_2$ intersect precisely in the single vertex (u_1, u_2). Any two distinct horizontal fibers are disjoint. We call such fibers *parallel horizontal fibers*. Similarly, any two distinct vertical fibers are disjoint, which are then called *parallel vertical fibers*. For any two horizontal fibers $G_1 \Box u_2$ and $G_1 \Box v_2$, the *distance* between the two fibers is just the distance between u_2 and v_2 in G_2. The distance between the two fibers equals the smallest distance between any vertex in the one fiber at any vertex in the other fiber. Similarly, we can define the distance between two vertical fibers. Two horizontal fibers $G_1 \Box u_2$ and $G_1 \Box v_2$ are *adjacent fibers* if their distance is 1, or, equivalently, if u_2 and v_2 are adjacent. The adjacency of vertical fibers is defined similarly.
Since the Cartesian product of two connected graphs is 2-connected, it does not contain any bridging path or any pendant path. By Theorems 1 and 2, we have the following observation.

Observation 4. *Let $G_1 \Box G_2$ be the Cartesian product of two connected graphs G_1 and G_2. Then any different-distance set S in $G_1 \Box G_2$ is either an edge or an independent set.*

Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be two vertices in $G = G_1 \Box G_2$. It is straightforward to check that the subgraph induced by the vertices in the interval $I(u, v)$ is $G(u, v) = G_1(u_1, v_1)\Box G_2(u_2, v_2)$, see [3] for a proof. Note that, if u and v are in the same horizontal fiber, then $G(u, v)$ is isomorphic to $G_1(u_1, v_1)$. Similarly, if u and v are in the same vertical fiber, then $G(u, v)$ is isomorphic to $G_2(u_2, v_2)$. If u and v are in different fibers, then u_1 and v_1 are distinct, as well as u_2 and v_2. So they are in parallel horizontal fibers as well as in parallel vertical fibers. Let R_1 be a u_1, v_1-geodesic in G_1 and let R_2 be a u_2, v_2-geodesic in G_2. Then the horizontal path $R_1\Box u_2$ followed by the vertical path $v_1\Box R_2$ is a $(u_1, u_2), (v_1, v_2)$-geodesic in $G(u, v)$. It is a boundary path in the grid $R_1\Box R_2$. This grid is an isometric subgraph of $G(u, v)$, that is, the distances in this grid are also the distances in $G(u, v)$ as well as in $G_1\Box G_2$. This implies that the middle of any two vertices at even distance in distinct fibers contains an opposite pair. This imposes quite some restrictions on different-distance sets in Cartesian products.

Lemma 7. *Let G_1 and G_2 be connected graphs of order at least 2, and let S be a different-distance set in $G_1\Box G_2$. Then S cannot contain two vertices at even distance in different fibers.*

Proof. Suppose, to the contrary, that u and v are two vertices in S in different fibers with $d(u, v)$ even. Our observations above produce an opposite pair in the middle of u and v. So this opposite pair lies in S, which is impossible by the Opposite-Pair Lemma. □

A different-distance set in a Cartesian product may contain vertices at even distance, as long as they are in the same fiber of the product. But also in this case there are restrictions, as given in the following lemma.

Lemma 8. *Let G_1 and G_2 be connected graphs of order at least 2, and let S be a different-distance set in $G_1\Box G_2$. If S lies in one fiber, then S cannot contain two vertices at even distance.*

Proof. Suppose to the contrary that S lies entirely in one fiber, and S contains two vertices u and v with $\mathrm{d}(u, v) = 2k$. Let x be a vertex in the middle of u and v, that is, $d(u, x) = k = d(x, v)$. Let z be a neighbor of x in an adjacent fiber. Then we have $d(z, u) = k + 1 = d(z, v)$. This implies that z is also in S. □

This lemma is essentially a consequence of the following fact. Let G_1 and G_2 be two connected graphs. Consider any fiber in $G_1\Box G_2$, say the horizontal fiber $F = G_1\Box z_2$. Let u and v be two vertices in F at even distance, and let x be in the middle of u and v. Clearly x lies in F, so $x = (x_1, z_2)$, for some vertex x_1 in G_1. Now u and v are equidistant from all vertices in the vertical fiber $x_1\Box G_2$.

4.1. The Cartesian Product of Bipartite Graphs

Note that the Cartesian product of two bipartite graphs is again bipartite.

