The Italian domatic number of a digraph

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Abstract: An Italian dominating function on a digraph $D$ with vertex set $V(D)$ is defined as a function $f: V(D) \rightarrow \{0, 1, 2\}$ such that every vertex $v \in V(D)$ with $f(v) = 0$ has at least two in-neighbors assigned 1 under $f$ or one in-neighbor $w$ with $f(w) = 2$. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct Italian dominating functions on $D$ with the property that $\sum_{i=1}^{d} f_i(v) \leq 2$ for each $v \in V(D)$, is called an Italian dominating family (of functions) on $D$. The maximum number of functions in an Italian dominating family on $D$ is the Italian domatic number of $D$, denoted by $d_I(D)$. In this paper we initiate the study of the Italian domatic number in digraphs, and we present some sharp bounds for $d_I(D)$. In addition, we determine the Italian domatic number of some digraphs.

Keywords: Digraphs, Italian dominating function, Italian domination number, Italian domatic number.

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1. Terminology and introduction

In this paper, $D$ is a simple digraph with vertex set $V = V(D)$ and arc set $A = A(D)$. The order $|V|$ of $D$ is denoted by $n = n(D)$. We write $d^+_D(v) = d^{+}(v)$ for the out-degree of a vertex $v$ and $d^-_D(v) = d^{-}(v)$ for its in-degree. The minimum and maximum in-degree and minimum and maximum out-degree of $D$ are denoted by $\delta^- = \delta^-(D)$, $\Delta^- = \Delta^{-}(D)$, $\delta^+ = \delta^+(D)$ and $\Delta^+ = \Delta^+(D)$, respectively. If $uv$ is an arc of $D$, then we also write $u \rightarrow v$, and we say that $v$ is an out-neighbor of $u$ and $u$ is an in-neighbor of $v$. A digraph $D$ is in-regular or $r$-in-regular when $\delta^-(D) = \Delta^-(D) = r$ and out-regular or $r$-out-regular when $\delta^+(D) = \Delta^+(D) = r$. If $D$ is $r$-in-regular and $r$-out-regular, then $D$ is called $r$-regular or regular. For a vertex $v$ of a digraph $D$, we denote the set of in-neighbors and out-neighbors of $v$ by $N^-_D(v) = N^-(v)$ and $N^+_D(v)$.
Let $D$ be a digraph of order $n$. Then $\gamma_I(D) \leq n$ and $\gamma_I(D) = n$ if and only if $\Delta^-(D), \Delta^+(D) \leq 1$. 

We make use of the following known results in this paper.

**Proposition 1** ([9]). If $D$ is a digraph of order $n$, then $\gamma_R(D) \leq n - \Delta^+(D) + 1$. 

**Proposition 2** ([13]). Let $D$ be a digraph of order $n$. Then $\gamma_I(D) \leq n$ and $\gamma_I(D) = n$ if and only if $\Delta^-(D), \Delta^+(D) \leq 1$. 

In this paper we continue the study of Roman and Italian dominating functions and domatic numbers in graphs and digraphs (see, for example, [1–4, 6, 8, 10–12]). We make use of the following known results in this paper.

A Roman dominating function on a digraph $D$ is defined in [9] as a function $f: V(D) \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex $v$ with $f(v) = 0$ has an in-neighbor $u$ with $f(u) = 2$. The weight of a Roman dominating function $f$ is the value $\sum_{v \in V(D)} f(v)$. The Roman domination number of a digraph $D$, denoted by $\gamma_R(D)$, is the minimum taken over the weights of all Roman dominating functions on $D$. 

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct Roman dominating functions on $D$ with $\sum_{i=1}^d f_i(v) \leq 2$ for each $v \in V(D)$, is called in [5] a Roman dominating family (of functions) on $D$. The maximum number of functions in a Roman dominating family on $D$ is the Roman domatic number of $D$, denoted by $d_R(D)$. 

