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The Italian domatic number of a digraph

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Abstract: An *Italian dominating function* on a digraph D with vertex set $V(D)$ is defined as a function $f\colon V(D) \to \{0,1,2\}$ such that every vertex $v \in V(D)$ with $f(v) = 0$ has at least two in-neighbors assigned 1 under f or one in-neighbor w with $f(w) = 2$. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct Italian dominating functions on D with the property that $\sum_{i=1}^{d} f_i(v) \leq 2$ for each $v \in V(D)$, is called an *Italian dominating family* (of functions) on D. The maximum number of functions in an Italian dominating family on D is the *Italian domatic number* of D, denoted by $d_I(D)$. In this paper we initiate the study of the Italian domatic number in digraphs, and we present some sharp bounds for $d_I(D)$. In addition, we determine the Italian domatic number of some digraphs.

Keywords: Digraphs, Italian dominating function, Italian domination number, Italian domatic number.

AMS Subject classification: 05C20, 05C69

1. Terminology and introduction

In this paper, D is a simple digraph with vertex set $V = V(D)$ and arc set $A = A(D)$. The order $|V|$ of D is denoted by $n = n(D)$. We write $d_D^+(v) = d^+(v)$ for the *out-degree* of a vertex v and $d_D^-(v) = d^-(v)$ for its *in-degree*. The *minimum* and *maximum in-degree* and *minimum* and *maximum out-degree* of D are denoted by $\delta^- = \delta^-(D)$, $\Delta^- = \Delta^-(D)$, $\delta^+ = \delta^+(D)$ and $\Delta^+ = \Delta^+(D)$, respectively. If uv is an arc of D, then we also write $u \to v$, and we say that v is an *out-neighbor* of u and u is an *in-neighbor* of v. A digraph D is *in-regular* or r*-in-regular* when $\delta^-(D) = \Delta^-(D) = r$ and *out-regular* or r*-out-regular* when $\delta^+(D) = \Delta^+(D) = r$. If D is r-in-regular and r-out-regular, then D is called r*-regular* or *regular*. For a vertex v of a digraph D, we denote the set of in-neighbors and out-neighbors of v by $N_D^-(v) = N^-(v)$ and $N_D^+(v)$

$= N^+(v)$, respectively. In addition, $N^-[v] = N^-(v) \cup \{v\}$ and $N^+[v] = N^+(v) \cup \{v\}$. If $X \subset V(D)$, then $N_D^+[X] = N^+[X] = \bigcup_{v \in X} N^+[v]$. We write K_n^* for the *complete digraph* of order n. For notation and graph theory terminology in general we follow Haynes, Hedetniemi and Slater [7].

A *Roman dominating function* on a digraph D is defined in [9] as a function $f: V(D) \to \{0,1,2\}$ satisfying the condition that every vertex v with $f(v) = 0$ has an in-neighbor u with $f(u) = 2$. The *weight* of a Roman dominating function f is the value $\sum_{v \in V(D)} f(v)$. The *Roman domination number* of a digraph D, denoted by $\gamma_R(D)$, is the minimum taken over the weights of all Roman dominating functions on D.

A set $\{f_1, f_2, \dots, f_d\}$ of distinct Roman dominating functions on D with $\sum_{i=1}^{d} f_i(v) \leq 2$ for each $v \in V(D)$, is called in [5] a *Roman dominating family* (of functions) on D. The maximum number of functions in a Roman dominating family on D is the *Roman domatic number* of D, denoted by $d_R(D)$.

An *Italian dominating function* (IDF) on a digraph D is defined in [13] as a function $f: V(D) \to \{0,1,2\}$ such that every vertex $v \in V(D)$ with $f(v) = 0$ has at least two in-neighbors assigned 1 under f or one in-neighbor w with $f(w) = 2$. The weight of an Italian dominating function f is the value $\omega(f) = f(V(D)) = \sum_{u \in V(D)} f(u)$. The *Italian domination number* of a digraph D, denoted by $\gamma_I(D)$, is the minimum taken over the weights of all ID functions on D. A $\gamma_I(D)$-*function* is an Italian dominating function on D with weight $\gamma_I(D)$. An Italian dominating function $f: V(D) \to \{0,1,2\}$ can be represented by the ordered partition (V_0, V_1, V_2) (or (V_0^f, V_1^f, V_2^f) to refer to f) of $V(D)$, where $V_i = V_i^f = \{v \in V(D) : f(v) = i\}$ for $i \in \{0,1,2\}$. In this representation, the weight of f is $\omega(f) = |V_1| + 2|V_2|$.