Theorem 7. *Let G_1 and G_2 be connected bipartite graphs of order at least 2. Then $dd(G_1 \Box G_2) = DD(G_1 \Box G_2) = 2$.*

Proof. By Observation 3, we know that $dd(G_1 \Box G_2) \geq 2$. Assume, to the contrary, that there is a different-distance set S with at least three vertices. Since the Cartesian product of bipartite graphs is again bipartite, no three vertices can be pairwise at odd distance. So S contains vertices at even distance. Let d be the distance function of $G_1 \Box G_2$. Choose two vertices u and v in S with $d(u, v) = 2k$ and k as small as possible. Note that this means that k is odd. By Lemma 7, u and v must lie in the same fiber. Without loss of generality, we may assume that this fiber is a horizontal fiber. Then we have, say, $u = (u_1, u_2)$ and $v = (v_1, u_2)$ in the fiber $F = G_1 \Box u_2$.
Let x be a vertex in the middle of u and v. Then x is in F, say $x = (x_1, u_2)$, with x_1 being a vertex in the middle of u_1 and v_1 in G_1. Take any neighbor y_2 of u_2 in G_2. Then $y = (x_1, y_2)$ is a neighbor of x in the vertical fiber $x_1 \Box G_2$. Due to the properties of distances in Cartesian products, we have $d(y, u) = 1 + d(x, u) = 1 + k = 1 + d(x, v) = d(y, v)$. So y must be in S. Moreover, k being odd, it follows that $d(y, u)$ is even. On the other hand, u and y are in different fibers. By Lemma 7, this is impossible. So S cannot contain more than two vertices. □

Since the Cartesian product of bipartite graphs is again bipartite, we have the following simple corollary.

Corollary 2. *Let $G_1, G_2, \ldots, G_n$ be connected bipartite graphs of order at least 2. Then $DD(G_1 \Box G_2 \Box \ldots \Box G_n) = 2$.*

The simplest connected bipartite graph of order at least 2 is K_2. The hypercube Q_n is the Cartesian product of n copies of K_2. For $n = 1$, we have $Q_1 = K_2$. So we also have the following theorem.

Theorem 8.

$$dd(Q_n) = DD(Q_n) = \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n > 1. \end{cases}$$

The *n-dimensional grid* is the product of n paths, all of length at least 1, with $n \geq 2$.

Corollary 3. *Let G be an n-dimensional grid. Then $dd(G) = DD(G) = 2$.*

As we observed in Section 2, any two vertices at odd distance in a bipartite graph form a different-distance set. So these are precisely the maximal ones in the above examples of Cartesian products of bipartite graphs.

4.2. The Cartesian Product of Complete Graphs

Let $q_1, q_2, \ldots, q_n$ be integers, all at least 2. The *Hamming graph* $H_{q_1,q_2,\ldots,q_n}$ has as vertex set the set of integer vectors $(a_1, a_2, \ldots, a_n)$ with $0 \le a_i < q_i$, for $i = 1, 2, \ldots, n$. Two vertices are adjacent if, as vectors, they differ in exactly one place (coordinate). We call q_i the *order* of the i-th coordinate. The definition of a Hamming graph can also be phrased in terms of products: it is the Cartesian product of complete graphs, viz. $H_{q_1,q_2,\ldots,q_n} = K_{q_1} \Box K_{q_2} \Box \ldots \Box K_{q_n}$. Hamming graphs were introduced in [4] as a generalization of hypercubes. The hypercube Q_n is the Hamming graph with $q_1 = q_2 = \ldots = q_n = 2$. If $n = 1$, then we get the complete graph K_{q_1}, which has different-distance number 1 (unless it is K_1, which, by definition, has no different-distance set). So, in the sequel we take $n \ge 2$.
The aim of this section is to determine the different-distance sets of a Hamming graph. A feature of a Hamming graph H is that, for two vertices u and v of H, the distance $d(u, v)$ between u and v is precisely the number of places (coordinates) in which their two vectors differ. This is the well-known *Hamming distance* between the two vectors. We would like to draw attention to the fact that a Hamming graph has many nice properties and symmetries. For example, if we prove something using two vertices, say at distance k, then we can choose these, without loss of generality, to be the all-zero vector and the vector with a 1 in the first k places and a 0 elsewhere. We use this below.

