Strong alliances in graphs

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Abstract: For any simple connected undirected graph $G = (V, E)$, a defensive alliance is a subset $S$ of $V$ satisfying the condition that every vertex $v \in S$ has at most one more neighbour in $V - S$ than it has in $S$. The minimum cardinality of any defensive alliance in $G$ is called the alliance number of $G$, and is denoted by $a_d(G)$. In this paper, we introduce a new type of alliance number called the $k$-strong alliance number and its varieties. The bounds for 1-strong alliance number in terms of different graphical parameters are determined and the characterizations of graphs with 1-strong alliance number 1, 2, and $n$ are obtained.

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1. Introduction

All the graphs considered in this article here are undirected, finite, connected and simple. We use the standard terminology, the terms not defined here may be found in [2]. Let $G = (V, E)$ be a connected graph and $v \in V$. Then $\Delta(G)$ is the maximum degree of a vertex in $G$, $N(v) = \{u \in V : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$.

The concept of an alliance and few of its variants was introduced by P. Kristiansen, S. T. Hedetniemi, and S. M. Hedetniemi in [10] and [9]. A defensive alliance in $G$ is a subset $S$ of $V$ such that $|N[x] \cap S| \geq |N[x] - S|$, for all $x \in S$. For each vertex $x \in S$, the vertices of $N[x] - S$ are termed as attackers of $x$ and those of $N[x] \cap S$ as defenders.

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of $x$. An alliance $S$ is a minimal (or critical) alliance if none of its proper subsets is an alliance of the same type. The minimum cardinality of a defensive alliance in $G$ is called the defensive alliance number of $G$ and is denoted by $a_d(G)$.

Alliances are formed to join forces if one or more of them are attacked. In any defensive alliance $S$ of $G$ each vertex $v \in S$ has at most one more neighbor in $V - S$ than it has in $S$. Further for any integer $k \in \{-\Delta(G), \ldots, \Delta(G)\}$, a nonempty subset $S \subset V$ is a defensive $k$-alliance in $G$ whenever $|N[x] \cap S| - |N[x] - S| \geq k$, for all $x \in S$. The minimum cardinality of a defensive $k$-alliance in $G$ is denoted by $a_k(G)$ and called the defensive $k$-alliance number of $G$. If $k$ is a positive integer, then every vertex in a defensive $k$-alliance has at least $k$ more defenders than its attackers. This generalization of defensive alliances was presented by Shafique and Dutton in [12, 13]. Till date, several other varieties of alliances have been introduced and studied. The related works can be found in [1, 3–5, 7–11].

In many practical situations like decision taken by board of directors of a company or no-confidence motion against the ruling government, 50% and above majority is required. Suppose a ruling party has 275 members in its alliance group in the house of 500 members. Even if one or two members leave the alliance, ruling party continues to win the majority. But if 25 people leave the alliance, then it has to yield to the opposite force. Therefore this ruling alliance has 24 surplus members in it. The situation like this may also appear in the board of directors of a company and other places. To deal with these situations, we introduce a new type of alliance called $k$-strong defensive alliances in graphs.

A defensive alliance $S$ of a nontrivial graph $G$ of order at least $k + 1$ is said to be a $k$-strong defensive alliance if $S$ remains a defensive alliance in $G$ by the omission of at most $k$ vertices from it. The minimum cardinality of a $k$-strong defensive alliance in $G$ is called the $k$-strong alliance number of $G$ and is denoted by $a^k(G)$. In particular, a 1-strong defensive alliance $S$ of $G$ is a defensive alliance in $G$ with the property that, for each vertex $v \in S$, $S - \{v\}$ is also a defensive alliance in $G$. The $k$-strong alliance number of $G$ is a natural generalization of alliance number, since $a_4(G) = a^0(G)$.

For the graph $G$ of Figure 1, $S_1 = \{x, y, z\}$ is a 2-strong defensive alliance, but not a defensive 2-alliance. Also the set $S = \{t, u, v, w\}$ is a defensive 3-alliance, but it is not a 2-strong defensive alliance in $G$. Further, it is easy to see that $a^1(G) = 2, a_1(G) = 3$; $a^2(G) = 3, a_2(G) = 4$; $a^3(G) = 7, a_3(G) = 4$.