A set $\{f_1, f_2, \dots, f_d\}$ of distinct Italian dominating functions on D with the property that $\sum_{i=1}^{d} f_i(v) \leq 2$ for each $v \in V(D)$, is called an *Italian dominating family* (of functions) on D (ID family on D). The maximum number of functions in an ID family on D is the *Italian domatic number* of D, denoted by $d_I(D)$.

We note that if D is a digraph, then $\gamma_I(D) \leq \gamma_R(D)$ (see [13]), and the definitions lead to $d_I(D) \geq d_R(D) \geq 1$.

In this paper we continue the study of Roman and Italian dominating functions and domatic numbers in graphs and digraphs (see, for example, [1–4, 6, 8, 10–12]). We initiate the study of the Italian domatic number in digraphs, and we present some sharp bounds for $d_I(D)$. In addition, we determine the Italian domatic number of some digraphs.

We make use of the following known results in this paper.

Proposition 1 ([9]). *If D is a digraph of order n, then $\gamma_R(D) \leq n - \Delta^+(D) + 1$.*

Proposition 2 ([13]). *Let D be a digraph of order n. Then $\gamma_I(D) \leq n$ and $\gamma_I(D) = n$ if and only if $\Delta^-(D), \Delta^+(D) \leq 1$.*

Proposition 3 ([13]). *If D is a directed cycle or a directed path of order n, then $\gamma_I(D) = n$.*

Proposition 4 ([13]). *If D is a digraph of order n, then*

$$\gamma_I(D) \geq \left\lceil \frac{2n}{\Delta^+(D)+2} \right\rceil .$$

2. Properties of the Italian domatic number

In this section we present basic properties and sharp bounds on the Italian domatic number of digraphs.

Theorem 1. *If D is a digraph of order n, then*

$$\gamma_I(D) \cdot d_I(D) \leq 2n.$$

Moreover, if $\gamma_I(D) \cdot d_I(D) = 2n$, then for each Italian dominating family $\{f_1, f_2, \ldots, f_d\}$ on D with $d = d_I(D)$, each function f_i is a $\gamma_I(D)$-function and $\sum_{i=1}^{d} f_i(v) = 2$ for all $v \in V(D)$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be an ID family on D such that $d = d_I(D)$ and let $v \in V(D)$. Then

$$d \cdot \gamma_I(D) = \sum_{i=1}^{d} \gamma_I(D) \leq \sum_{i=1}^{d} \sum_{v \in V(D)} f_i(v) = \sum_{v \in V(D)} \sum_{i=1}^{d} f_i(v) \leq \sum_{v \in V(D)} 2 = 2n.$$

If $\gamma_I(D) \cdot d_I(D) = 2n$, then the two inequalities occurring in inequality chain above become equalities. Hence, for the ID family $\{f_1, f_2, \ldots, f_d\}$ on D and for each i, $\sum_{v \in V(D)} f_i(v) = \gamma_I(D)$. Thus, each function f_i is a $\gamma_I(D)$-function, and $\sum_{i=1}^{d} f_i(v) = 2$ for all $v \in V(D)$. □

Since $\gamma_I(D) \geq 2$ for each digraph D of order $n \geq 2$, Theorem 1 implies the next result immediately.

Corollary 1. *If D is a digraph of order n, then $d_R(D) \leq d_I(D) \leq n$.*

Theorem 2. *If D is a digraph, then $d_I(D) = 1$ if and only if $\Delta^-(D) \leq 1$ and D has no directed cycle of even length.*

