t-Pancyclic arcs in tournaments

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Dedicated to Prof. Dr. Lutz Volkmann, on the occasion of his 75th birthday.

Abstract: Let $T$ be a non-trivial tournament. An arc is $t$-pancyclic in $T$, if it is contained in a cycle of length $\ell$ for every $t \leq \ell \leq |V(T)|$. Let $p^t(T)$ denote the number of $t$-pancyclic arcs in $T$ and $h^t(T)$ the maximum number of $t$-pancyclic arcs contained in the same Hamiltonian cycle of $T$. Moon (J. Combin. Inform. System Sci., 19 (1994), 207-214) showed that $h^3(T) \geq 3$ for any non-trivial strong tournament $T$ and characterized the tournaments with $h^3(T) = 3$. In this paper, we generalize Moon’s theorem by showing that $h^t(T) \geq t$ for every $3 \leq t \leq |V(T)|$ and characterizing all tournaments which satisfy $h^t(T) = t$. We also present all tournaments which fulfill $p^t(T) = t$.

Keywords: Tournament, pancyclicity, $t$-pancyclic arc

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1. Terminology and introduction

In this paper we consider only finite and simple digraphs. Let $D$ be a digraph with vertex set $V(D)$ and arc set $A(D)$. Denote $|V(D)|$ the order of $D$. If $xy$ is an arc of $D$, then we write $x \rightarrow y$ and say $x$ dominates $y$. More generally, if $X$ and $Y$ are two disjoint subdigraphs of $D$ (or subsets of $V(D)$) such that every vertex of $X$ dominates every vertex of $Y$, then we say that $X$ dominates $Y$ and denote it by $X \rightarrow Y$. In
addition, we denote the set of arcs from $X$ to $Y$ by $A(X,Y)$. Let $W \subseteq V(D)$. Then $D[W]$ is a subdigraph of $D$ induced by $W$ and $D - W = D[V(D)\setminus W]$.

A strong component $H$ of a digraph $D$ is a maximal subdigraph of $D$ such that for any two distinct vertices $x, y \in V(H)$, the subdigraph $H$ contains a path from $x$ to $y$ and a path from $y$ to $x$. A digraph $D$ is strong if it has only one strong component. A reductor of $D$ is a smallest subdigraph $X$ such that $D - V(X)$ is not strong.

A path from $x$ to $y$ is called an $(x,y)$-path. A cycle of length $\ell$ is said to be an $\ell$-cycle. A path (resp. cycle) in $D$ is a Hamiltonian path (resp. Hamiltonian cycle) if it contains all the vertices of $D$. An arc of a digraph $D$ is $t$-pancyclic ($t \geq 3$) if it is contained in an $\ell$-cycle for every $t \leq \ell \leq |V(D)|$. Instead of 3-pancyclic we just say pancyclic. It is immediate that each $s$-pancyclic arc is also $t$-pancyclic for $s \leq t \leq |V(D)|$. If $xy \in A(D)$ is $t$-pancyclic in $D$, then $yx$ is $t$-pancyclic in $D^{-1}$, where $D^{-1} = (V(D),\{yx \mid xy \in A(D)\})$ is the converse digraph of $D$.

The number of pancyclic (resp. $t$-pancyclic) arcs in a digraph $D$ is denoted by $p(D)$ (resp. $p^t(D)$) and $h(D)$ (resp. $h^t(D)$) is the maximum number of pancyclic (resp. $t$-pancyclic) arcs belonging to the same Hamiltonian cycle of $D$.

A tournament $T$ is a digraph with exactly one arc between every pair of distinct vertices. A tournament without any cycles is called transitive.

In 1994, Moon [4] showed that every non-trivial strong tournament contains at least three pancyclic arcs. Actually, he proved a somewhat stronger result:

**Theorem 1.** (Moon [4]) Let $T$ be a strong tournament with order $n \geq 3$. Then $h(T) \geq 3$ with equality holding if and only if $T \in \mathcal{P}_3$, where $\mathcal{P}_3$ is the set of tournaments $T$ containing a vertex $v$ such that $T - v$ is a transitive tournament with a unique Hamiltonian path $t_1t_2\ldots t_{n-1}$ and $\{t_1,\ldots, t_{n-1}\} \rightarrow v \rightarrow \{t_1,\ldots, t_{i-1}\}$ for some $2 \leq i \leq n - 1$.

