Communications in Combinatorics and Optimization
Vol. 4 No. 2, 2019 pp.123-130
DOI: 10.22049/CCO.2019.26333.1097

CCO
Commun. Comb. Optim.

# $t$-Pancyclic arcs in tournaments

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*Received: 1 September 2018; Accepted: 8 March 2019*
*Published Online: 11 March 2019*

Dedicated to Prof. Dr. Lutz Volkmann, on the occasion of his 75th birthday.

**Abstract:** Let $T$ be a non-trivial tournament. An arc is *t-pancyclic* in $T$, if it is contained in a cycle of length $\ell$ for every $t \leq \ell \leq |V(T)|$. Let $p^t(T)$ denote the number of $t$-pancyclic arcs in $T$ and $h^t(T)$ the maximum number of $t$-pancyclic arcs contained in the same Hamiltonian cycle of $T$. Moon (*J. Combin. Inform. System Sci.*, **19** (1994), 207-214) showed that $h^3(T) \geq 3$ for any non-trivial strong tournament $T$ and characterized the tournaments with $h^3(T) = 3$. In this paper, we generalize Moon's theorem by showing that $h^t(T) \geq t$ for every $3 \leq t \leq |V(T)|$ and characterizing all tournaments which satisfy $h^t(T) = t$. We also present all tournaments which fulfill $p^t(T) = t$.

**Keywords:** Tournament, pancyclicity, $t$-pancyclic arc

**AMS Subject classification:** 05C20, 05C38

## 1. Terminology and introduction

In this paper we consider only finite and simple digraphs. Let $D$ be a digraph with vertex set $V(D)$ and arc set $A(D)$. Denote $|V(D)|$ the *order* of $D$. If $xy$ is an arc of $D$, then we write $x \to y$ and say $x$ *dominates* $y$. More generally, if $X$ and $Y$ are two disjoint subdigraphs of $D$ (or subsets of $V(D)$) such that every vertex of $X$ dominates every vertex of $Y$, then we say that $X$ *dominates* $Y$ and denote it by $X \to Y$. In

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addition, we denote the set of arcs from $X$ to $Y$ by $A(X,Y)$. Let $W \subseteq V(D)$. Then $D[W]$ is a subdigraph of $D$ induced by $W$ and $D - W = D[V(D)\backslash W]$.
A *strong component* $H$ of a digraph $D$ is a maximal subdigraph of $D$ such that for any two distinct vertices $x, y \in V(H)$, the subdigraph $H$ contains a path from $x$ to $y$ and a path from $y$ to $x$. A digraph $D$ is *strong* if it has only one strong component.
A *reductor* of $D$ is a smallest subdigraph $X$ such that $D - V(X)$ is not strong.
A path from $x$ to $y$ is called an $(x,y)$-path. A cycle of length $\ell$ is said to be an $\ell$-cycle. A path (resp. cycle) in $D$ is a *Hamiltonian path* (resp. *Hamiltonian cycle*) if it contains all the vertices of $D$. An arc of a digraph $D$ is *$t$-pancyclic* ($t \geq 3$) if it is contained in an $\ell$-cycle for every $t \leq \ell \leq |V(D)|$. Instead of 3-pancyclic we just say *pancyclic*. It is immediate that each $s$-pancyclic arc is also $t$-pancyclic for $s \leq t \leq |V(D)|$. If $xy \in A(D)$ is $t$-pancyclic in $D$, then $yx$ is $t$-pancyclic in $D^{-1}$, where $D^{-1} = (V(D), \{yx \mid xy \in A(D)\})$ is the *converse digraph* of $D$.
The number of pancyclic (resp. $t$-pancyclic) arcs in a digraph $D$ is denoted by $p(D)$ (resp. $p^t(D)$) and $h(D)$ (resp. $h^t(D)$) is the maximum number of pancyclic (resp. $t$-pancyclic) arcs belonging to the same Hamiltonian cycle of $D$.
A *tournament* $T$ is a digraph with exactly one arc between every pair of distinct vertices. A tournament without any cycles is called *transitive*.