Proof. Assume first that $\Delta^-(D) \geq 2$, and let w be a vertex with $d^-(w) \geq 2$. Define $f, g\colon V(D) \to \{0,1,2\}$ by $f(x) = 1$ for each $x \in V(D)$ and $g(w) = 0$ and $g(v) = 1$ for $v \in V(D) \setminus \{w\}$. Since $d^-(w) \geq 2$, we observe that f and g are Italian dominating functions on D with the property that $f(x) + g(x) \leq 2$ for each $x \in V(G)$. Therefore $\{f, g\}$ is an ID family on D and so $d_I(D) \geq 2$. Assume next that D has a directed cycle $C = v_1v_2 \cdots v_{2p}v_1$ for an integer $p \geq 1$. Then the functions $f\colon V(D) \to \{0,1,2\}$ with $f(v_1) = f(v_3) = \cdots = f(v_{2p-1}) = 2$, $f(v_2) = f(v_4) = \cdots = f(v_{2p}) = 0$ and $f(x) = 1$ for each $x \in V(D) - V(C)$ and $g\colon V(D) \to \{0,1,2\}$ with $g(v_1) = g(v_3) = \cdots = g(v_{2p-1}) = 0$, $g(v_2) = g(v_4) = \cdots = g(v_{2p}) = 2$ and $g(x) = 1$ for each $x \in V(D) - V(C)$ are Italian dominating functions on D. This implies that $\{f, g\}$ is an ID family on D, and therefore $d_I(D) \geq 2$.
Conversely assume that $d_I(D) \geq 2$. Suppose to the contrary that $\Delta^-(D) \leq 1$ and D has no directed cycle of even length. Let f and g be two distinct ID functions of an ID family on D. Since $\Delta^-(D) \leq 1$, we note that $V_0^f, V_2^f, V_0^g, V_2^g \neq \emptyset$. By definition, $f(v) + g(v) \leq 2$ for every vertex v. It follows that $V_2^f \subset V_0^g$ and $V_2^g \subset V_0^f$. Since f is an IDF, every vertex $v \in V_2^g \subset V_0^f$ has an in-neighbor in V_2^f. Likewise, every vertex $w \in V_2^f \subset V_0^g$ has an in-neighbor in V_2^g. Hence the bipartite subdigraph D' with vertex set $V_2^f \cup V_2^g$ and the arcs of D between V_2^f and V_2^g has minimum in-degree at least 1. It follows that D' has a directed cycle and, obviously, this directed cycle has even length, a contradiction. □

Corollary 2. *If P_n is a directed path of order n, then $d_I(P_n) = 1$. If C_n is a directed cycle of order n, then $d_I(C_n) = 1$ if n is odd and $d_I(C_n) = 2$ if n is even.*

Proof. Theorem 2 implies $d_I(P_n) = 1$, $d_I(C_n) = 1$ if n is odd and $d_I(C_n) \geq 2$ if n is even. Using Proposition 3 and Theorem 1, we obtain $d_I(C_n) \leq 2$ and thus $d_I(C_n) = 2$ if n is even. □

Proposition 5. *Let D be a digraph of order $n \geq 2$. Then $\gamma_I(D) = n$ and $d_I(D) = 2$ if and only if $\Delta^-(D) = \Delta^+(D) = 1$ and D contains a directed cycle of even length.*

Proof. Assume first that $\gamma_I(D) = n$ and $d_I(D) = 2$. It follows from Proposition 2 that $\Delta^-(D), \Delta^+(D) \leq 1$. Since $d_I(D) = 2$, Theorem 2 implies that D has a directed cycle of even length and so $\Delta^-(D) = \Delta^+(D) = 1$.
Conversely, assume that $\Delta^-(D) = \Delta^+(D) = 1$ and D contains a directed cycle of even length. Proposition 2 leads to $\gamma_I(D) = n$, and Theorem 2 shows that $d_I(D) \geq 2$. Therefore Theorem 1 implies $d_I(D) \leq 2$ and thus $d_I(G) = 2$. □

Next we show that the upper bound in Corollary 1 is attained only for complete digraphs.

Theorem 3. *If D is a digraph of order $n \geq 2$, then $d_I(D) = n$ if and only if D is the complete digraph on n vertices.*