Further results on pancyclicity in tournaments can be found in [1], [5]-[6]. In this paper we consider the number of $t$-pancyclic arcs for $t \geq 3$ instead of pancyclic arcs in tournaments. According to the definitions of $p^t(D)$ and $h^t(D)$, we immediately have $p^t(D) \leq h^t(D) \leq |V(D)|$. Moreover, if $D$ contains a unique Hamiltonian cycle, which therefore has to contain all $t$-pancyclic arcs, then $p^t(D) = h^t(D)$. Note that all tournaments of $\mathcal{P}_3$ contain exactly one Hamiltonian cycle. So $p(T) = h(T) = 3$ for $T \in \mathcal{P}_3$.

In the next section we generalize Theorem 1 by showing that $h^t(T) \geq t$ for every $3 \leq t \leq |V(T)|$ and characterizing all tournaments which satisfy $h^t(T) = t$. Additionally, we present all tournaments which fulfill $p^t(T) = t$.

2. Main Results

The following important lemma will be used frequently in the proofs of our main results. The parts (1)-(3) and (8)-(9) of Lemma 1 can be seen in [3], the other parts (4)-(7) are very easy, so we omit their proofs here.
Lemma 1. Let $T$ be a non-trivial strong tournament and $X$ a reductor of $T$. Then the following statements hold.

1. There is a unique sequence $T_1, T_2, \ldots, T_m$ $(m \geq 2)$ of the strong components of $T - V(X)$ satisfying $T_i \rightarrow T_j$ for every $1 \leq i < j \leq m$. We call it a strong decomposition of $T - V(X)$. Similarly, there is a strong decomposition $X_1, X_2, \ldots, X_\ell$ $(\ell \geq 1)$ of $X$.

2. Every vertex of $X$ dominates a vertex of $T_1$ and is dominated by a vertex of $T_m$.

3. Every arc from $T_m$ to $X_1$ is pancyclic and every arc from $X_\ell$ to $T_1$ is also pancyclic.

4. Each arc in $X$ that lies on a Hamiltonian path of $X$ is 4-pancyclic.

5. If $m \geq 4$, then every arc from $T_i$ to $T_{i+1}$ is 5-pancyclic for $i = 2, 3, \ldots, m - 2$.

6. If $m \geq 3$ and $|V(T_i)| = 1$ (resp. $|V(T_m)| = 1$), then every arc in $A(T_1, T_2)$ (resp. $A(T_{m-1}, T_m)$) is 4-pancyclic.

7. If $|V(T_i)| \geq 3$ for some $1 < i < m$, then every arc, which lies on a Hamiltonian cycle of $T_i$, is 5-pancyclic in $T$.

8. If $|V(T_i)| \geq 4$ for some $1 < i < m$, then every $t$-pancyclic arc in $T_i$ is also $t$-pancyclic in $T$ for $3 \leq t \leq |V(T_i)|$.

9. If $|V(T_i)| = 3$ for some $1 < i < m$, then at least two arcs of $T_i$ are pancyclic in $T$.

Building upon the results above, we can prove the first main result, which is a generalization of the first part of Theorem 1.

Theorem 2. Let $T$ be a strong tournament with order $n \geq 3$. Then

$$h^i(T) \geq t$$

for every $3 \leq t \leq n$.

Proof. We prove this theorem by induction on $n$. For $n = 3$, $T$ is a 3-cycle, and clearly, $h^3(T) = 3$. For $n = 4$, it is easy to check that $h^3(T) = 3$ and $h^4(T) = 4$. Suppose now $n \geq 5$ and it is true for all strong tournaments with less than $n$ vertices. By Theorem 1, $h^3(T) \geq 3$, and clearly, $h^n(T) = n$. So we only need to consider the cases $t = 4, 5, \ldots, n - 1$.