In 1994, Moon [4] showed that every non-trivial strong tournament contains at least three pancyclic arcs. Actually, he proved a somewhat stronger result:

**Theorem 1.** (Moon [4]) *Let $T$ be a strong tournament with order $n \geq 3$. Then $h(T) \geq 3$ with equality holding if and only if $T \in \mathcal{P}_3$, where $\mathcal{P}_3$ is the set of tournaments $T$ containing a vertex $v$ such that $T - v$ is a transitive tournament with a unique Hamiltonian path $t_1 t_2 \ldots t_{n-1}$ and $\{t_i, \ldots, t_{n-1}\} \rightarrow v \rightarrow \{t_1, \ldots, t_{i-1}\}$ for some $2 \leq i \leq n-1$.*

Further results on pancyclicity in tournaments can be found in [1], [5]-[6]. In this paper we consider the number of $t$-pancyclic arcs for $t \geq 3$ instead of pancyclic arcs in tournaments. According to the definitions of $p^t(D)$ and $h^t(D)$, we immediately have $p^t(D) \geq h^t(D)$ and $h^t(D) \leq |V(D)|$. Moreover, if $D$ contains a unique Hamiltonian cycle, which therefore has to contain all $t$-pancyclic arcs, then $p^t(D) = h^t(D)$. Note that all tournaments of $\mathcal{P}_3$ contain exactly one Hamiltonian cycle. So $p(T) = h(T) = 3$ for $T \in \mathcal{P}_3$.
In the next section we generalize Theorem 1 by showing that $h^t(T) \geq t$ for every $3 \leq t \leq |V(T)|$ and characterizing all tournaments which satisfy $h^t(T) = t$. Additionally, we present all tournaments which fulfill $p^t(T) = t$.

## 2. Main Results

The following important lemma will be used frequently in the proofs of our main results. The parts (1)-(3) and (8)-(9) of Lemma 1 can be seen in [3], the other parts (4)-(7) are very easy, so we omit their proofs here.

**Lemma 1.** *Let $T$ be a non-trivial strong tournament and $X$ a reductor of $T$. Then the following statements hold.*

(1) *There is a unique sequence $T_1, T_2, \dots, T_m$ ($m \geq 2$) of the strong components of $T - V(X)$ satisfying $T_i \to T_j$ for every $1 \leq i < j \leq m$. We call it a* strong decomposition *of $T - V(X)$. Similarly, there is a strong decomposition $X_1, X_2, \dots, X_\ell$ ($\ell \geq 1$) of $X$.*

(2) *Every vertex of $X$ dominates a vertex of $T_1$ and is dominated by a vertex of $T_m$.*

(3) *Every arc from $T_m$ to $X_1$ is pancyclic and every arc from $X_\ell$ to $T_1$ is also pancyclic.*

(4) *Each arc in $X$ that lies on a Hamiltonian path of $X$ is 4-pancyclic.*

(5) *If $m \geq 4$, then every arc from $T_i$ to $T_{i+1}$ is 5-pancyclic for $i = 2, 3, \dots, m-2$.*

(6) *If $m \geq 3$ and $|V(T_1)| = 1$ (resp. $|V(T_m)| = 1$), then every arc in $A(T_1, T_2)$ (resp. $A(T_{m-1}, T_m)$) is 4-pancyclic.*

(7) *If $|V(T_i)| \geq 3$ for some $1 < i < m$, then every arc, which lies on a Hamiltonian cycle of $T_i$, is 5-pancyclic in $T$.*

(8) *If $|V(T_i)| \geq 4$ for some $1 < i < m$, then every $t$-pancyclic arc in $T_i$ is also $t$-pancyclic in $T$ for $3 \leq t \leq |V(T_i)|$.*

(9) *If $|V(T_i)| = 3$ for some $1 < i < m$, then at least two arcs of $T_i$ are pancyclic in $T$.*

Building upon the results above, we can prove the first main result, which is a generalization of the first part of Theorem 1.