Proof. Let $D = K_n^*$ be the complete digraph on n vertices with vertex set $\{v_1, v_2, \ldots, v_n\}$. Define the functions $f_i \colon V(D) \to \{0,1,2\}$ by $f_i(v_i) = 2$ and $f_i(v_j) = 0$ for $j \neq i$ and $1 \leq i, j \leq n$. Then f_i is an IDF on D for each $1 \leq i \leq n$ such that $\sum_{i=1}^{n} f_i(v) = 2$ for each $v \in V(D)$. Therefore $\{f_1, f_2, \ldots, f_n\}$ is an ID family on D. Combining this with Corollary 1, we deduce that $d_I(K_n^*) = n$.
Conversely, assume that $d_I(D) = n$. If $n = 2$, then the result is immediate. Assume next that $n \geq 3$. Then $\gamma_I(D) \geq 2$ and it follows from Theorem 1 that $\gamma_I(D) = 2$. Let $\{f_1, f_2, \ldots, f_n\}$ be an ID family on D. We deduce from Theorem 1 that f_i is a $\gamma_I(D)$-function for each i, and $\sum_{i=1}^{d} f_i(v) = 2$ for all $v \in V(D)$. Since $n \geq 3$ and $\gamma_I(D) = 2$, we conclude that for each i, there exists a vertex $x \in V(D)$ such that $f_i(x) \geq 1$. Assume, without loss of generality, that $f_i(v_i) \geq 1$ for each $i \in \{1, 2, \ldots, n\}$.
Suppose, without loss of generality, that there doesn't exist the arc $v_n v_1$. If $f_n(v_n) = 2$, then $f_n(v_1) = f_n(v_2) = \ldots = f_n(v_{n-1}) = 0$, and we obtain the contradiction $f_n(N^-[v_1]) = 0$. Therefore $f_n(v_n) = 1$ and thus $f_n(v_1) = 1$. Since $f_1(v_1) \geq 1$, we deduce that $f_1(v_1) = 1$. As f_1 and f_n are distinct, we deduce that $f_1(v_n) = 0$. Since $\sum_{i=1}^{n} f_i(v_n) = 2$, there exists an index $j \in \{2, 3, \ldots, n-1\}$ with $f_j(v_n) = 1$ and $f_j(v_1) = 0$. This leads to the contradiction $f_j(N^-[v_1]) \leq 1$. This completes the proof. □

Since $d_R(D) \leq d_I(D)$, Theorem 3 and the first part of its proof lead to the next result.

Corollary 3. *([5]) If D is a digraph of order $n \geq 2$, then $d_R(D) = n$ if and only if D is the complete digraph on n vertices.*

The upper bound on the product $\gamma_I(D) \cdot d_I(D)$ leads to an upper bound on the sum of these terms.

Theorem 4. *If D is a digraph of order $n \geq 2$, then*

$$\gamma_I(D) + d_I(D) \leq n + 2. \tag{1}$$

Moreover, equality holds if and only if $\Delta^+(D) = \Delta^-(D) = 1$ and D has a directed cycle of even length or D is the complete digraph.

Proof. If $d_I(D) \leq 1$, then obviously $\gamma_I(D) + d_I(D) \leq n + 1$. Assume now that $d_I(D) \geq 2$. According to Corollary 1, we have $d_I(D) \leq n$. Theorem 1 implies that

$$\gamma_I(D) + d_I(D) \leq \frac{2n}{d_I(D)} + d_I(D).$$

Using the fact that the function $g(x) = x + 2n/x$ is decreasing for $2 \leq x \leq \sqrt{2n}$ and increasing for $\sqrt{2n} \leq x \leq n$, this inequality leads to

$$\gamma_I(D) + d_I(D) \leq \frac{2n}{d_I(D)} + d_I(D) \leq \max\{n + 2, 2 + n\} = n + 2, \tag{2}$$

and this is the desired bound.
If D is the complete digraph on n vertices, then $\gamma_I(D) = 2$ and by Theorem 3, $d_I(D) = n$. If $\Delta^+(D) = \Delta^-(D) = 1$ and D contains an even cycle, then it follows from Proposition 5 that $\gamma_I(D) = n$ and $d_I(D) = 2$. Thus $\gamma_I(D) + d_I(D) = n + 2$ in both cases.
Conversely, let equality hold in (1). It follows from (2) that

$$n + 2 = \gamma_I(D) + d_I(D) \leq \frac{2n}{d_I(D)} + d_I(D) \leq n + 2,$$

which implies that $\gamma_I(D) = 2n/d_I(D)$ and $d_I(D) = 2$ or $d_I(D) = n$. If $d_I(D) = n$, then D is the complete digraph by Theorem 3. If $d_I(D) = 2$, then $\gamma_I(D) = n$, and it follows from Proposition 5 that $\Delta^+(D) = \Delta^-(D) = 1$ and D contains a directed cycle of even length. This completes the proof. □

The Italian domatic number of a digraph may also be bounded from above by its minimum in-degree plus 2.

Theorem 5. *For every digraph D,*

$$d_I(D) \leq \delta^-(D) + 2$$

and this bound is sharp.