Let $X_1, X_2, \ldots, X_\ell$, $\ell \geq 1$, be the strong decomposition of a reductor $X$ of $T$ and $T_1, T_2, \ldots, T_m$, $m \geq 2$, be the strong decomposition of $T - V(X)$ with $n_i = |V(T_i)|$ for $1 \leq i \leq m$. Because of $n \geq 5$ we have $|V(T)\setminus V(X)| \geq 3$. Note that every component $T_i$, if not consisting of a single vertex, contains a Hamiltonian cycle $C_i = t_1^i t_2^i \ldots t_n^i t_1^i$ for $1 \leq i \leq m$.

Let $C$ be a Hamiltonian cycle in $T$ of the form $w_1 Q w_2 P$, where $w_1 \in A(X_\ell, T_1)$, $w_2 \in A(T_m, X_1)$, $P$ is a Hamiltonian path of $X$ and $Q$ is a Hamiltonian path of $T - V(X)$. To prove this theorem we only need to find at least $t$ arcs on $C$ which are
t-pancyclic in $T$ for $t = 4, 5, \ldots, n - 1$. Note that the two arcs $w_1$ and $w_2$ on $C$ are always pancyclic in $T$ by Lemma 1 (3).

Below we give a claim concerning the number of $t$-pancyclic arcs in $T_1$.

**Claim 1.** If $n_1 \geq 3$, then there is a Hamiltonian path $P_1$ of $T_1$ on which at least $\min\{t - 1, n_1 - 1\}$ arcs are $t$-pancyclic in $T$ and $P_1$ is the first part of $Q$.

**Proof of Claim 1.** By the induction hypothesis for $T_1$, there is a Hamiltonian cycle in $T_1$, say $C = t_1^1 t_2^1 \ldots t_{n_1}^1 t_1^1$, containing $h^t(T_1) \geq t$ arcs which are $t$-pancyclic in $T_1$. Let $x_\ell$ be an arbitrary vertex of $X_\ell$. By Lemma 1 (2) we may assume without loss of generality that $t_m \to x_\ell \to t_1^1$ for some $t_m \in V(T_m)$.

If $x_\ell \to t_{n_1}^1$, then let $P_1 = t_{n_1}^1 t_{n_1-1}^1 \ldots t_1^1$. The two cycles $t_1^1 t_2^1 \ldots t_{n_1-1}^1 t_m x t_1^1$ and $t_{n_1}^1 t_2^1 \ldots t_{n_1-2}^1 t_m x t_1^1$ yield that every arc on the Hamiltonian path $P_1$ of $T_1$ is contained in an $(n_1 + 1)$-cycle. Furthermore, the $(n_1 + 2)$-cycle $t_{n_1}^1 t_1^1 t_2^1 \ldots t_{n_1-1}^1 t_m x t_1^1$ can be successively extended to a Hamiltonian cycle $C = w_1 Q w_2 Q w_3$ in $T$ such that $P_1$ is the first part of $Q$. So every arc on $P_1$ is $n_1$-pancyclic in $T$. In the case when $t \geq n_1$, we immediately have that every arc on $P_1$ is $t$-pancyclic in $T$; In the other case when $t < n_1$, we deduce that least $t - 1$ arcs on $P_1$ are $t$-pancyclic in $T$.

If $t_{n_1}^1 \to x_\ell$, then let $P_1 = t_1^1 \ldots t_{n_1}^1$ and from the cycles $t_1^1 t_2^1 \ldots t_{n_1}^1 x t_1^1$ and $t_1^1 t_2^1 \ldots t_{n_1}^1 t_m x t_1^1$ we can deduce the same conclusion as above. So we are done.

Analogously, Claim 1 also holds for $T_m$. We distinguish the following two cases according to the value of $t$.

**Case 1.** $t = 4$.

If $|V(T_1)| = |V(T_m)| = 1$, then $m \geq 3$ and the two arcs of $A(T_1, T_2)$ and $A(T_m-1, T_m)$ on $Q$ are 4-pancyclic by Lemma 1 (6). So $h^4(T) \geq 4$.