An Italian dominating function (IDF) on a digraph $D$ is defined in [13] as a function $f: V(D) \rightarrow \{0, 1, 2\}$ such that every vertex $v \in V(D)$ with $f(v) = 0$ has at least two in-neighbors assigned 1 under $f$ or one in-neighbor $w$ with $f(w) = 2$. The weight of an Italian dominating function $f$ is the value $\omega(f) = f(V(D)) = \sum_{u \in V(D)} f(u)$. The Italian domination number of a digraph $D$, denoted by $\gamma_I(D)$, is the minimum taken over the weights of all ID functions on $D$. A $\gamma_I(D)$-function is an Italian dominating function on $D$ with weight $\gamma_I(D)$. An Italian dominating function $f: V(D) \rightarrow \{0, 1, 2\}$ can be represented by the ordered partition $(V_0, V_1, V_2)$ (or $(V_0^f, V_1^f, V_2^f)$ to refer to $f$) of $V(D)$, where $V_i = V_i^f = \{v \in V(D): f(v) = i\}$ for $i \in \{0, 1, 2\}$. In this representation, the weight of $f$ is $\omega(f) = |V_1| + 2|V_2|$. 

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct Italian dominating functions on $D$ with the property that $\sum_{i=1}^d f_i(v) \leq 2$ for each $v \in V(D)$, is called an Italian dominating family (of functions) on $D$ (ID family on $D$). The maximum number of functions in an ID family on $D$ is the Italian domatic number of $D$, denoted by $d_I(D)$. 

We note that if $D$ is a digraph, then $\gamma_I(D) \leq \gamma_R(D)$ (see [13]), and the definitions lead to $d_I(D) \geq d_R(D) \geq 1$. 

In this paper we continue the study of Italian domatic number in digraphs, and we present some sharp bounds for $d_I(D)$. In addition, we determine the Italian domatic number of some digraphs.
Proposition 3 ([13]). If $D$ is a directed cycle or a directed path of order $n$, then \( \gamma_I(D) = n \).

Proposition 4 ([13]). If $D$ is a digraph of order $n$, then

\[
\gamma_I(D) \geq \left\lceil \frac{2n}{\Delta^+(D) + 2} \right\rceil.
\]

2. Properties of the Italian domatic number

In this section we present basic properties and sharp bounds on the Italian domatic number of digraphs.

Theorem 1. If $D$ is a digraph of order $n$, then

\[
\gamma_I(D) \cdot d_I(D) \leq 2n.
\]

Moreover, if $\gamma_I(D) \cdot d_I(D) = 2n$, then for each Italian dominating family \( \{f_1, f_2, \ldots, f_d\} \) on $D$ with $d = d_I(D)$, each function $f_i$ is a $\gamma_I(D)$-function and $\sum_{i=1}^{d} f_i(v) = 2$ for all $v \in V(D)$.

Proof. Let \( \{f_1, f_2, \ldots, f_d\} \) be an ID family on $D$ such that $d = d_I(D)$ and let $v \in V(D)$. Then

\[
d \cdot \gamma_I(D) = \sum_{i=1}^{d} \gamma_I(D) \leq \sum_{i=1}^{d} \sum_{v \in V(D)} f_i(v) = \sum_{v \in V(D)} \sum_{i=1}^{d} f_i(v) \leq \sum_{v \in V(D)} 2 = 2n.
\]

If $\gamma_I(D) \cdot d_I(D) = 2n$, then the two inequalities occurring in inequality chain above become equalities. Hence, for the ID family \( \{f_1, f_2, \ldots, f_d\} \) on $D$ and for each $i$, $\sum_{v \in V(D)} f_i(v) = \gamma_I(D)$. Thus, each function $f_i$ is a $\gamma_I(D)$-function, and $\sum_{i=1}^{d} f_i(v) = 2$ for all $v \in V(D)$.

Since $\gamma_I(D) \geq 2$ for each digraph $D$ of order $n \geq 2$, Theorem 1 implies the next result immediately.

Corollary 1. If $D$ is a digraph of order $n$, then \( d_R(D) \leq d_I(D) \leq n \).

