Eternal $m$-security subdivision numbers in trees

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Abstract: An eternal $m$-secure set of a graph $G = (V,E)$ is a set $S_0 \subseteq V$ that can defend against any sequence of single-vertex attacks by means of multiple-guard shifts along the edges of $G$. A suitable placement of the guards is called an eternal $m$-secure set. The eternal $m$-security number $\sigma_m(G)$ is the minimum cardinality among all eternal $m$-secure sets in $G$. An edge $uv \in E(G)$ is subdivided if we delete the edge $uv$ from $G$ and add a new vertex $x$ and two edges $ux$ and $vx$. The eternal $m$-security subdivision number $sd\sigma_m(G)$ of a graph $G$ is the minimum cardinality of a set of edges that must be subdivided (where each edge in $G$ can be subdivided at most once) in order to increase the eternal $m$-security number of $G$. In this paper, we study the eternal $m$-security subdivision number in trees. In particular, we show that the eternal $m$-security subdivision number of trees is at most 2 and we characterize all trees attaining this bound.

Keywords: eternal $m$-secure set, eternal $m$-security number, eternal $m$-security subdivision number

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1. Introduction

Throughout this paper, $G$ is a simple connected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The numbers of vertices and edges are called the order and size of the $G$, respectively. For every vertex $v \in V$, the open neighborhood of $v$ is the set $N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the closed neighborhood of $v$ is the set $N[v] = N(v) \cup \{v\}$. The degree $\deg(v)$ of $v$ is the number of edges incident with $v$ or equivalently $\deg(v) = |N(v)|$. The minimum and maximum degree of $G$ are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A leaf of $G$ is a vertex of degree 1 and a support
vertex of $G$ is a vertex adjacent to a leaf. A support vertex is called strong support vertex if it is adjacent to at least two leaves. We denote the set of leaves of a graph $G$ and the set of leaves adjacent to $v \in V(G)$ by $L(G)$ and $L_v$, respectively. For a vertex $v$ in a rooted tree $T$, let $D(v)$ denote the set of descendants of $v$. An edge $uv \in E(G)$ is subdivided if the edge $uv$ is deleted and a new vertex $x$ and two new edges $ux$ and $vx$ are added in $G$.

The concept of domination in graphs was first defined by Ore in 1962 [7]. A set $S$ of vertices in a graph $G$ is called a dominating set if every vertex in $V$ is either an element of $S$ or is adjacent to an element of $S$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality among all dominating sets of $G$. A $\gamma(G)$-set is a dominating set of $G$ of size $\gamma(G)$.

The domination subdivision number $sd_\gamma(G)$ of a graph $G$ is the minimum cardinality of a set of edges of $G$ that must be subdivided (where each edge in $G$ can be subdivided at most once) in order to increase the domination number of $G$. This concept was first introduced by Velammal in his Ph.D. thesis [8] and since then many results have been obtained on some domination parameters (see for instance [2, 5]).

An eternal 1-secure set of a graph $G$ is a set $S_0 \subseteq V$ that can defend against any sequence of single-vertex attacks by means of single-guard shifts along the edges of $G$. That is, for any $k$ and any sequence $v_1, v_2, \ldots, v_k$ of vertices, there exists a sequence of guards $u_1, u_2, \ldots, u_k$ with $u_i \in S_{i-1}$ and either $u_i = v_i$ or $u_i v_i \in E$, such that each set $S_i = (S_{i-1} \setminus \{u_i\}) \cup \{v_i\}$ is a dominating set. It follows that each $S_i$ can be chosen to be an eternal 1-secure set. The eternal 1-security number of $G$, denoted by $\sigma_1(G)$, is the minimum cardinality among all eternal 1-secure set. The eternal 1-security number was introduced by Burger et al. [3] using the notation $\gamma_\infty$. In order to reduce the number of guards needed in an eternal secure set, Goddard et al. [4] considered allowing more guards to move. Suppose that in responding to each attack, every guard may shift along an incident edge. The eternal $m$-security number $\sigma_m(G)$ is the minimum number of guards to handle an arbitrary sequence of single attacks using multiple-guard shifts. A suitable placement of the guards is called an eternal $m$-secure set (EmSS) of $G$. An EmSS of size $\sigma_m(G)$ is called a $\sigma_m(G)$-set.