**Theorem 2.** *Let $T$ be a strong tournament with order $n \geq 3$. Then*

$$h^t(T) \geq t$$

*for every $3 \leq t \leq n$.*

*Proof.* We prove this theorem by induction on $n$. For $n = 3$, $T$ is a 3-cycle, and clearly, $h^3(T) = 3$. For $n = 4$, it is easy to check that $h^3(T) = 3$ and $h^4(T) = 4$. Suppose now $n \geq 5$ and it is true for all strong tournaments with less than $n$ vertices. By Theorem 1, $h^3(T) \geq 3$, and clearly, $h^n(T) = n$. So we only need to consider the cases $t = 4, 5, \dots, n-1$.
Let $X_1, X_2, \dots, X_\ell$, $\ell \geq 1$, be the strong decomposition of a reductor $X$ of $T$ and $T_1, T_2, \dots, T_m, m \geq 2$, be the strong decomposition of $T - V(X)$ with $n_i = |V(T_i)|$ for $1 \leq i \leq m$. Because of $n \geq 5$ we have $|V(T) \backslash V(X)| \geq 3$. Note that every component $T_i$, if not consisting of a single vertex, contains a Hamiltonian cycle $C_i = t_1^i t_2^i \dots t_{n_i}^i t_1^i$ for $1 \leq i \leq m$.
Let $C$ be a Hamiltonian cycle in $T$ of the form $w_1 Q w_2 P$, where $w_1 \in A(X_\ell, T_1)$, $w_2 \in A(T_m, X_1)$, $P$ is a Hamiltonian path of $X$ and $Q$ is a Hamiltonian path of $T - V(X)$. To prove this theorem we only need to find at least $t$ arcs on $C$ which are

$t$-pancyclic in $T$ for $t = 4, 5, \ldots, n-1$. Note that the two arcs $w_1$ and $w_2$ on $C$ are always pancyclic in $T$ by Lemma 1 (3).
Below we give a claim concerning the number of $t$-pancyclic arcs in $T_1$.

*Claim 1. If $n_1 \geq 3$, then there is a Hamiltonian path $P_1$ of $T_1$ on which at least* $\min\{t-1, n_1-1\}$ *arcs are $t$-pancyclic in $T$ and $P_1$ is the first part of $Q$.*

*Proof of Claim 1.* By the induction hypothesis for $T_1$, there is a Hamiltonian cycle in $T_1$, say $C_1 = t_1^1 t_2^1 \ldots t_{n_1}^1 t_1^1$, containing $h^t(T_1) \geq t$ arcs which are $t$-pancyclic in $T_1$.
Let $x_\ell$ be an arbitrary vertex of $X_\ell$. By Lemma 1 (2) we may assume without loss of generality that $t_m \to x_\ell \to t_1^1$ for some $t_m \in V(T_m)$.
If $x_\ell \to t_{n_1}^1$, then let $P_1 = t_{n_1}^1 t_1^1 t_2^1 \ldots t_{n_1-1}^1$. The two cycles $t_1^1 t_2^1 \cdots t_{n_1-1}^1 t_m x_\ell t_1^1$ and $t_{n_1}^1 t_1^1 t_2^1 \ldots t_{n_1-2}^1 t_m x_\ell t_{n_1}^1$ yield that every arc on the Hamiltonian path $P_1$ of $T_1$ is contained in an $(n_1+1)$-cycle. Furthermore, the $(n_1+2)$-cycle $t_{n_1}^1 t_1^1 t_2^1 \ldots t_{n_1-1}^1 t_m x_\ell t_{n_1}^1$ can be successively extended to a Hamiltonian cycle $C = w_1 Q w_2 P$ in $T$ such that $P_1$ is the first part of $Q$. So every arc on $P_1$ is $n_1$-pancyclic in $T$. In the case when $t \geq n_1$, we immediately have that every arc on $P_1$ is $t$-pancyclic in $T$; In the other case when $t < n_1$, we deduce that least $t-1$ arcs on $P_1$ are $t$-pancylic in $T$.
If $t_{n_1}^1 \to x_\ell$, then let $P_1 = t_1^1 t_2^1 \ldots t_{n_1}^1$ and from the cycles $t_1^1 t_2^1 \ldots t_{n_1}^1 x_\ell t_1^1$ and $t_1^1 t_2^1 \ldots t_{n_1}^1 t_m x_\ell t_1^1$ we can deduce the same conclusion as above. So we are done. □

Analogously, Claim 1 also holds for $T_m$. We distinguish the following two cases according to the value of $t$.