Theorem 2. If $D$ is a digraph, then $d_I(D) = 1$ if and only if $\Delta^-(D) \leq 1$ and $D$ has no directed cycle of even length.
Proof. Assume first that $\Delta^{-}(D) \geq 2$, and let $w$ be a vertex with $d^{-}(w) \geq 2$. Define $f, g : V(D) \to \{0, 1, 2\}$ by $f(x) = 1$ for each $x \in V(D)$ and $g(w) = 0$ and $g(v) = 1$ for $v \in V(D) \setminus \{w\}$. Since $d^{-}(w) \geq 2$, we observe that $f$ and $g$ are Italian dominating functions on $D$ with the property that $f(x) + g(x) \leq 2$ for each $x \in V(G)$. Therefore $\{f, g\}$ is an ID family on $D$ and so $d_{I}(D) \geq 2$. Assume next that $D$ has a directed cycle $C = v_{1}v_{2} \cdots v_{2p}v_{1}$ for an integer $p \geq 1$. Then the functions $f : V(D) \to \{0, 1, 2\}$ with $f(v_{1}) = f(v_{3}) = \cdots = f(v_{2p-1}) = 2$, $f(v_{2}) = f(v_{4}) = \cdots = f(v_{2p}) = 0$ and $f(x) = 1$ for each $x \in V(D) - V(C)$ and $g : V(D) \to \{0, 1, 2\}$ with $g(v_{1}) = g(v_{3}) = \cdots = g(v_{2p-1}) = 0$, $g(v_{2}) = g(v_{4}) = \cdots = g(v_{2p}) = 2$ and $g(x) = 1$ for each $x \in V(D) - V(C)$ are Italian dominating functions on $D$. This implies that $\{f, g\}$ is an ID family on $D$, and therefore $d_{I}(D) \geq 2$. Conversely assume that $d_{I}(D) \geq 2$. Suppose to the contrary that $\Delta^{-}(D) \leq 1$ and $D$ has no directed cycle of even length. Let $f$ and $g$ be two distinct ID functions of an ID family on $D$. Since $\Delta^{-}(D) \leq 1$, we note that $V_{0}^{f}, V_{0}^{g}, V_{2}^{f}, V_{2}^{g} \neq \emptyset$. By definition, $f(v) + g(v) \leq 2$ for every vertex $v$. It follows that $V_{2}^{f} \subseteq V_{0}^{g}$ and $V_{2}^{g} \subseteq V_{0}^{f}$. Since $f$ is an IDF, every vertex $v \in V_{2}^{g} \subseteq V_{0}^{f}$ has an in-neighbor in $V_{2}^{f}$. Likewise, every vertex $w \in V_{2}^{f} \subseteq V_{0}^{g}$ has an in-neighbor in $V_{2}^{g}$. Hence the bipartite subdigraph $D'$ with vertex set $V_{2}^{f} \cup V_{2}^{g}$ and the arcs of $D$ between $V_{2}^{f}$ and $V_{2}^{g}$ has minimum in-degree at least $1$. It follows that $D'$ has a directed cycle and, obviously, this directed cycle has even length, a contradiction.

Corollary 2. If $P_{n}$ is a directed path of order $n$, then $d_{I}(P_{n}) = 1$. If $C_{n}$ is a directed cycle of order $n$, then $d_{I}(C_{n}) = 1$ if $n$ is odd and $d_{I}(C_{n}) = 2$ if $n$ is even.

Proof. Theorem 2 implies $d_{I}(P_{n}) = 1$, $d_{I}(C_{n}) = 1$ if $n$ is odd and $d_{I}(C_{n}) \geq 2$ if $n$ is even. Using Proposition 3 and Theorem 1, we obtain $d_{I}(C_{n}) \leq 2$ and thus $d_{I}(C_{n}) = 2$ if $n$ is even.

Proposition 5. Let $D$ be a digraph of order $n \geq 2$. Then $\gamma_{I}(D) = n$ and $d_{I}(D) = 2$ if and only if $\Delta^{-}(D) = \Delta^{+}(D) = 1$ and $D$ contains a directed cycle of even length.