Obviously, any EmSS of $G$ is a dominating set of $G$. So we have $\gamma(G) \leq \sigma_m(G)$. When an edge $uv \in E(G)$ is subdivided with a vertex $x$, then the eternal $m$-security number of $G$ can not decrees. The eternal $m$-security subdivision number $sd_{\sigma_m}(G)$ of a graph $G$ is the minimum cardinality of a set of edges of $G$ that must be subdivided (where each edge in $G$ can be subdivided at most once) in order to increase the eternal $m$-security number of $G$. Since in the study of eternal $m$-security subdivision number, the assumption $\sigma_m(G) < n$ is necessary, we always assume that when we discuss $sd_{\sigma_m}(G)$, all graphs involved satisfy $\sigma_m(G) < n$, i.e., all graphs are nonempty. In this paper, we study of the eternal $m$-security subdivision number in trees. In particular, we prove that the eternal $m$-security subdivision number of a tree is at most 2 and we characterize all trees attaining this bound. For a more thorough treatment of domination parameters and for terminology not presented here see [6, 9]. The proof of the following results can be found in [4].
**Theorem A.** For any graph $G$, $\gamma(G) \leq \sigma_m(G)$.

**Theorem B.** For any graph $G$, $\sigma_m(G) \leq \alpha(G)$.

**Theorem C.**
1. $\sigma_m(K_n) = 1$.
2. $\sigma_m(P_n) = \lceil \frac{n}{2} \rceil$.
3. $\sigma_m(C_n) = \lceil \frac{n}{3} \rceil$.

**Theorem D.** For any graph $G$, $\sigma_m(G) \geq (\text{diam}(G) + 1)/2$.

Next results are immediate consequence of Propositions C and D.

**Corollary 1.** For any graph $G$, $\sigma_m(G) = 1$ if and only if $G \cong K_n$.

**Corollary 2.** For $n \geq 2$, $\text{sd}_{\sigma_m}(K_n) = 1$.

**Corollary 3.** For $n \geq 2$, $\text{sd}_{\sigma_m}(P_n) = \begin{cases} 1, & \text{if } n \text{ is even} \\ 2, & \text{if } n \text{ is odd} \end{cases}$.

### 2. Main Results

In this section, we show that for any tree $T$, $\text{sd}_{\sigma_m}(T) \leq 2$ and we characterize all trees attaining this bound. We start with two propositions.

**Proposition 1.** Let $G$ be a connected graph. If $G$ has a vertex $u$ with $|L_u| \geq 3$, then $\text{sd}_{\sigma_m}(G) = 1$.

**Proof.** Let $w_1, w_2, w_3 \in L_u$ and let $G'$ be the graph obtained from $G$ by subdividing the edge $uw_1$ by subdivision vertex $x$. Let $S$ be a $\sigma_m(G')$—set containing $w_2$ (we may assume that $S$ is a response to an attack on $w_2$). To dominate $w_3$, we may assume that $u \in S$. On the other hand, to dominate $w_1$, we must have $|S \cap \{w_1, x\}| \geq 1$. It is easy to see that $S \setminus \{w_1, x\}$ is an EmSS of $G$. This implies that $\text{sd}_{\sigma_m}(T) = 1$. □

**Proposition 2.** Let $G$ be a connected graph. If $G$ has a vertex $u$ with $|L_u| = 2$, then $\text{sd}_{\sigma_m}(G) \leq 2$.

**Proof.** Let $w_1, w_2 \in L_u$ and let $G'$ is the graph obtained from $G$ by subdividing the edges $uw_1$ and $uw_2$ by subdivision vertices $x$ and $y$ respectively. Let $S$ be a $\sigma_m(G')$—set containing $u$ (we may assume that $S$ is a response to an attack on $u$). To dominate $w_1$ and $w_2$, we must have $|S \cap \{w_1, x\}| \geq 1$ and $|S \cap \{w_1, y\}| \geq 1$, respectively. It is easy to see that $(S \setminus \{w_1, x, y\}) \cup \{w_2\}$ is an EmSS of $G$. This implies that $\text{sd}_{\sigma_m}(T) \leq 2$. □
**Theorem 1.** For any tree $T$, $sd_{\sigma_m}(T) \leq 2$.