Assume without loss of generality that $|V(T_1)| \geq 3$. According to Claim 1, at least two arcs of $T_1$ are 4-pancyclic in $T$ which are contained in the Hamiltonian cycle $C$. So $h^4(T) \geq 4$.

**Case 2.** $5 \leq t \leq n - 1$.

If $|V(T_1)| = |V(T_m)| = 1$, then $m \geq 3$ and every arc on the Hamiltonian cycle $C = w_1 Q w_2 P$ is $t$-pancyclic in $T$ by Lemma 1 (3)-(7). So $h^t(T) = n > t$. Assume without loss of generality that $|V(T_1)| \geq 3$.

If $|V(T_1)| \geq t$ or $|V(T_m)| \geq t$, then by Claim 1 we have $h^t(T) \geq t - 1 + |\{w_1, w_2\}| = t + 1$. So assume in the following that $3 \leq |V(T_1)| \leq t - 1$ and $1 \leq |V(T_m)| \leq t - 1$.

If $|V(T_m)| = 1$, then by Claim 1 and Lemma 1 (3)-(7) only the arc of $A(T_1, T_2)$ on $C = w_1 Q w_2 P$ is possibly not $t$-pancyclic. So $h^t(T) \geq n - 1 \geq t$.

If $3 \leq |V(T_m)| \leq t - 1$, then by Claim 1 and Lemma 1 (3)-(7) only the arcs $e_C \in A(T_1, T_2)$ and $e'_C \in A(T_m-1, T_m)$ on $C = w_1 Q w_2 P$ are possibly not $t$-pancyclic. So $h^t(T) \geq n - 2$. For $t \leq n - 2$ we are done obviously. For $m = 2$, we are also done.
with $e_C = e'_C$ and $h^t(T) \geq n - 1 \geq t$. For the remaining case $t = n - 1$ and $m \geq 3$, it is easy to see that the arc $e_C$ (resp. $e'_C$) is on an $(n-1)$-cycle just by skipping one vertex of $T_m$ (resp. $T_1$). Therefore, $h^t(T) = n > t$.

To characterize all tournaments with $h^t(T) = t$, we need the following definition.

**Definition 1.** Let $H^n$ be the strong tournament on $n$ vertices with a Hamiltonian path $P = x_1x_2 \ldots x_n$ such that $x_j \rightarrow x_i$ for all $3 \leq i + 2 \leq j \leq n$. Instead of $H^n$ we often write $H^n_2$ or $H^n_3$ to mark the path $P$ or its initial vertex $x_1$.

**Lemma 2.** Let $T$ be a strong tournament of order $n \geq 3$ and $x \in V(T)$. Then $T = H^n_2$ if and only if for every Hamiltonian path of $T$ with initial vertex $x$ there is no path of length $n - 2$ from $x$ to the end vertex of such Hamiltonian path.

**Proof.** The necessity is clear and we prove the sufficiency by using induction on $n$. If $n = 3$, then $T$ is a 3-cycle and therefore $T = H^3$. If $n = 4$, then let $P = x_1x_2x_3x_4$ be a Hamiltonian path of $T$ with $x_1 = x$. Since there is no $(x_1, x_2)$-path of length 2, we have $x_3 \rightarrow x_1$ and $x_4 \rightarrow x_2$. If $x_1 \rightarrow x_4$, then $P' = x_1x_4x_2x_3$ is another Hamiltonian path starting at $x_1$, but $x_1x_2x_3$ is an $(x_1, x_3)$-path of length 2, a contradiction. So $x_4 \rightarrow x_1$ and $T = H^4_2$. Assume $n \geq 5$ and the claim holds for all strong tournaments with less than $n$ vertices.