Proof. Assume first that $\gamma_{I}(D) = n$ and $d_{I}(D) = 2$. It follows from Proposition 2 that $\Delta^{-}(D), \Delta^{+}(D) \leq 1$. Since $d_{I}(D) = 2$, Theorem 2 implies that $D$ has a directed cycle of even length and so $\Delta^{-}(D) = \Delta^{+}(D) = 1$. Conversely, assume that $\Delta^{-}(D) = \Delta^{+}(D) = 1$ and $D$ contains a directed cycle of even length. Proposition 2 leads to $\gamma_{I}(D) = n$, and Theorem 2 shows that $d_{I}(D) \geq 2$. Therefore Theorem 1 implies $d_{I}(D) \leq 2$ and thus $d_{I}(D) = 2$.

Next we show that the upper bound in Corollary 1 is attained only for complete digraphs.

Theorem 3. If $D$ is a digraph of order $n \geq 2$, then $d_{I}(D) = n$ if and only if $D$ is the complete digraph on $n$ vertices.
Proof. Let $D = K_n^*$ be the complete digraph on $n$ vertices with vertex set \{v_1, v_2, \ldots, v_n\}. Define the functions $f_i: V(D) \rightarrow \{0, 1, 2\}$ by $f_i(v_i) = 2$ and $f_i(v_j) = 0$ for $j \neq i$ and $1 \leq i, j \leq n$. Then $f_i$ is an IDF on $D$ for each $1 \leq i \leq n$ such that $\sum_{i=1}^{n} f_i(v) = 2$ for each $v \in V(D)$. Therefore $\{f_1, f_2, \ldots, f_n\}$ is an ID family on $D$. Combining this with Corollary 1, we deduce that $d_I(K_n^*) = n$.

Conversely, assume that $d_I(D) = n$. If $n = 2$, then the result is immediate. Assume next that $n \geq 3$. Then $\gamma_I(D) \geq 2$ and it follows from Theorem 1 that $\gamma_I(D) = 2$. Let $\{f_1, f_2, \ldots, f_n\}$ be an ID family on $D$. We deduce from Theorem 1 that $f_i$ is a $\gamma_I(D)$-function for each $i$, and $\sum_{d=1}^{n} f_i(v) = 2$ for all $v \in V(D)$. Since $n \geq 3$ and $\gamma_I(D) = 2$, we conclude that for each $i$, there exists a vertex $x \in V(D)$ such that $f_i(x) \geq 1$. Assume, without loss of generality, that $f_i(v_i) \geq 1$ for each $i \in \{1, 2, \ldots, n\}$.

Suppose, without loss of generality, that there doesn’t exist the arc $v_n v_1$. If $f_n(v_n) = 2$, then $f_n(v_1) = f_n(v_2) = \ldots = f_n(v_{n-1}) = 0$, and we obtain the contradiction $f_n(N^-[v_1]) = 0$. Therefore $f_n(v_n) = 1$ and thus $f_n(v_1) = 1$. Since $f_1(v_1) \geq 1$, we deduce that $f_1(v_1) = 1$. As $f_1$ and $f_n$ are distinct, we deduce that $f_1(v_n) = 0$. Since $\sum_{i=1}^{n} f_i(v_n) = 2$, there exists an index $j \in \{2, 3, \ldots, n-1\}$ with $f_j(v_n) = 1$ and $f_j(v_1) = 0$. This leads to the contradiction $f_j(N^-[v_1]) \leq 1$. This completes the proof.

Since $d_R(D) \leq d_I(D)$, Theorem 3 and the first part of its proof lead to the next result.

**Corollary 3.** ([5]) If $D$ is a digraph of order $n \geq 2$, then $d_R(D) = n$ if and only if $D$ is the complete digraph on $n$ vertices.

The upper bound on the product $\gamma_I(D) \cdot d_I(D)$ leads to an upper bound on the sum of these terms.

**Theorem 4.** If $D$ is a digraph of order $n \geq 2$, then

$$\gamma_I(D) + d_I(D) \leq n + 2.$$  \hspace{1cm} (1)

Moreover, equality holds if and only if $\Delta^+(D) = \Delta^-(D) = 1$ and $D$ has a directed cycle of even length or $D$ is the complete digraph.