**Proof.** The result is obvious for $n(T) \leq 3$. Let $n(T) \geq 4$. If $T$ is a star, then the result follows from Proposition 1. Assume that $T$ is not a star and $v_1v_2\ldots v_k$ be a diametrical path in $T$. Root $T$ at $v_k$. If $\deg(v_2) \geq 3$, then the result follows from Proposition 2. Suppose that $\deg(v_2) = 2$ and $T'$ is the tree obtained from $T$ by subdividing the edges $v_1v_2$ and $v_2v_3$ by subdivision vertices $x$ and $y$, respectively. Let $S$ be a $\sigma_m(T')$-set containing $v_2$. To dominate $v_1$, we may assume that $x \in S$. Let $S' = S \setminus \{x\}$ if $y \notin S$, $S' = (S \setminus \{x, y\}) \cup \{v_3\}$ if $v_3 \notin S$ and $y \in S$ and $S' = (S \setminus \{x, y\}) \cup \{w\}$ if $v_3, y \in S$, where $w \in N_T(v_3) \setminus \{v_2\}$. Clearly, $S'$ is an EmSS of $T$ of size $|S| - 1$ and this completes the proof.

Now we give a constructive characterization of trees $T$ for which $sd_{\sigma_m}(T) = 2$. For this purpose, we describe a procedure to build a family $\mathcal{T}$ of trees as follows. Let $T$ be the family of trees that: A path $P_3$ is a tree in $\mathcal{T}$ and if $T$ is a tree in $\mathcal{T}$, then the tree $T'$ obtained from $T$ by the following four operations which extend the tree $T$ by attaching a tree to a vertex $v \in V(T)$, called an attacher, is also a tree in $\mathcal{T}$.

**Operation $\mathcal{T}_1$.** If $v \in V(T)$, then $\mathcal{T}_1$ adds a path $vxy$ to $T$.

**Operation $\mathcal{T}_2$.** If $v \in V(T)$, then $\mathcal{T}_2$ adds a star $K_{1,3}$ with center $y$ and leaves $x, w, z$ and joins $x$ to $v$.

**Operation $\mathcal{T}_3$.** If $v \in V(T)$ is a leaf of $T$, then $\mathcal{T}_3$ adds a pendant edge $vw$ and a star $K_{1,2}$ with center $x$ and leaves $y, z$ and joins $x$ to $v$.

**Operation $\mathcal{T}_4$.** If $v$ is a leaf of $T$, then $\mathcal{T}_4$ adds two new stars $K_{1,2}$ with centers $x_1$ and $x_2$ and joins $v$ to $x_1$ and $x_2$ (see Fig. 1).

![Fig. 1. The four operations](image-url)

The proof of the following Lemmas can be found in [1].

**Lemma 1.** Let $T'$ be a tree, $v \in V(T')$ and $T$ be obtained from $T'$ by Operation $\mathcal{T}_1$. Then $\sigma_m(T) = \sigma_m(T') + 1$. 

Lemma 2. Let $T'$ be a tree and $v \in V(T')$. If $T$ is the tree obtained from $T'$ by Operation $\Xi_2$, then $\sigma_m(T) = \sigma_m(T') + 2$.

Lemma 3. Let $T'$ be a tree and $v \in L(T')$. If $T$ is the tree obtained from $T'$ by Operation $\Xi_3$, then $\sigma_m(T) = \sigma_m(T') + 2$.

Lemma 4. Let $T'$ be a tree and let $v \in L(T')$. If $T$ is the tree obtained from $T'$ by Operation $\Xi_4$, then $\sigma_m(T) = \sigma_m(T') + 3$.

Lemma 5. Let $T' \in \mathcal{T}$ and $u \in V(T')$. If $T$ is a tree obtained from $T'$ by adding a pendant edge $uu'$, then $\sigma_m(T) = \sigma_m(T')$.

Observation 2. Let $T'$ be a tree and $T$ be obtained from $T'$ by an operation from the set $\{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}$. Then $sd_{\sigma_m}(T) \leq sd_{\sigma_m}(T')$.