Let $P = x_1x_2 \ldots x_n$ be a Hamiltonian path in $T$ with $x_1 = x$. As there is no $(x_1, x_n)$-path of length $n - 2$, we have $x_{i+2} \rightarrow x_i$ for all $1 \leq i \leq n - 2$. Consider the strong subdigraph $T - x_1$ of $T$. For any Hamiltonian path $Q$ of $T - x_1$ starting at $x_2$, there is no path $S$ of length $n - 3$ from $x_2$ to the end vertex of $Q$. As otherwise we can extend $S$ and $Q$ to $S' = x_1S$ and $P' = x_1Q$, a contradiction. Therefore, $T - x_1 = H^{n-1}_{x_2}$. If there exists an index $i \in \{4, \ldots, n\}$ such that $x \rightarrow x_i$, then $xx_1 \ldots x_nx_2 \ldots x_{i-1}$ is a Hamiltonian path of $T$ and $xx_1 \ldots x_nx_3 \ldots x_{i-1}$ is an $(x, x_{i-1})$-path of length $n - 2$, a contradiction. So $T = H^n_2$.

Now we are ready to generalize the second part of Theorem 1 by Moon.

**Theorem 3.** Let $T$ be a strong tournament with order $n$ and $t \geq 4$. Then $h^t(T) = t$ if and only if $n = t$ or $T = H^{t+1}$.

**Proof.** First we assume $n = t$ or $T = H^{t+1}$. If $n = t$, then the desired result is obvious. If $T = H^{t+1} = H^t_Q$ with $Q = x_1 \ldots x_{t+1}$, then this tournament has exactly one Hamiltonian cycle and every arc of $Q$ is contained in the cycles $x_1 \ldots x_1 x_1$ or $x_2 \ldots x_{t+1} x_2$ and therefore $t$-pancyclic. By Lemma 2, there is no $(x_1, x_{t+1})$-path of length $t - 1$, and therefore, the arc $x_{t+1} x_1$ cannot be contained in any $t$-cycle. So $h^t(T) = t$.

To prove the other direction, let $X_1, X_2, \ldots, X_\ell$, $\ell \geq 1$, be the strong decomposition of a reductor $X$ of $T$ and $T_1, T_2, \ldots, T_m$, $m \geq 2$, be the strong decomposition of $T - V(X)$ with $n_i = |V(T_i)|$ for $1 \leq i \leq m$. Like in the proof of Theorem 2 we
distinguish two cases $t = 4$ and $t > 4$. In both cases we assume $h^i(T) = t$ and $n > t$. Therefore we always have $n \geq 5$ and $|V(T) \setminus V(X)| \geq 3$. Again we consider a Hamiltonian cycle $C = w_1Qw_2P$ of $T$, where $w_1 \in A(X_t, T_1)$, $w_2 \in A(T_m, X_1)$, $P$ is a Hamiltonian path of $X$ and $Q$ is a Hamiltonian path of $T - V(X)$. Note that $w_1, w_2$ are always pancyclic in $T$, and Claim 1 in the proof of Theorem 2 also holds here.

Case 1. $t = 4$.

If $|V(T_1)| = |V(T_m)| = 1$, then $m \geq 3$ and by Lemma 1 (3) and (6), there are already four 4-pancyclic arcs $w_1, w_2, e_C \in A(T_1, A_2)$ and $e'_C \in A(T_{m-1}, T_m)$ on $C$. Since $h^4(T) = 4$, by Lemma 1 (4), (8), (9) and $n \geq 5$ we have $|V(X)| = 1, |V(T_i)| = 1$ for $i = 2, 3, \ldots, m - 1$, and $m \geq 4$. Let $V(X) = \{x\}$ and $V(T_i) = \{t_i\}$ for $i = 1, 2, \ldots, m$. If $m \geq 5$, then either $t_2t_3$ is 4-pancyclic in $T$ when $t_3 \rightarrow x$ or $t_3t_4$ is 4-pancyclic when $x \rightarrow t_3$. It is a contradiction. So $m = 4$ and $\{t_2, t_4\} \rightarrow x \rightarrow \{t_1, t_3\}$. This means $T = H_{T_2}^1$, with $P^* = t_3t_4xt_1t_2$.