*Case 1.* $t = 4$.

If $|V(T_1)| = |V(T_m)| = 1$, then $m \geq 3$ and the two arcs of $A(T_1, T_2)$ and $A(T_{m-1}, T_m)$ on $Q$ are 4-pancyclic by Lemma 1 (6). So $h^4(T) \geq 4$.
Assume without loss of generality that $|V(T_1)| \geq 3$. According to Claim 1, at least two arcs of $T_1$ are 4-pancyclic in $T$ which are contained in the Hamiltonian cycle $C$. So $h^4(T) \geq 4$.

*Case 2.* $5 \leq t \leq n-1$.

If $|V(T_1)| = |V(T_m)| = 1$, then $m \geq 3$ and every arc on the Hamiltonian cycle $C = w_1 Q w_2 P$ is $t$-pancyclic in $T$ by Lemma 1 (3)-(7). So $h^t(T) = n > t$. Assume without loss of generality that $|V(T_1)| \geq 3$.
If $|V(T_1)| \geq t$ or $|V(T_m)| \geq t$, then by Claim 1 we have $h^t(T) \geq t - 1 + |\{w_1, w_2\}| = t + 1$. So assume in the following that $3 \leq |V(T_1)| \leq t-1$ and $1 \leq |V(T_m)| \leq t-1$.
If $|V(T_m)| = 1$, then by Claim 1 and Lemma 1 (3)-(7) only the arc of $A(T_1, T_2)$ on $C = w_1 Q w_2 P$ is possibly not $t$-pancyclic. So $h^t(T) \geq n - 1 \geq t$.
If $3 \leq |V(T_m)| \leq t-1$, then by Claim 1 and Lemma 1 (3)-(7) only the arcs $e_C \in A(T_1, T_2)$ and $e'_C \in A(T_{m-1}, T_m)$ on $C = w_1 Q w_2 P$ are possibly not $t$-pancyclic. So $h^t(T) \geq n - 2$. For $t \leq n-2$ we are done obviously. For $m = 2$, we are also done

with $e_C = e'_C$ and $h^t(T) \geq n-1 \geq t$. For the remaining case $t = n-1$ and $m \geq 3$, it is easy to see that the arc $e_C$ (resp. $e'_C$) is on an $(n-1)$-cycle just by skipping one vertex of $T_m$ (resp. $T_1$). Therefore, $h^t(T) = n > t$. □

To characterize all tournaments with $h^t(T) = t$, we need the following definition.

**Definition 1.** Let $H^n$ be the strong tournament on $n$ vertices with a Hamiltonian path $P = x_1x_2 \dots x_n$ such that $x_j \to x_i$ for all $3 \leq i+2 \leq j \leq n$. Instead of $H^n$ we often write $H^n_P$ or $H^n_{x_1}$ to mark the path $P$ or its initial vertex $x_1$.

**Lemma 2.** *Let $T$ be a strong tournament of order $n \geq 3$ and $x \in V(T)$. Then $T = H^n_x$ if and only if for every Hamiltonian path of $T$ with initial vertex $x$ there is no path of length $n-2$ from $x$ to the end vertex of such Hamiltonian path.*