Lemma 9. *Let H be a Hamming graph, and let S be a different-distance set in H. Then S does not contain a pair of vertices at even distance.*

Proof. Let x and y be two vertices at even distance, say $d(x, y) = 2k$, for some $k \ge 1$. Without loss of generality we may take $x = (0, \ldots, 0, 0, \ldots, 0)$ and $y = (1, \ldots, 1, 0, \ldots, 0)$, where x is the all-zero vector and y has a 1 in the first $2k$ places and a 0 elsewhere. Consider the vertex $z = (1, \ldots, 1, 0, \ldots, 0)$ with a 1 in the first k places and a 0 elsewhere. Then $d(z, x) = d(z, y) = k$. Next consider the vertex $z' = (0, 1, \ldots, 1, 1, 0, \ldots 0)$, with a 0 in the first place, a 1 in the places 1 up to $k + 1$ and a 0 elsewhere. Also for this vertex we have $d(z', x) = d(z', y) = k$. So z and z' are in the middle of x and y, whence they should both be in S. On the other hand, z and z' are an opposite pair, so they cannot both be in S. Hence we conclude that x and y cannot be both in S. □

Theorem 9. *Let H be a Hamming graph, and let S be a different-distance set containing two vertices u and v at odd distance. Then $S = \{u, v\}$, and all coordinates in which u and v differ, have order 2.*

Proof. By Observation 4, either uv is an edge and $S = \{u, v\}$, or S is independent. Suppose that uv is an edge. By the Triangle Lemma, uv is not in a triangle, so the coordinate in which u and v differ is of order 2, and we are done.

Next we assume that S is independent, whence u and v are not adjacent. So let $d(u,v) = 2k+1$, with $k \geq 1$. Without loss of generality, we may take $u = (0,0,\ldots,0)$, and $v = (1,\ldots,1,0,\ldots,0)$ with a 1 in the first $2k+1$ places and a 0 elsewhere.
Assume that $q_1 \geq 3$. Let $v_1 = (2,1,\ldots,1,0,\ldots,0)$, with a 1 in the places 2 up to $k+1$. Then we have $d(v_1,u) = k+1 = d(v_1,v)$. So v_1 is in S. Due to Lemma 9, $k+1$ must be odd, say $k = 2k_1$. Thus we have found a vertex v_1 in S with $d(u,v_1) = 2k_1+1$, where $k_1 = \frac{1}{2}k$. Moreover, u and v_1 differ in the first coordinate. Hence we can repeat this argument to find a vertex v_2 in S with $d(u,v_2) = 2k_2+1$, where $k_2 = \frac{1}{2}k_1$, such that u and v_2 differ in the first coordinate. We can repeat this indefinitely, which is, of course, impossible. So the conclusion is that $q_1 = 2$.
A similar argument shows that $q_i = 2$, for $1 \leq i \leq 2k+1$.
Next we show that $|S| = 2$. Assume, to the contrary, that S contains a vertex w distinct from u and v, with $d(w,u) = 2\ell+1$. Note that all coordinates, where v and w differ from u, have order 2. So w has a 1 in $2\ell+1$ places, and a 0 elsewhere. Without loss of generality, we may assume that $w = (1,\ldots,1,0,\ldots,0,1,\ldots,1,0,\ldots,0)$, with a 1 in the first t places, a 0 in the next $2k+1-t$ places, a 1 in the next $2\ell+1-t$ places, and a 0 in the remaining places. We list the three vertices below.

$$u = (\overbrace{0,\ldots,0}^{t},\overbrace{0,\ldots,0}^{2k+1-t},\overbrace{0,\ldots,0}^{2\ell+1-t},0,\ldots,0),$$

$$v = (\overbrace{1,\ldots,1}^{t},\overbrace{1,\ldots,1}^{2k+1-t},\overbrace{0,\ldots,0}^{2\ell+1-t},0,\ldots,0),$$

$$w = (\overbrace{1,\ldots,1}^{t},\overbrace{0,\ldots,0}^{2k+1-t},\overbrace{1,\ldots,1}^{2\ell+1-t},0,\ldots,0).$$

We have $d(w,v) = (2k+1-t)+(2\ell+1-t) = 2\ell+2k+2-2t$, which is even. This is impossible. This contradiction shows that $S = \{u,v\}$, which completes the proof. $\square$