Proof. If $d_I(D) \leq 1$, then obviously $\gamma_I(D) + d_I(D) \leq n + 1$. Assume now that $d_I(D) \geq 2$. According to Corollary 1, we have $d_I(D) \leq n$. Theorem 1 implies that

$$\gamma_I(D) + d_I(D) \leq \frac{2n}{d_I(D)} + d_I(D).$$

Using the fact that the function $g(x) = x + 2n/x$ is decreasing for $2 \leq x \leq \sqrt{2n}$ and increasing for $\sqrt{2n} \leq x \leq n$, this inequality leads to

$$\gamma_I(D) + d_I(D) \leq \frac{2n}{d_I(D)} + d_I(D) \leq \max\{n + 2, 2 + n\} = n + 2,$$  \hspace{1cm} (2)
and this is the desired bound.

If $D$ is the complete digraph on $n$ vertices, then $\gamma_I(D) = 2$ and by Theorem 3, $d_I(D) = n$. If $\Delta^+(D) = \Delta^-(D) = 1$ and $D$ contains an even cycle, then it follows from Proposition 5 that $\gamma_I(D) = n$ and $d_I(D) = 2$. Thus $\gamma_I(D) + d_I(D) = n + 2$ in both cases.

Conversely, let equality hold in (1). It follows from (2) that

$$n + 2 = \gamma_I(D) + d_I(D) \leq \frac{2n}{d_I(D)} + d_I(D) \leq n + 2,$$

which implies that $\gamma_I(D) = 2n/d_I(D)$ and $d_I(D) = 2$ or $d_I(D) = n$. If $d_I(D) = n$, then $D$ is the complete digraph by Theorem 3. If $d_I(D) = 2$, then $\gamma_I(D) = n$, and it follows from Proposition 5 that $\Delta^+(D) = \Delta^-(D) = 1$ and $D$ contains a directed cycle of even length. This completes the proof.

The Italian domatic number of a digraph may also be bounded from above by its minimum in-degree plus 2.

**Theorem 5.** For every digraph $D$,

$$d_I(D) \leq \delta^-(D) + 2$$

and this bound is sharp.

**Proof.** If $d_I(D) \leq 2$, then the bound is immediate. Let now $d_I(D) \geq 3$ and let $\{f_1, f_2, \ldots, f_d\}$ be an ID family on $D$ such that $d = d_I(D)$. Assume that $v$ is a vertex of minimum in-degree $\delta^-(D)$. Since the equality $\sum_{x \in N^-[v]} f_i(x) = 1$ holds for at most two indices $i \in \{1, 2, \ldots, d\}$, we have

$$2d - 2 = \sum_{i=1}^{d} \sum_{x \in N^-[v]} f_i(x) = \sum_{x \in N^-[v]} \sum_{i=1}^{d} f_i(x) \leq \sum_{x \in N^-[v]} 2 = 2(\delta^-(D) + 1).$$

This implies the desired bound $d_I(D) \leq \delta^-(D) + 2$.

To prove sharpness, let $k \geq 2$ be an integer. In addition, let $D_i$ be a copy of the complete digraph $K^*_k$ with vertex set $V(D_i) = \{v_1^i, v_2^i, \ldots, v_{k+3}^i\}$ for $1 \leq i \leq k$, and let $D$ be the digraph obtained from $\bigcup_{i=1}^{k} D_i$ by adding a new vertex $v$ and the arcs $vv_1^1$ as well as $v_j^i v_1^i$ for $1 \leq i \leq k$. Define the ID functions $f_1, f_2, \ldots, f_{k+2}$ as follows:

\[
\begin{align*}
    f_i(v_1^i) &= 2, \\
    f_i(v_{i+1}^j) &= 2 & \text{if } j \in \{1, 2, \ldots, k\} - \{i\} & \text{and } f(x) = 0 \text{ otherwise } (1 \leq i \leq k), \\
    f_{k+1}(v) &= 1, \\
    f_{k+1}(v_{k+2}^j) &= 2 & \text{if } j \in \{1, 2, \ldots, k\} & \text{and } f(x) = 0 \text{ otherwise},
\end{align*}
\]

...
and
\[ f_{k+2}(v) = 1, f_{k+2}(v'_{k+3}) = 2, \text{ if } j \in \{1, 2, \ldots, k\} \text{ and } f(x) = 0 \text{ otherwise.} \]

It is easy to see that every \( f_i \) is an IDF on \( D \) and that \( \{f_1, f_2, \ldots, f_{k+2}\} \) is an ID family on \( D \). Since \( \delta^{-}(D) = k \), we deduce that \( d_I(D) = \delta^{-}(D) + 2. \)

Since \( d_R(D) \leq d_I(D) \), Theorem 5, and the example in the proof of Theorem 5 yield to the next result.