*Proof.* The necessity is clear and we prove the sufficiency by using induction on $n$. If $n = 3$, then $T$ is a 3-cycle and therefore $T = H^3$. If $n = 4$, then let $P = x_1x_2x_3x_4$ be a Hamiltonian path of $T$ with $x_1 = x$. Since there is no $(x_1, x_4)$-path of length 2, we have $x_3 \to x_1$ and $x_4 \to x_2$. If $x_1 \to x_4$, then $P' = x_1x_4x_2x_3$ is another Hamiltonian path starting at $x_1$, but $x_1x_2x_3$ is an $(x_1, x_3)$-path of length 2, a contradiction. So $x_4 \to x_1$ and $T = H^4_x$. Assume $n \geq 5$ and the claim holds for all strong tournaments with less than $n$ vertices.
Let $P = x_1x_2 \dots x_n$ be a Hamiltonian path in $T$ with $x_1 = x$. As there is no $(x_1, x_n)$-path of length $n-2$, we have $x_{i+2} \to x_i$ for all $1 \leq i \leq n-2$. Consider the strong subdigraph $T - x_1$ of $T$. For any Hamiltonian path $Q$ of $T - x_1$ starting at $x_2$, there is no path $S$ of length $n-3$ from $x_2$ to the end vertex of $Q$. As otherwise we can extend $S$ and $Q$ to $S' = x_1S$ and $P' = x_1Q$, a contradiction. Therefore, $T - x_1 = H^{n-1}_{x_2}$. If there exists an index $i \in \{4, \dots, n\}$ such that $x \to x_i$, then $xx_i \dots x_nx_2 \dots x_{i-1}$ is a Hamiltonian path of $T$ and $xx_i \dots x_nx_3 \dots x_{i-1}$ is an $(x, x_{i-1})$-path of length $n-2$, a contradiction. So $T = H^n_x$. □

Now we are ready to generalize the second part of Theorem 1 by Moon.

**Theorem 3.** *Let $T$ be a strong tournament with order $n$ and $t \geq 4$. Then $h^t(T) = t$ if and only if $n = t$ or $T = H^{t+1}$.*

*Proof.* First we assume $n = t$ or $T = H^{t+1}$. If $n = t$, then the desired result is obvious. If $T = H^{t+1} = H^{t+1}_Q$ with $Q = x_1 \dots x_{t+1}$, then this tournament has exactly one Hamiltonian cycle and every arc of $Q$ is contained in the cycles $x_1 \dots x_tx_1$ or $x_2 \dots x_{t+1}x_2$ and therefore $t$-pancyclic. By Lemma 2, there is no $(x_1, x_{t+1})$-path of length $t-1$, and therefore, the arc $x_{t+1}x_1$ cannot be contained in any $t$-cycle. So $h^t(T) = t$.
To prove the other direction, let $X_1, X_2, \dots, X_\ell,\ \ell \geq 1$, be the strong decomposition of a reductor $X$ of $T$ and $T_1, T_2, \dots, T_m,\ m \geq 2$, be the strong decomposition of $T - V(X)$ with $n_i = |V(T_i)|$ for $1 \leq i \leq m$. Like in the proof of Theorem 2 we

distinguish two cases $t = 4$ and $t > 4$. In both cases we assume $h^t(T) = t$ and $n > t$. Therefore we always have $n \geq 5$ and $|V(T)\backslash V(X)| \geq 3$. Again we consider a Hamiltonian cycle $C = w_1 Q w_2 P$ of $T$, where $w_1 \in A(X_\ell, T_1)$, $w_2 \in A(T_m, X_1)$, $P$ is a Hamiltonian path of $X$ and $Q$ is a Hamiltonian path of $T - V(X)$. Note that $w_1, w_2$ are always pancyclic in $T$, and Claim 1 in the proof of Theorem 2 also holds here.

*Case 1.* $t = 4$.

If $|V(T_1)| = |V(T_m)| = 1$, then $m \geq 3$ and by Lemma 1 (3) and (6), there are already four 4-pancyclic arcs $w_1$, $w_2$, $e_C \in A(T_1, A_2)$ and $e'_C \in A(T_{m-1}, T_m)$ on $C$. Since $h^4(T) = 4$, by Lemma 1 (4), (8), (9) and $n \geq 5$ we have $|V(X)| = 1$, $|V(T_i)| = 1$ for $i = 2, 3, \ldots, m-1$, and $m \geq 4$. Let $V(X) = \{x\}$ and $V(T_i) = \{t_i\}$ for $i = 1, 2, \ldots, m$. If $m \geq 5$, then either $t_2 t_3$ is 4-pancyclic in $T$ when $t_3 \to x$ or $t_3 t_4$ is 4-pancyclic when $x \to t_3$. It is a contradiction. So $m = 4$ and $\{t_2, t_4\} \to x \to \{t_1, t_3\}$. This means $T = H^5_{P^*}$ with $P^* = t_3 t_4 x t_1 t_2$.
Assume without loss of generality that $|V(T_1)| \geq 3$. Since $h^4(T) = 4$, by Claim 1 and Lemma 1 it is not difficult to deduce that $|V(T_1)| = 3$, $|V(T_m)| = 1$, $|V(X)| = 1$ and $m = 2$. Let $t_1 t_2 t_3 t_1$ be the Hamiltonian cycle of $T_1$, $V(X) = \{x\}$, $V(T_2) = \{y\}$ and assume without loss of generality that $x \to t_1$. Then $\{t_2, t_3\} \to x$ and $T = H^5_{P*}$ with $P^* = y x t_1 t_2 t_3$.