Proof. Let $F$ be a set of edges in $T'$ where subdividing the edges in $F$ increases the eternal $m-$ security number of $T'$. Let $T_1$ and $T_2$ be the trees obtained from $T'$ and $T$, by subdividing the edges in $F$, respectively. Then $T_2$ is obtained from $T_1$ by one of the Operations $\Xi_1, \ldots, \Xi_4$ and the result follows from Lemmas 1, 2, 3 and 4. □

Theorem 3. Let $T \in \mathcal{T}$ and let $T'$ be a tree obtained from $T$ by subdividing an edge of $T$. Then $\sigma_m(T') = \sigma_m(T)$.

Proof. Let $T \in \mathcal{T}$, $e \in E(T)$ and let $T'$ be the tree obtained from $T$ by subdividing the edge $e$ by subdivision vertex $u$. First note that $\sigma_m(T') \geq \sigma_m(T)$. Let $T$ be obtained from a path $P_3$ by successive operations $\Xi_1, \ldots, \Xi_m$, respectively, where $\Xi_i \in \{\Xi_1, \Xi_2, \Xi_3, \Xi_4\}$ for $1 \leq i \leq m$ if $m \geq 1$ and $T = P_3$ if $m = 0$. We proceed by induction on $m$. If $m = 0$, then clearly the statement is true by Corollary 3. Assume $m \geq 1$ and that the statement holds for all trees which are obtained from $P_3$ by applying at most $m-1$ operations. Suppose $T_{m-1}$ is a tree obtained by applying the first $m-1$ operations $\Xi_1, \ldots, \Xi^{m-1}$. When $e \in E(T_{m-1})$, let $T'_{m-1}$ be obtained from $T_{m-1}$ by subdividing the edge $e$. We consider the following cases:

Case 1. $\Xi^m = \Xi_1$. Then $T$ is obtained from $T_{m-1}$ by attaching a path $vxy$ to $v \in V(T_{m-1})$. If $e \in E(T_{m-1})$, then by the inductive hypothesis, $\sigma_m(T''_{m-1}) = \sigma_m(T_{m-1})$ and by Lemma 1,

$$\sigma_m(T') = \sigma_m(T''_{m-1}) + 1 = \sigma_m(T_{m-1}) + 1 = \sigma_m(T).$$

Assume that $e = xy$ (the case $e = vx$ is similar). Let $T^* = T' - \{u, y\}$. Then $T^*$ is obtained from $T_{m-1}$ by attaching a pendant edge $vy$. By Lemma 5, $\sigma_m(T^*) = \sigma_m(T_{m-1})$ and by Lemma 1, we have

$$\sigma_m(T') = \sigma_m(T^*) + 1 = \sigma_m(T_{m-1}) + 1 = \sigma_m(T).$$
**Case 2.** \( \mathfrak{T}^m = \mathfrak{T}_2 \). Then \( T \) is obtained from \( T_{m-1} \) by adding a star \( K_{1,3} \) centered at \( y \) and leaves \( x, w, z \) and joining \( x \) to \( v \). If \( e \in E(T_{m-1}) \), then by the inductive hypothesis, \( \sigma_m(T_{m-1}) = \sigma_m(T_{m-1}) \) and by Lemma 2,

\[
\sigma_m(T') = \sigma_m(T'_{m-1}) + 2 = \sigma_m(T_{m-1}) + 2 = \sigma_m(T).
\]

If \( e = xy \) (the case \( e = vx \) is similar), then let \( T^* = T' - \{u, y, z, w\} \). Then \( T^* \) is obtained from \( T_{m-1} \) by attaching a pendant edge \( vx \). By Lemma 5, \( \sigma_m(T^*) = \sigma_m(T_{m-1}) \) and by Lemma 2, we have

\[
\sigma_m(T') = \sigma_m(T^*) + 2 = \sigma_m(T_{m-1}) + 2 = \sigma_m(T).
\]

Assume that \( e = yz \) (the case \( e = yw \) is similar). Let \( T^* = T' - z \). Then \( T^* \cong T \) and \( T' \) is obtained from \( T^* \) by attaching a pendant edge \( uz \). It follows from Lemma 5 that \( \sigma_m(T') = \sigma_m(T^*) = \sigma_m(T) \).