**Corollary 4.** ([5]) For every digraph \( D \),
\[ d_R(D) \leq \delta^{-}(D) + 2 \]
and this bound is sharp.

### 3. Nordhaus-Gaddum type results

The complement \( \overline{D} \) of a digraph \( D \) is the digraph with vertex set \( V(D) \) such that for any two distinct vertices \( u, v \) the arc \( uv \) belongs to \( \overline{D} \) if and only if \( uv \) does not belong to \( D \). Results of Nordhaus-Gaddum type study extreme values of the sum or the product of a parameter on a digraph and its complement. We establish such inequalities for the Italian domination number.

**Theorem 6.** For every digraph \( D \) of order \( n \),
\[ d_I(D) + d_I(\overline{D}) \leq n + 3. \]
If \( d_I(D) + d_I(\overline{D}) = n + 3 \), then \( D \) is in-regular.

**Proof.** Since \( \Delta^{-}(D) + \delta^{-}(\overline{D}) + 1 = n \), Theorem 5 implies
\[
\begin{align*}
d_I(D) + d_I(\overline{D}) & \leq (\delta^{-}(D) + 2) + (\delta^{-}(\overline{D}) + 2) \\
& = (\delta^{-}(D) + 2) + (n - \Delta^{-}(D) - 1) + 2 \\
& = n - (\Delta^{-}(D) - \delta^{-}(D)) + 3 \leq n + 3,
\end{align*}
\]
and this is the desired bound. If \( D \) is not in-regular, then \( \Delta^{-}(D) - \delta^{-}(D) \geq 1 \), and the inequality chain above leads to the better bound \( d_I(D) + d_I(\overline{D}) \leq n + 2. \)

For a lot of regular digraphs we will improve first Theorem 5 and then Theorem 6.

**Theorem 7.** Let \( D \) be a \( \delta \)-regular digraph of order \( n \) with \( \delta \geq 1 \), and let \( n = p(\delta + 2) + r \) with integers \( p \geq 0 \) and \( 0 \leq r \leq \delta + 1 \). If \( 1 \leq r < (\delta + 2)/2 \) or \( (\delta + 2)/2 < r \leq \delta + 1 \), then \( d_I(D) \leq \delta + 1 \).
Proof. If \(1 \leq r < (\delta + 2)/2\), then Proposition 4 implies that
\[
\gamma_I(D) \geq \left\lceil \frac{2n}{\delta + 2} \right\rceil = \left\lceil \frac{2p(\delta + 2) + 2r}{\delta + 2} \right\rceil \geq 2p + 1.
\]

Using Theorem 1, we obtain
\[
d_I(D) \leq \frac{2n}{\gamma_I(D)} \leq \frac{2p(\delta + 2) + 2r}{2p + 1} < \frac{2p(\delta + 2) + \delta + 2}{2p + 1} = \delta + 2,
\]
and therefore \(d_I(D) \leq \delta + 1\) in this case.

If \((\delta + 2)/2 < r \leq \delta + 1\), then Proposition 4 implies that
\[
\gamma_I(D) \geq \left\lceil \frac{2n}{\delta + 2} \right\rceil = \left\lceil \frac{2p(\delta + 2) + 2r}{\delta + 2} \right\rceil \geq 2p + 2.
\]

Using Theorem 1, we obtain
\[
d_I(D) \leq \frac{2n}{\gamma_I(D)} \leq \frac{2p(\delta + 2) + 2r}{2p + 2} < \frac{2p(\delta + 2) + 2(\delta + 2)}{2p + 2} = \delta + 2,
\]
and therefore \(d_I(D) \leq \delta + 1\) also in this case.

\[\square\]

Theorem 8. If \(D\) is a \(\delta\)-regular digraph of order \(n\), then
\[
d_I(D) + d_I(\overline{D}) \leq n + 2,
\]

with exception of the cases that \(D\) is 4-regular of order 9, 7-regular of order 18 or 16-regular of order 45.