The invariant $a^k(G)$ introduced in the paper is incomparable with $a_k(G)$ in general. In fact, $a_1(C_3) = 2 < 3 = a^1(C_3)$; $a_1(K_{1, 4}) = 3 > 2 = a^1(K_{1, 4})$; and $a_1(K_4) = 3 = a^1(K_4)$.

We recall the following theorems for immediate reference.

**Theorem 1 (P.Kristiansen, S.T.Hedetniemi, and S.M.Hedetniemi [10]).**

The subgraph induced by a minimal defensive alliance of a connected graph $G$ is connected.

**Theorem 2 (P.Kristiansen, S.T.Hedetniemi, and S.M.Hedetniemi [10]).**

For any graph $G$ of order $n \geq 2$, $a_d(G) \leq \lceil \frac{n}{2} \rceil$. 
If $S$ is any defensive alliance of $G$, then for each vertex $v \in S$, $S$ should contain at least $\left\lfloor \frac{\deg(v)}{2} \right\rfloor$ neighbors of $v$. Thus;

**Theorem 3.** If $S$ is any defensive alliance set of the graph $G$ and $v \in S$, then

$$|S| \geq 1 + \left\lfloor \frac{\deg(v)}{2} \right\rfloor = \left\lceil \frac{1 + \deg(v)}{2} \right\rceil \geq \left\lceil \frac{1 + \delta(G)}{2} \right\rceil$$

## 2. 1-Strong Alliances in Graphs

It is clear from the definition that 1-strong alliance number is defined for non-trivial graphs. For any graph $G$ of order 2, $a^1(G) = 2$. If $G$ is a disconnected graph with components $G_1, G_2, \ldots, G_k$ such that $\delta(G_i) \geq 2$ for every component $G_i$ except possible for one $i$, then $a^1(G) = \min\{a^1(G_i) : 1 \leq i \leq k\}$. But, if at least two components, say $G_i$ and $G_j$, contain pendent vertices, then a 2-element set $S'$ containing pendent vertices one each from $G_i$ and $G_j$ is a minimal 1-strong defensive alliance. Also, $S'$ may have lesser cardinality of every minimal 1-strong defensive of each component $G_i$, for $1 \leq i \leq k$. Therefore, here onwards we consider simple connected graphs of order at least 3.

The following is a direct consequence of the definition.

**Proposition 1.** For any positive integer $k$ and a graph $G$ of order at least $k + 1$, $a^k(G) \geq a^{k-1}(G) \geq a_d(G) \geq 1$.

We now see that $a^1(G)$ need not be equal to $a_d(G) + 1$. In fact, for the cycle $C_4$, on 4 vertices, $a_d(C_4) = 2$, but $a^1(C_4) = 4$. In [9], Hedetniemi and Kristiansen proved that the problem of finding the alliance number of a graph is NP-complete. Thus the problem of finding 1-strong alliance number of a graph is also NP-complete. For any graph $G(V, E)$ of order at least 3, the set $S = V - \{v\}$ is a defensive alliance. Thus the following proposition is trivial.

**Proposition 2.** For any graph $G$ on $n \ (\geq 3)$ vertices, $2 \leq a^1(G) \leq n$. 
Proposition 3. For any connected non-trivial graph $G$, $a^1(G) = 2$ if and only if $G$ has at least two pendant vertices.

Proof. Let $G$ be a graph with $a^1(G) = 2$. Then there exists a defensive alliance $S = \{u, v\}$, such that both the subsets $S_1 = \{u\}$ and $S_2 = \{v\}$ are defensive alliances of $G$. This is possible only if $\deg_G(u) = 1$ and $\deg_G(v) = 1$. Hence $u$ and $v$ are pendant vertices in $G$. The converse is trivial. \hfill $\Box$

The above Propositions 2 and 3 imply the following:

Corollary 1. For any graph $G$ of order at least 3 having at most one pendant vertex, $a^1(G) \geq 3$.

Corollary 2. For any tree $T$, $a^1(T) = 2$.

Proposition 4. For any connected graph $G$, $a^1(G) = 3$ if and only if $a^1(G) \neq 2$ and there exist three vertices each of degree at most three that induce the graph $K_3$.