*Case 2.* $5 \leq t \leq n - 1$.

If $|V(T_1)| = |V(T_m)| = 1$, then $m \geq 3$ and every arc on the Hamiltonian cycle $C = w_1 Q w_2 P$ is $t$-pancyclic by Lemma 1 (3)-(7). That is to say $h^t(T) = n \neq t$, a contradiction. So assume without loss of generality that $|V(T_1)| \geq 3$.
In addition, we have $|V(T_1)|, |V(T_m)| \leq t-1$, as otherwise $h^t(T) \geq t-1+|\{w_1, w_2\}| \geq t+1$ by Claim 1 in the proof of Theorem 2, a contradiction.
Now by Claim 1 and Lemma 1 only the arcs $e_C \in A(T_1, T_2)$ and $e'_C \in A(T_{m-1}, T_m)$ on $C = w_1 Q w_2 P$ are possibly not $t$-pancyclic. So $n - 1 \geq t = h^t(T) \geq n - 2$.

*Subcase 2.1.* $3 \leq |V(T_1)|, |V(T_m)| \leq t - 1$.

If $m \geq 3$, then the arc $e_C$ (resp. $e'_C$) is on cycles of length $n - 1$ and $n - 2$ just by skipping one or two vertices in $T_m$ (resp. $T_1$). So whenever $t = n - 1$ or $t = n - 2$, $e_C$ and $e'_C$ are $t$-pancyclic. Therefore $h^t(T) = n \neq t$, a contradiction.
Assume in the following that $m = 2$. Then $e_C = e'_C$ is the unique arc which is not $t$-pancyclic in $T$. So $t = h^t(T) = n - 1$ and $e_C$ is not on any $(n - 1)$-cycle. Hence, $|V(X)| = 1$, as otherwise, $e_C$ is on an $(n - 1)$-cycle by skipping one vertex of $V(X)$ on $C$.
Let $V(X) = \{x\}$ and $C_i = t^i_1 t^i_2 \ldots t^i_{n_i} t^i_1$ be a Hamiltonian cycle of $T_i$ for $i = 1, 2$. By Lemma 1 (2) we may assume without loss of generality that $t^2_{n_2} \to x \to t^1_1$.
In $T_1$, for any Hamiltonian path with the initial vertex $t^1_1$, there is no $(n_1 - 2)$-path from $t^1_1$ to the end vertex of such Hamiltonian path, as otherwise $e_C$ lies on an $(n-1)$-

cycle, a contradiction. By Lemma 2, $T_1 = H^{n_1}_{P_1}$ for $P_1 = t^1_1 t^1_2 \dots t^1_{n_1}$. Using $T^{-1}$, we can similarly deduce that $T_2 = H^{n_2}_{P_2}$ for $P_2 = t^2_1 t^2_2 \dots t^2_{n_2}$. Now our aim is to show $\{t^1_2, \dots, t^1_{n_1}\} \to x \to \{t^2_1, \dots, t^2_{n_2-1}\}$, and then, $T = H^{t+1}_{P^*}$ for $P^* = P_2 x P_1$.
If $t^2_1 \to x$, then from this cycle $t^1_1 t^1_2 \dots t^1_{n_1} t^2_2 t^2_3 \dots t^2_{n_2} (t^2_1) x t^1_1$ we can see that $e_C = t^1_{n_1} t^2_2$ is on an $(n-1)$-cycle, a contradiction. So $x \to t^2_1$.
If $t^2_2 \to x$ and $n_2 \geq 4$, then from this cycle $t^1_1 t^1_2 \dots t^1_{n_1} t^2_3 \dots (t^2_{n_2}) t^2_1 t^2_2 x t^1_1$ we can see that $e_C = t^1_{n_1} t^2_3$ is on an $(n-1)$-cycle, a contradiction. If $t^2_2 \to x$ and $n_2 = 3$, then from this cycle $t^1_1 \dots t^1_{n_1} t^2_1 t^2_2 (t^2_3) x t^1_1$ we can see that $e_C = t^1_{n_1} t^2_1$ is on an $(n-1)$-cycle, a contradiction. So $x \to t^2_2$.
Successively, we can show that $x \to \{t^2_3, \dots, t^2_{n_2-1}\}$. Considering $T^{-1}$, we can further deduce that $\{t^1_2, \dots, t^1_{n_1}\} \to x$. Altogether, $T = H^{t+1}_{P^*}$.