Assume without loss of generality that $|V(T_1)| \geq 3$. Since $h^4(T) = 4$, by Claim 1 and Lemma 1 it is not difficult to deduce that $|V(T_1)| = 3, |V(T_m)| = 1, |V(X)| = 1$ and $m = 2$. Let $t_1t_2t_3t_1$ be the Hamiltonian cycle of $T_1$, $V(X) = \{x\}, V(T_2) = \{y\}$ and assume without loss of generality that $x \rightarrow t_1$. Then $\{t_2, t_3\} \rightarrow x \rightarrow t_1$ and $T = H_{T_2}^1$. with $P^* = yxt_1t_2t_3$.

Case 2. $5 \leq t \leq n - 1$.

If $|V(T_1)| = |V(T_m)| = 1$, then $m \geq 3$ and every arc on the Hamiltonian cycle $C = w_1Qw_2P$ is $t$-pancyclic by Lemma 1 (3)-(7). That is to say $h^i(T) = n \neq t$, a contradiction. So assume without loss of generality that $|V(T_1)| \geq 3$.

In addition, we have $|V(T_1)|, |V(T_m)| \leq t - 1$, as otherwise $h^i(T) \geq t - 1 + |\{w_1, w_2\}| \geq t + 1$ by Claim 1 in the proof of Theorem 2, a contradiction.

Now by Claim 1 and Lemma 1 only the arcs $e_C \in A(T_1, T_2)$ and $e'_C \in A(T_{m-1}, T_m)$ on $C = w_1Qw_2P$ are possibly not t-pancyclic. So $n - 1 \geq t = h^i(T) \geq n - 2$.

Subcase 2.1. $3 \leq |V(T_1)|, |V(T_m)| \leq t - 1$.

If $m \geq 3$, then the arc $e_C$ (resp. $e'_C$) is on cycles of length $n - 1$ and $n - 2$ just by skipping one or two vertices in $T_m$ (resp. $T_1$). So whenever $t = n - 1$ or $t = n - 2$, $e_C$ and $e'_C$ are $t$-pancyclic. Therefore $h^i(T) = n \neq t$, a contradiction.

Assume in the following that $m = 2$. Then $e_C = e'_C$ is the unique arc which is not t-pancyclic in $T$. So $t = h^i(T) = n - 1$ and $e_C$ is not on any $(n - 1)$-cycle. Hence, $|V(X)| = 1$, as otherwise, $e_C$ is on an $(n - 1)$-cycle by skipping one vertex of $V(X)$ on $C$.

Let $V(X) = \{x\}$ and $C_i = t_1^it_2^i \ldots t_{n_i}^it_1^i$ be a Hamiltonian cycle of $T_i$ for $i = 1, 2$. By Lemma 1 (2) we may assume without loss of generality that $t_2^i \rightarrow x \rightarrow t_1^i$.

In $T_1$, for any Hamiltonian path with the initial vertex $t_1^1$, there is no $(n_1 - 2)$-path from $t_1^1$ to the end vertex of such Hamiltonian path, as otherwise $e_C$ lies on an $(n - 1)$-
cycle, a contradiction. By Lemma 2, $T_1 = H_{P_1}^{n_1}$ for $P_1 = t_1^1 t_2^1 \ldots t_{n_1}^1$. Using $T^{-1}$, we can similarly deduce that $T_2 = H_{P_2}^{n_2}$ for $P_2 = t_1^2 t_2^2 \ldots t_{n_2}^2$. Now our aim is to show \{t_2^1, \ldots, t_{n_1}^1 \} → x → \{t_2^1, \ldots, t_{n_2}^2, t_{n_2}^1 \}$, and then, $T = H_{P_2}^{n_1} P_{P_1}$.