The last part of the proof, where we show that $|S| = 2$, can also be given in a different way. Let $H = H_{q_1,q_2,\ldots,q_n}$ be a Hamming graph, in which the first m coordinates have order at least 3, and $q_i = 2$, for $m+1 \leq i \leq n$. Then we can also write H as the Cartesian product $H_{q_1,\ldots,q_m} \Box Q_{n-m}$. Due to Lemma 9, any two vertices in S have odd distance. Due to the first part of the proof, the vertices in S must lie in the same vertical fiber $x \Box Q_{n-m}$, with x a vertex in $H_{q_1,\ldots,q_m}$. Since this fiber is a bipartite graph, there can be at most two vertices in S.
This theorem enables us to determine the different-distance numbers of any Hamming graph.

Theorem 10. *Let $H = H_{q_1,q_2,\ldots,q_n}$ be a Hamming graph with all coordinate orders at least 3. Then $dd(H) = DD(H) = 1$.*

Proof. Let S be a different-distance set of H. By Lemma 9 and Theorem 9, the set S cannot contain two distinct vertices. So $|S| = 1$. $\square$

Theorem 11. *Let $H = H_{q_1,q_2,\ldots,q_n}$ be a Hamming graph, in which at least one coordinate has order 2. Then $dd(H) = DD(H) = 2$, and any pair of vertices at odd distance differing only in coordinates of order 2 is a different-distance set.*

Proof. Let u and v be vertices with $d(u, v) = 2k + 1$, for some $k \geq 0$, such that u and v differ only in coordinates of order 2. Without loss of generality we may assume that $u = (0, 0, \ldots, 0)$ is the all-zero vector, and that $v = (1, \ldots, 1, 0, \ldots, 0)$, with a 1 in the first $2k + 1$ places. Note that the first $2k + 1$ coordinates all have order 2.
Let x be any other vertex. In any of the first $2k + 1$ places, x has either a 0 or a 1. In these first $2k + 1$ places, let x have a 0 in t places and a 1 in $2k + 1 - t$ places. Let x have r non-zero's in the remaining places. Then we have $d(x, u) = 2k + 1 - t + r$ and $d(x, v) = t + r$. Clearly we have $2k + 1 - t \neq t$. So x has different distances to u and v. Therefore $S = \{u, v\}$ is a different-distance set, and by Theorem 9, it is a maximum different-distance set. Moreover, due to Theorem 9, there are no other non-trivial different-distance sets in H. □

Theorem 8 on hypercubes is also a simple corollary of the previous theorem.

4.3. Cartesian Products with a Different-Distance Set of Order k

Since Cartesian products contain many induced 4-cycles, the different-distance numbers will be low compared to the order of the graph. The first question that arises is: given any positive integer k, is there a Cartesian product that has a different-distance set of order k? We present a simple construction that gives an affirmative answer to this question.
Let G be a connected graph. Let S be a difference-distance set in G. We call S a *strong different-distance set* if, in addition, no vertex in S is equidistant to any two vertices in S.

Theorem 12. *Let G and H be connected graphs of order at least 2, and let S be a strong different-distance set in G. Then $S \times \{x\}$ is a strong different-distance set in $G \Box H$, for any vertex x in H.*

Proof. Note that $S' = S \times \{x\}$ is just a copy of the set S in the horizontal fiber $F = G \Box x$. No vertex in F is equidistant to two vertices in S'. Let z be any vertex in another fiber. Let D be the vertical fiber containing z, and let y be the common vertex of D and F. Then $d(z, w) = d(z, y) + d(y, w)$, for any vertex w in F. Now y is not equidistant to any two vertices in S'. So z cannot be equidistant from any two vertices in S'. □

Next we construct a graph that has a strong different-distance set of order k, for any $k \geq 1$. The case $k = 3$ is depicted in Figure 3. Let $P = u_1 \to u_2 \to \ldots \to u_{2^k-1} \to u_{2^k}$ be a path of length $2^k - 1$. We construct the graph R_k by adding the vertex v_i adjacent to u_{2^i-1} and u_{2^i}, for $i = 1, 2, \ldots, k$. Let $S = \{v_1, v_2, \ldots, v_k\}$. It is straightforward to check that all distances in S are odd, and that they are pairwise distinct. Moreover,