*Subcase 2.2.* $3 \leq |V(T_1)| \leq t-1$ and $|V(T_m)| = 1$.

If $m \geq 3$, then $e'_C$ is $t$-pancyclic by Lemma 1 (6), and if $m = 2$, then $e_C = e'_C$. All of these yield $t = n-1$ and $e_C$ is not on any $(n-1)$-cycle. So $|V(X)| = 1$, as otherwise, $e_C$ is on an $(n-1)$-cycle by skipping one vertex of $V(X)$ on $C$. Similarly, we get $m \leq 3$, $|V(T_2)| = 1$ and $X \twoheadrightarrow T_1$. We also have $X \to T_2$ when $m = 3$.
If $m = 3$, then it can be transferred to the case $m = 2$ by choosing another reductor $X' = T_3$, where $T'_1 = T[T_1 \cup V(X)]$, $T'_2 = T_2$ is the strong decomposition of $T - V(X')$. So we only need to consider the case $m = 2$.
Let $V(X) = \{x\}$, $T_2 = \{y\}$ and $C_1 = t^1_1 t^1_2 \dots t^1_{n_1} t^1_1$ be a Hamiltonian cycle of $T_1$. Assume without loss of generality that $x \to t^1_1$. Then by a similar argument as in Subcase 2.1 we can deduce that $T_1 = H^{n_1}_{P_1}$ with $P_1 = t^1_1 t^1_2 \dots t^1_{n_1}$ and $T = H^{t+1}_{P^*}$ with $P^* = y x t^1_1 t^1_2 \dots t^1_{n_1}$. □

All the previous results deal with the maximum number of $t$-pancyclic arcs on the same Hamiltonian cycle. As we have a characterisation of all tournaments with $h^t(T) = t$, we naturally look for all tournaments with $p^t(T) = t$. We have seen that tournaments which achieve $h(T) = 3$ have been characterised by Moon [4] and these are the same tournaments with $p(T) = 3$. The important fact is that these tournaments contain exactly one Hamiltonian cycle. As this is also the key in the following theorem, we refer to an earlier work by Douglas [2] which gives valuable information about the structure of tournaments containing exactly one Hamiltonian cycle.

**Theorem 4.** *Let $T$ be a strong tournament with order $n$.*

(1) *If $4 \leq t \leq n-1$, then $p^t(T) = t$ if and only if $T = H^{t+1}$;*

(2) *If $t = n$, then $p^t(T) = t$ if and only if there is exactly one Hamiltonian cycle in $T$.*

*Proof.* From the definitions of $p^t(T)$ and $h^t(T)$ and Theorem 2, we have $p^t(T) \geq h^t(T) \geq t$.

(1) If $p^t(T) = t$, then from the inequality above we have $h^t(T) = t$. By Theorem 3 and $n \neq t$ we deduce that $T = H^{t+1}$. To prove the other direction, let $T = H^{t+1}$. Then there is exactly one Hamiltonian cycle in $T$, which implies $p^t(T) = h^t(T) = t$.

(2) Note that every arc on a Hamiltonian cycle of $T$ is $n$-pancyclic. So this statement obviously holds. □

## Acknowledgements

This work was partially supported by the National Natural Science Foundation for Young Scientists of China under Grant 11701349 and 11501341.

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