Proof. Since \(D\) is \(\delta\)-regular, \(\overline{D}\) is \(\overline{\delta}\)-regular with \(\overline{\delta} = n - \delta - 1\). Assume, without loss of generality, that \(\delta \leq \overline{\delta}\).

If \(\delta = 0\), then it follows from Corollary 1 and Theorem 2 that \(d_I(D) + d_I(\overline{D}) \leq n + 1\).

If \(\delta = 1\), then Corollary 2 and Theorem 3 lead to \(d_I(D) + d_I(\overline{D}) \leq 2 + n - 1 = n + 1\).

Thus let now \(\delta \geq 2\) and \(n = p(\delta + 2) + r\) with integers \(p \geq 0\) and \(0 \leq r \leq \delta + 1\). If \(r \neq 0, (\delta + 2)/2\), then Theorem 7 implies \(d_I(D) \leq \delta + 1\), and we obtain \(d_I(D) + d_I(\overline{D}) \leq n + 2\) as in the proof of Theorem 6. Next we discuss the cases \(r = 0\) or \(r = (\delta + 2)/2\).

a) Let \(r = 0\) and therefore \(n = p(\delta + 2)\). We observe that \(n = (\overline{\delta} + 2) + (\delta - 1)\) with \(\delta - 1 \geq 1\). If \(\delta - 1 \neq (\overline{\delta} + 2)/2\), then Theorem 7 implies \(d_I(\overline{D}) \leq \overline{\delta} + 1\), and we obtain \(d_I(D) + d_I(\overline{D}) \leq n + 2\) as in the proof of Theorem 6. Let now \(\delta - 1 = (\overline{\delta} + 2)/2\). Then
\[
n = (\overline{\delta} + 2) + \frac{\overline{\delta} + 2}{2} = \frac{3}{2}(\overline{\delta} + 2) = \frac{3}{2}(n + 1 - \delta)
\]
and thus \( n = 3\delta - 3 \). Hence \( p(\delta + 2) = 3\delta - 3 \), and this leads to \( p = 2 \). We deduce that \( \delta = 7 \) and \( n = 18 \).

b) Let \( r = (\delta + 2)/2 \) and therefore \( n = p(\delta + 2) + (\delta + 2)/2 \). As in case a), there remains the case that \( n = 3\delta - 3 \). Hence \( (p + 1/2)(\delta + 2) = 3\delta - 3 = 3(\delta + 2) - 9 \), and this yields to \( p \leq 2 \).

If \( p = 1 \), then we observe that \( \delta = 4 \) and \( n = 9 \).
If \( p = 2 \), then we obtain \( \delta = 16 \) and \( n = 45 \).

\[ \square \]

4. Open problems

We conclude by mentioning some conjectures suggested by this research.

**Conjecture 1.** If \( D \) is a \( \delta \)-regular digraph, then \( d_I(D) \leq \delta + 1 \).

The next conjecture is a consequence of Conjecture 1.

**Conjecture 2.** If \( D \) is a \( \delta \)-regular digraph of order \( n \), then \( d_I(D) + d_I(D) \leq n + 1 \).

Conjecture 2 is valid for \( \delta = 0 \) and for \( \delta = 1 \). If \( n = p_1(\delta + 2) + r_1 \) with \( r_1 \neq 0, (\delta + 2)/2 \) and \( n = p_2(\delta + 2) + r_2 \) with \( r_2 \neq 0, (\delta + 2)/2 \), then Theorem 7 shows that Conjecture 2 is also valid. I even think that the bound in Conjecture 2 is valid for all digraphs.

**Conjecture 3.** If \( D \) is a digraph of order \( n \), then \( d_I(D) + d_I(D) \leq n + 1 \).

References