**Case 3.** \( \mathfrak{T}^m = \mathfrak{T}_3 \). Then \( T \) is obtained from \( T_{m-1} \) by attaching a pendant edge \( vw \) to the leaf \( v \in V(T_{m-1}) \) and adding a star \( K_{1,2} \) with center \( x \) and leaves \( y, z \) and joining \( x \) to \( v \). If \( e \in E(T_{m-1}) \), then by the inductive hypothesis, \( \sigma_m(T'_{m-1}) = \sigma_m(T_{m-1}) \) and by Lemma 3,

\[
\sigma_m(T') = \sigma_m(T'_{m-1}) + 2 = \sigma_m(T_{m-1}) + 2 = \sigma_m(T).
\]

If \( e = xv \), then let \( T^* = T' - \{u, x, y, z\} \). Then \( T' \) is obtained from \( T^* \) by adding a star with center \( x \) and leaves \( u, y, z \) and joining \( u \) to \( v \). On the other hand, \( T^* \) is obtained from \( T_{m-1} \) by attaching a pendant edge \( vw \). By Lemmas 5 and 2,

\[
\sigma_m(T') = \sigma_m(T^*) + 2 = \sigma_m(T_{m-1}) + 2 = \sigma_m(T).
\]

If \( e = vw \), then let \( T^* = T' - w \). Then \( T^* \cong T \) and \( T' \) is obtained from \( T^* \) by attaching a pendant edge \( u'w \). By Lemma 5, \( \sigma_m(T') = \sigma_m(T^*) = \sigma_m(T) \). If \( e = xy \) (the case \( e = xz \) is similar), then let \( T^* = T' - y \). Then \( T^* \cong T \) and \( T' \) is obtained from \( T^* \) by attaching a pendant edge \( uy \). By Lemma 5, \( \sigma_m(T') = \sigma_m(T^*) = \sigma_m(T) \).

**Case 4.** \( \mathfrak{T}^m = \mathfrak{T}_4 \). Then \( T \) is obtained from \( T_{m-1} \) by adding two stars \( K_{1,2} \) with centers \( x_1 \) and \( x_2 \) and joining \( x_1, x_2 \) to \( v \). Let \( y_i, z_i \) be the leaves adjacent to \( x_i \), for \( i = 1, 2 \). If \( e \in E(T_{m-1}) \), then by the inductive hypothesis, \( \sigma_m(T'_{m-1}) = \sigma_m(T_{m-1}) \) and by Lemma 4,

\[
\sigma_m(T') = \sigma_m(T'_{m-1}) + 3 = \sigma_m(T_{m-1}) + 3 = \sigma_m(T).
\]
By Proposition 1, three copies adjacent to the center of a star \( n \) holds for any tree of order less than \( n \). According to Corollary 4, we only need to prove the necessity. We proceed by Proof.

Now we are ready to prove the main theorem of this section. Let \( T \) be a tree and \( v \) be a strong support vertex in \( T \) such that \( v \) is adjacent to the center of a star \( K_{1,2} \). Then \( \text{sd}_{\sigma_m}(T) = 1 \).

**Proposition 3.** Let \( T \) be a tree and \( v \) be a strong support vertex in \( T \) such that \( v \) is adjacent to the center of a star \( K_{1,2} \). Then \( \text{sd}_{\sigma_m}(T) = 1 \).

**Proof.** Let \( v \) be adjacent to the vertex \( u \) which is a center of a star \( K_{1,2} \) and let \( T' \) be the tree obtained from \( T \) by subdividing the edge \( vu \) with the subdivision vertex \( z \). Assume that \( S \) is a \( \sigma_m(T') \)-set containing \( z \) (we may assume \( S \) as a response to an attack on \( z \)). To dominate the leaves in \( L_u \) and \( L_v \), we have \(|(N_{T'}[u] \cup N_{T'}[v]) \cap S| \geq 4 \) and we may assume that \( v, u \in S \). It is easy to see that \( S \setminus \{z\} \) is an EmSS of \( T \) of size \( |S| - 1 \). This completes the proof.

**Proposition 4.** Let \( T \) be a tree and \( v \in V(T) \). If \( T' \) is a tree obtained from \( T \) by adding three copies \( x_iy_iz_i \) \((1 \leq i \leq 3)\) of \( P_3 \) and joining \( v \) to \( y_1, y_2, y_3 \), then \( \text{sd}_{\sigma_m}(T') = 1 \).