If $t_2^1 → x$, then from this cycle $t_1^1 t_2^1 \ldots t_{n_1}^1 t_2^3 \ldots t_{n_2}^2 (t_1^2) x t_1^1$ we can see that $e_C = t_1^1 t_2^1$ is on an $(n-1)$-cycle, a contradiction. By Lemma 2, $W. Meng, S. Grütter, Y. Guo, M. Kapolke, S. Meesker 129

if $t_2^2 → x$ and $n_2 ≥ 4$, then from this cycle $t_1^2 t_2^2 \ldots t_{n_1}^1 t_2^3 \ldots (t_{n_2}^2) t_1^1 t_2^1 xt_1^1$ we can see that $e_C = t_1^1 t_2^3$ is on an $(n-1)$-cycle, a contradiction. If $t_2^2 → x$ and $n_2 = 3$, then from this cycle $t_1^2 t_2^2 t_3^2 (t_3^2) xt_1^1$ we can see that $e_C = t_1^1 t_2^1$ is on an $(n-1)$-cycle, a contradiction. So $x → t_2^2$.

Successively, we can show that $x → \{t_3^2, \ldots, t_{n_2}^2 \}$. Considering $T^{-1}$, we can further deduce that $\{t_2^1, \ldots, t_{n_1}^1 \} → x$. Altogether, $T = H_{P_2}^{n_1}$.

Subcase 2.2. $3 ≤ |V(T_1)| ≤ t − 1$ and $|V(T_m)| = 1$.

If $m ≥ 3$, then $e_C'$ is $t$-pancyclic by Lemma 1 (6), and if $m = 2$, then $e_C = e_C'$. All of these yield $t = n − 1$ and $e_C$ is not on any $(n-1)$-cycle. So $|V(X)| = 1$, as otherwise, $e_C$ is on an $(n-1)$-cycle by skipping one vertex of $V(X)$ on $C$. Similarly, we get $m ≤ 3$, $|V(T_2)| = 1$ and $X → T_1$. We also have $X → T_2$ when $m = 3$.

If $m = 3$, then it can be transferred to the case $m = 2$ by choosing another reductor $X' = T_3$, where $T_1 = T [T_1 ∪ V(X)]$, $T_2 = T_2$ is the strong decomposition of $T − V(X')$. So we only need to consider the case $m = 2$.

Let $V(X) = \{x\}$, $T_2 = \{y\}$ and $C_1 = t_1^1 t_2^1 \ldots t_{n_1}^1 t_1^1$ be a Hamiltonian cycle of $T_1$. Assume without loss of generality that $x → t_1^1$. Then by a similar argument as in Subcase 2.1 we can deduce that $T_1 = H_{P_1}^{n_1}$ with $P_1 = t_1^1 t_2^1 \ldots t_{n_1}^1$ and $T = H_{P_2}^{n_1}$ with $P^* = yxt_1^1 t_2^1 \ldots t_{n_1}^1$.

All the previous results deal with the maximum number of $t$-pancyclic arcs on the same Hamiltonian cycle. As we have a characterisation of all tournaments with $h^i(T) = t$, we naturally look for all tournaments with $p^i(T) = t$. We have seen that tournaments which achieve $h(T) = 3$ have been characterised by Moon [4] and these are the same tournaments with $p(T) = 3$. The important fact is that these tournaments contain exactly one Hamiltonian cycle. As this is also the key in the following theorem, we refer to an earlier work by Douglas [2] which gives valuable information about the structure of tournaments containing exactly one Hamiltonian cycle.

**Theorem 4.** Let $T$ be a strong tournament with order $n$.

1. If $4 ≤ t ≤ n − 1$, then $p^i(T) = t$ if and only if $T = H_{P_1}^{t+1}$;
2. If $t = n$, then $p^i(T) = t$ if and only if there is exactly one Hamiltonian cycle in $T$.

**Proof.** From the definitions of $p^i(T)$ and $h^i(T)$ and Theorem 2, we have $p^i(T) ≥ h^i(T) ≥ t$. 

(1) If $p^t(T) = t$, then from the inequality above we have $h^t(T) = t$. By Theorem 3 and $n \neq t$ we deduce that $T = H^{t+1}$. To prove the other direction, let $T = H^{t+1}$. Then there is exactly one Hamiltonian cycle in $T$, which implies $p^t(T) = h^t(T) = t$.

(2) Note that every arc on a Hamiltonian cycle of $T$ is $n$-pancyclic. So this statement obviously holds. □

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References