Proof. If $d_I(D) \leq 2$, then the bound is immediate. Let now $d_I(D) \geq 3$ and let $\{f_1, f_2, \ldots, f_d\}$ be an ID family on D such that $d = d_I(D)$. Assume that v is a vertex of minimum in-degree $\delta^-(D)$. Since the equality $\sum_{x \in N^-[v]} f_i(x) = 1$ holds for at most two indices $i \in \{1, 2, \ldots, d\}$, we have

$$2d - 2 \leq \sum_{i=1}^{d} \sum_{x \in N^-[v]} f_i(x) = \sum_{x \in N^-[v]} \sum_{i=1}^{d} f_i(x) \leq \sum_{x \in N^-[v]} 2 = 2(\delta^-(D) + 1).$$

This implies the desired bound $d_I(D) \leq \delta^-(D) + 2$.
To prove sharpness, let $k \geq 2$ be an integer. In addition, let D_i be a copy of the complete digraph K^*_{k+3} with vertex set $V(D_i) = \{v^i_1, v^i_2, \ldots, v^i_{k+3}\}$ for $1 \leq i \leq k$, and let D be the digraph obtained from $\bigcup_{i=1}^{k} D_i$ by adding a new vertex v and the arcs vv^i_1 as well as $v^i_1 v$ for $1 \leq i \leq k$. Define the ID functions $f_1, f_2, \ldots, f_{k+2}$ as follows:

$$f_i(v^i_1) = 2,\ f_i(v^j_{i+1}) = 2 \text{ if } j \in \{1, 2, \ldots, k\} - \{i\} \text{ and } f(x) = 0 \text{ otherwise } (1 \leq i \leq k),$$

$$f_{k+1}(v) = 1, f_{k+1}(v^j_{k+2}) = 2, \text{ if } j \in \{1, 2, \ldots, k\} \text{ and } f(x) = 0 \text{ otherwise},$$

and

$$f_{k+2}(v) = 1, f_{k+2}(v^j_{k+3}) = 2, \text{ if } j \in \{1, 2, \ldots, k\} \text{ and } f(x) = 0 \text{ otherwise.}$$

It is easy to see that every f_i is an IDF on D and that $\{f_1, f_2, \ldots, f_{k+2}\}$ is an ID family on D. Since $\delta^-(D) = k$, we deduce that $d_I(D) = \delta^-(D) + 2$. □

Since $d_R(D) \leq d_I(D)$, Theorem 5, and the example in the proof of Theorem 5 yield to the next result.

Corollary 4. *([5]) For every digraph D,*

$$d_R(D) \leq \delta^-(D) + 2$$

and this bound is sharp.

3. Nordhaus-Gaddum type results

The *complement* $\overline{D}$ of a digraph D is the digraph with vertex set $V(D)$ such that for any two distinct vertices u, v the arc uv belongs to $\overline{D}$ if and only if uv does not belong to D. Results of Nordhaus-Gaddum type study extreme values of the sum or the product of a parameter on a digraph and its complement. We establish such inequalities for the Italian domination number.

Theorem 6. *For every digraph D of order n,*

$$d_I(D) + d_I(\overline{D}) \leq n + 3.$$

If $d_I(D) + d_I(\overline{D}) = n + 3$, then D is in-regular.

Proof. Since $\Delta^-(D) + \delta^-(\overline{D}) + 1 = n$, Theorem 5 implies

$$\begin{aligned} d_I(D) + d_I(\overline{D}) &\leq (\delta^-(D) + 2) + (\delta^-(\overline{D}) + 2) \\ &= (\delta^-(D) + 2) + (n - \Delta^-(D) - 1) + 2 \\ &= n - (\Delta^-(D) - \delta^-(D)) + 3 \leq n + 3, \end{aligned}$$

and this is the desired bound. If D is not in-regular, then $\Delta^-(D) - \delta^-(D) \geq 1$, and the inequality chain above leads to the better bound $d_I(D) + d_I(\overline{D}) \leq n + 2$. □

For a lot of regular digraphs we will improve first Theorem 5 and then Theorem 6.

Theorem 7. *Let D be a δ-regular digraph of order n with $\delta \geq 1$, and let $n = p(\delta + 2) + r$ with integers $p \geq 0$ and $0 \leq r \leq \delta + 1$. If $1 \leq r < (\delta + 2)/2$ or $(\delta + 2)/2 < r \leq \delta + 1$, then $d_I(D) \leq \delta + 1$.*

Proof. If $1 \le r < (\delta+2)/2$, then Proposition 4 implies that

$$\gamma_I(D) \ge \left\lceil \frac{2n}{\delta+2} \right\rceil = \left\lceil \frac{2p(\delta+2)+2r}{\delta+2} \right\rceil \ge 2p+1.$$