Proof. Let $G$ be a graph with $a^1(G) = 3$. Let $S = \{u, v, w\}$ be a 1-strong defensive alliance in $G$. Then, by Proposition 3, $G$ can have at most one pendant vertex. Without loss of generality, we assume $\deg(v) \geq 2$ and $\deg(w) \geq 2$. Since $S$ is a 1-strong defensive alliance in $G$, $S_1 = S - \{u\} = \{v, w\}$, $S_2 = S - \{v\} = \{u, w\}$, and $S_3 = S - \{w\} = \{u, v\}$ are defensive alliances of $G$. So, by Theorem 1, $\langle S_1 \rangle$, $\langle S_2 \rangle$, and $\langle S_3 \rangle$ are connected and hence each is isomorphic to $K_2$. Thus $u, v, w$ are mutually adjacent. Further if $\deg(u) \geq 4$, then $|N[u] - (S - \{v\})| \geq 3 > 2 = |N[u] \cap (S - \{v\})|$, a contradiction to the fact that $S$ is a 1-strong defensive alliance. The other cases follow similarly.

Conversely, let $a^1(G) \neq 2$ and $G$ has three mutually adjacent vertices, say $u, v, w$, each of degree at most three. Then by Proposition 2, $a^1(G) \geq 3$ and $G$ has at most one pendant vertex. The set $S = \{u, v, w\}$ is a 1-strong defensive alliance in $G$. \hfill $\Box$

3. Bounds and Characterizations

In this section, we give bound for a 1-strong alliance number of the graph in terms of graphical parameters. Also, we characterize the graphs having 1-strong alliance number $n$. First we estimate the bounds for 1-strong alliance number of a graph in terms of degree of a vertex.

Lemma 1. For a graph $G$, $a^1(G) \geq \left\lceil \frac{\delta(G) + 1}{2} \right\rceil + 1$. 

Proof. Let $S$ be a 1-strong defensive alliance of minimum cardinality in $G$ and $v \in S$. Then $S' = S - \{v\}$ is defensive alliance of $G$. Therefore, by Theorem 3, we get $|S| = 1 + |S'| \geq 1 + \left\lceil \frac{\delta(G) + 1}{2} \right\rceil$.

Remark 1. For any 1-strong defensive alliance $S$ of a graph $G$ and $v \in S$, $|S| \geq \left\lceil \frac{\deg(v) + 1}{2} \right\rceil + 1$ or equivalently $\deg(v) \leq 2|S| - 3$.

Lemma 2 (P. Kristiansen, S.T. Hedetniemi, and S.M. Hedetniemi [9]). If $S$ is any defensive alliance in a graph $G$ with minimum cardinality, then $\langle S \rangle$ is connected.

We obtain a similar result for the 1-strong defensive alliance with minimum cardinality in a graph $G$. In view of Proposition 3, we see that if $S$ is a 1-strong defensive alliance with cardinality 2, then $\langle S \rangle$ is not connected. Thus we consider only the graphs having at most one pendant vertex for further discussions.

Proposition 5. If a graph $G$ of order $n \geq 3$ has a pendant vertex, then $a_1(G) \leq n - 1$.

Proof. If $G$ has more than one pendent vertex, then $a_1(G) = 2 \leq n - 1$. We now suppose that $G$ has exactly one pendent vertex $v$. Let $a_1(G) = n$. Then every $(n - 1)$-element subset of $V$ is not a 1-strong defensive alliance. Let $S = V - \{v\}$. Then $S$ contains some $x \in S$ such that $S - \{x\}$ is not a defensive alliance. This is possible only if $x$ is adjacent to some $y$ in $S$ such that $y$ is adjacent to $v$ and $\deg_G(y) = 2$. Let $S' = S - \{y\}$. Then clearly $S'$ is a defensive alliance. But then, as $S'$ cannot be a 1-strong defensive alliance (since $|S'| < a_1(G)$), there is a vertex $z \in S'$ with $\deg_G(z) = 2$ and that is adjacent to $x$ and $\deg_G(x) = 2$. Then $S'' = S' - \{x\}$ is a defensive alliance. Continuing this we end up with a set containing only one vertex $u$, which is a defensive alliance in $G$. Then this vertex $u$ should be a pendant vertex, a contradiction to the fact that $G$ has only one pendant vertex.