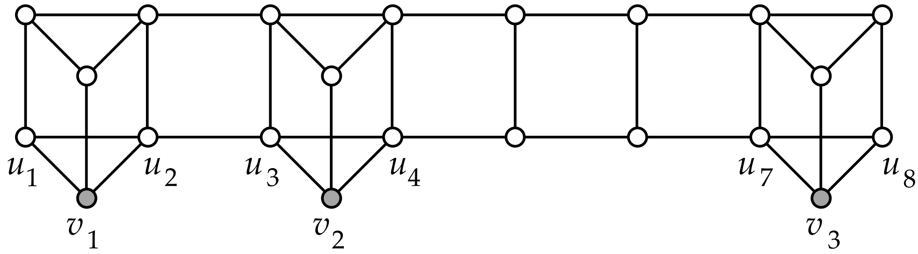

Figure 3. $DD(R_3 \Box K_2) \geq 3$.

no vertex u_j is equidistant from two vertices in S. So S is a strong different-distance set in R_k. By Theorem 12, we have $DD(R_k \Box H) \geq k$, for any graph H of order at least 2.

The strongness of the different-distance set is a necessary condition in Theorem 12. This is shown by the following example. Take the path P_n with different-distance set $S = \{s_1, s_2, \ldots, s_k\}$ of order at least 3, equally spaced at odd distance r. Take the Cartesian product $P_n \Box K_2$, with K_2 being the edge xy. Then $S \Box x$ is *not* a different distance set in $P_n \Box K_2$. For, (s_2, y) in the fiber $P_n \Box y$ is adjacent to (s_2, x) in the set $S \times \{x\}$. Clearly, (s_2, y) is equidistant from (s_1, x) and (s_3, x) in $S \times \{x\}$. We can say even more, see the next theorem. Loosely speaking it says that the vertices of a different-distance set lying in a fiber is a strong different-distance set in that fiber.

Theorem 13. *Let G and H be connected graphs, and let S be a different-distance set in $G \Box H$. Let y be a vertex of H, and let $S_y = \{(w_1, y), (w_2, y), \ldots, (w_\ell, y))\}$ be the set of the vertices of S lying in the fiber $G \Box y$. Then $S'_y = \{w_1, w_2, \ldots, w_\ell\}$ is a strong different-distance set in G.*

Proof. Let z be any neighbor of y in H. Then (w_i, z) is a neighbor of (w_i, y) that lies in the fiber $G \Box z$ adjacent to the fiber $G \Box z$. Since S is independent, (w_i, z) is not in S. Hence it is not equidistant from any two vertices in S_y. Since the distance from (w_i, y) to a vertex in $G \Box y$ is one less than the distance from (w_i, z) to that vertex, it follows that also (w_i, y) is not equidistant from any two vertices in S_y. So S_y is a strong different-distant set in $G \Box y$, whence S'_y is a strong different-distant set in G. □

Of course, the same assertion holds when we consider the vertices of S lying in a vertical fiber.

5. Concluding Remarks and Open Problems

In this paper we introduce the concept of a different-distance set S in a connected graph G. This gives rise to two parameters, the lower and upper different-distance numbers $dd(G)$ and $DD(G)$. We establish the structure of the subgraph $G[S]$ induced by S: it is either an independent set or an edge or a bridging path or a pendant path. In a tree the independent sets consist of vertices equally spaced along a path at an odd distance. This also holds for cycles. Thus the different-distance numbers of paths and cycles can be easily computed. Finally, we use the Triangle Lemma and the Opposite Pair Lemma to determine the different-distance numbers of the Cartesian product of two bipartite graphs, and the Cartesian products of complete graphs. The hypercubes are the graphs that belong to both these classes.
Some open questions and problems come to mind. We mention the following ones.

1. For which graphs G, do we have $dd(G) = DD(G) = 1$?
2. For which graphs G, do we have $dd(G) = DD(G) = 2$?
3. For which graphs are the maximal different-distance sets precisely those that consist of the ends of an edge?
4. What are the different-distance numbers of trees? Are there simple bounds?
5. What is the complexity of determining the different-distance numbers of a tree, or of an arbitrary connected graph?
6. What are the different-distance numbers of the products of cycles, where at least one of the cycles is odd?
7. Let S be an independent different distance set in a 2-connected graph G of order n. What can be said about the ratio $\frac{|S|}{n}$?

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