Using Theorem 1, we obtain

$$d_I(D) \le \frac{2n}{\gamma_I(D)} \le \frac{2p(\delta+2)+2r}{2p+1} < \frac{2p(\delta+2)+\delta+2}{2p+1} = \delta+2,$$

and therefore $d_I(D) \le \delta+1$ in this case.
If $(\delta+2)/2 < r \le \delta+1$, then Proposition 4 implies that

$$\gamma_I(D) \ge \left\lceil \frac{2n}{\delta+2} \right\rceil = \left\lceil \frac{2p(\delta+2)+2r}{\delta+2} \right\rceil \ge 2p+2.$$

Using Theorem 1, we obtain

$$d_I(D) \le \frac{2n}{\gamma_I(D)} \le \frac{2p(\delta+2)+2r}{2p+2} < \frac{2p(\delta+2)+2(\delta+2)}{2p+2} = \delta+2,$$

and therefore $d_I(D) \le \delta+1$ also in this case. □

Theorem 8. *If D is a δ-regular digraph of order n, then*

$$d_I(D) + d_I(\overline{D}) \le n+2,$$

with exception of the cases that D is 4-regular of order 9, 7-regular of order 18 or 16-regular of order 45.

Proof. Since D is δ-regular, $\overline{D}$ is $\overline{\delta}$-regular with $\overline{\delta} = n-\delta-1$. Assume, without loss of generality, that $\delta \le \overline{\delta}$.
If $\delta = 0$, then it follows from Corollary 1 and Theorem 2 that $d_I(D)+d_I(\overline{D}) \le n+1$. If $\delta = 1$, then Corollary 2 and Theorem 3 lead to $d_I(D)+d_I(\overline{D}) \le 2+n-1 = n+1$. Thus let now $\delta \ge 2$ and $n = p(\delta+2)+r$ with integers $p \ge 0$ and $0 \le r \le \delta+1$. If $r \ne 0, (\delta+2)/2$, then Theorem 7 implies $d_I(D) \le \delta+1$, and we obtain $d_I(D)+d_I(\overline{D}) \le n+2$ as in the proof of Theorem 6. Next we discuss the cases $r = 0$ or $r = (\delta+2)/2$.
a) Let $r = 0$ and therefore $n = p(\delta+2)$. We observe that $n = (\overline{\delta}+2)+(\delta-1)$ with $\delta-1 \ge 1$. If $\delta-1 \ne (\overline{\delta}+2)/2$, then Theorem 7 implies $d_I(\overline{D}) \le \overline{\delta}+1$, and we obtain $d_I(D)+d_I(\overline{D}) \le n+2$ as in the proof of Theorem 6. Let now $\delta-1 = (\overline{\delta}+2)/2$. Then

$$n = (\overline{\delta}+2) + \frac{\overline{\delta}+2}{2} = \frac{3}{2}(\overline{\delta}+2) = \frac{3}{2}(n+1-\delta)$$

and thus $n = 3\delta - 3$. Hence $p(\delta + 2) = 3\delta - 3$, and this leads to $p = 2$. We deduce that $\delta = 7$ and $n = 18$.
b) Let $r = (\delta + 2)/2$ and therefore $n = p(\delta + 2) + (\delta + 2)/2$. As in case a), there remains the case that $n = 3\delta - 3$. Hence $(p + 1/2)(\delta + 2) = 3\delta - 3 = 3(\delta + 2) - 9$, and this yields to $p \leq 2$.
If $p = 1$, then we observe that $\delta = 4$ and $n = 9$.
If $p = 2$, then we obtain $\delta = 16$ and $n = 45$. □

4. Open problems

We conclude by mentioning some conjectures suggested by this research.

Conjecture 1. If D is a δ-regular digraph, then $d_I(D) \leq \delta + 1$.

The next conjecture is a consequence of Conjecture 1.

Conjecture 2. If D is a δ-regular digraph of order n, then $d_I(D) + d_I(\overline{D}) \leq n + 1$.

Conjecture 2 is valid for $\delta = 0$ and for $\delta = 1$. If $n = p_1(\delta+2)+r_1$ with $r_1 \neq 0, (\delta+2)/2$ and $n = p_2(\overline{\delta} + 2) + r_2$ with $r_2 \neq 0, (\overline{\delta} + 2)/2$, then Theorem 7 shows that Conjecture 2 is also valid. I even think that the bound in Conjecture 2 is valid for all digraphs.

Conjecture 3. If D is a digraph of order n, then $d_I(D) + d_I(\overline{D}) \leq n + 1$.

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