**Proof.** Let \( T_1 \) be the tree obtained from \( T \) by subdividing the edge \( vy_1 \) by subdivision vertex \( w \). Let \( S \) be a \( \sigma_m(T_1) \)-set containing \( w \) (we may consider a response to an attack on \( w \)). To dominate \( x_1 \) and \( z_1 \), we may assume that \( y_1 \in S \). If \( v \notin S \), then we may assume that \( z_i, y_i \in S \) for \( i = 2, 3 \) and the set \( S' = (S \setminus \{w, y_2\}) \cup \{v\} \) is clearly an EmSS of \( T' \) of size less than \( \sigma_m(T_1) \). If \( v \in S \), then it is not hard to see that the set \( S' = S \setminus \{w\} \) is an EmSS of \( T \) of size \( |S| - 1 \). This implies that \( \text{sd}_{\sigma_m}(T) = 1 \).

Now we are ready to prove the main theorem of this section.

**Theorem 4.** For any tree \( T \) of order \( n \geq 3 \), \( \text{sd}_{\sigma_m}(T) = 2 \) if and only if \( T \in \mathcal{X} \).

**Proof.** According to Corollary 4, we only need to prove the necessity. We proceed by induction \( n \). The result is trivial for \( n = 3 \). Let \( n \geq 4 \) and assume that the statement holds for any tree of order less than \( n \). Let \( T \) be a tree of order \( n \) and \( \text{sd}_{\sigma_m}(T) = 2 \). By Proposition 1, \( T \) is not a star and so \( \text{diam}(T) \geq 3 \). Assume \( P := v_1, v_2, \ldots, v_k \) is the
diametrical path in $T$ such that $\deg(v_2)$ is as small as possible. Suppose that $T$ is rooted at $v_k$. It follows from Proposition 1 and the assumption $\text{sd}_{\sigma_m}(T) = 2$ that $\deg(v_2) \leq 3$. If $\deg(v_2) = 2$, then let $T' = T - \{v_1, v_2\}$. It follows from Lemma 1 and Observation 2, that $\text{sd}_{\sigma_m}(T') = 2$. By the inductive hypothesis, $T' \in \mathcal{T}$. Now $T$ can be obtained from $T'$ by Operation $\mathcal{T}_1$ and so $T \in \mathcal{T}$. Let $\deg(v_2) = 3$. Suppose that $w \neq v_1$ is a leaf adjacent to $v_2$. If $\deg(v_3) = 2$, then let $T' = T - \{v_1, v_2, v_3, w\}$. It follows from Lemma 1 and Observation 2 that $\text{sd}_{\sigma_m}(T') = 2$. By the inductive hypothesis, $T' \in \mathcal{T}$. Now $T$ can be obtained from $T'$ by Operation $\mathcal{T}_2$ and so $T \in \mathcal{T}$.

Let $\deg(v_3) \geq 3$. It follows from the assumption about $v_2$ and Propositions 3 and 4 that $\deg(v_3) = 3$ and there is only two possible cases.

**Case 1.** $v_3$ is adjacent to a leaf $x$. Let $T' = T - \{v_1, v_2, x, w\}$. It follows from Lemma 3 and Observation 2 that $\text{sd}_{\sigma_m}(T') = 2$. By the inductive hypothesis, $T' \in \mathcal{T}$. Now $T$ can be obtained from $T'$ by Operation $\mathcal{T}_3$ and so $T \in \mathcal{T}$.

**Case 2.** $v_3$ is adjacent to the center of a star $K_{1,2}$ other than $v_2$. Let $T' = T - D(v_3)$. It follows from Lemma 4 and Observation 2 that $\text{sd}_{\sigma_m}(T') = 2$. By the inductive hypothesis, $T' \in \mathcal{T}$. Now $T$ can be obtained from $T'$ by Operation $\mathcal{T}_4$ and so $T \in \mathcal{T}$.

This completes the proof.

We conclude this paper with the following problem.

**Problem.** Prove or disprove: For any nonempty graph $G$, $1 \leq \text{sd}_{\sigma_m}(G) \leq 3$.

References