If $S_1$ and $S_2$ are any two disjoint defensive alliances in $G$, then $S_1 \cup S_2$ is also a defensive alliance in $G$. Moreover, if $S_1$ and $S_2$ are any two 1-strong defensive alliances in $G$, then for any $x \in S_1$ and $y \in S_2$ the subsets $S_1 - \{x\}$ and $S_2 - \{y\}$ are defensive alliances of $G$. So, for any vertex $z \in S_1 \cup S_2$, we observe $(S_1 \cup S_2) - \{z\} = (S_1 - \{z\}) \cup S_2$ if $z \in S_1$, otherwise $(S_1 \cup S_2) - \{z\} = S_1 \cup (S_2 - \{z\})$. Thus $(S_1 \cup S_2) - \{z\}$ is a defensive alliance in $G$. Thus we have proved the following:

Lemma 3. If $S_1$ and $S_2$ are any two disjoint 1-strong defensive alliances in $G$, then so is $S_1 \cup S_2$.

Now we prove the following result similar to Lemma 2.

Theorem 4. Let $G$ be a graph with $\delta(G) \geq 2$. If $S$ is any 1-strong defensive alliance with minimum cardinality in $G$, then the graph $\langle S \rangle$ is connected and has no pendant vertex.
Now to prove the induced subgraph \( \langle S \rangle \) is disconnected. Suppose to contrary that \( \langle S \rangle \) is connected. Without loss of generality we may assume \( \langle S \rangle \) has exactly two components say \( \langle S_1 \rangle \) and \( \langle S_2 \rangle \) (result follows by induction hypothesis if \( \langle S \rangle \) has more than two components). Then \( S_1 \cap S_2 = \emptyset \) and \( S_1 \cup S_2 = S \). Since \( S = S_1 \cup S_2 \) is a 1-strong defensive alliance in \( G \), we have for every \( x \in S \), \( S - \{x\} \) is a defensive alliance. Since \( S_1 \cap S_2 = \emptyset \), for every \( x \in S \), either \( x \in S_1 \) or \( x \in S_2 \). Thus both \( S_1 - \{x\} \) and \( S_2 - \{x\} \) are defensive alliance in \( G \). Hence both \( S_1 \) and \( S_2 \) are 1-strong defensive alliances in \( G \). But we have \( |S_1| < |S| \) and \( |S_2| < |S| \), a contradiction to the fact that \( S \) is a 1-strong defensive alliance in \( G \) with minimum cardinality. Therefore, \( \langle S \rangle \) is connected.

Now to prove the induced subgraph \( \langle S \rangle \) has no pendant vertex, we use method of contradiction. Suppose that the graph \( \langle S \rangle \) has a pendant vertex \( v \) adjacent to the vertex \( w \) in \( S \). Since \( v \in S \subseteq V(G) \) and \( \delta(G) \geq 2 \), the vertex \( v \) is not a pendant vertex of \( G \), therefore \( \deg(v) \geq 2 \) in \( G \). But we have \( N[v] \cap S = \{v, w\} \) and hence \( |N[v] \cap (S - \{w\})| = \{v\} \). So \( |N[v] \cap (S - \{w\})| = 1 \) and as \( \deg(v) \geq 2 \) in \( G \), \( |N[v] - (S - \{w\})| \) \( \geq 2 \). Therefore, \( |N[v] \cap (S - \{w\})| < |N[v] - (S - \{w\})| \), a contradiction to the fact that \( S \) is a 1-strong defensive alliance in \( G \).

Let \( G \) be a graph of order at least 3. If \( \delta(G) \geq 2 \), then every vertex of \( G \) lies on some cycle of \( G \). Hence by using Theorem 4, we state the following corollary.

**Corollary 3.** Let \( G \) be a graph with \( \delta(G) \geq 2 \). If \( S \) is any 1-strong defensive alliance in \( G \) with minimum cardinality, then every vertex of \( S \) lies on some cycle of \( \langle S \rangle \).

The *girth* of a graph \( G \), denoted by \( g(G) \) (or simply \( g \)), is the length of a shortest cycle (if any) in \( G \). If \( G \) is a connected graph with \( \delta(G) \geq 2 \), then \( G \) contains a cycle and hence the girth is at least 3. Thus, in general for any graph with at most one pendant vertex, \( S \) may include the pendant vertex or not, so by above discussions, the following corollary is straightforward.

**Corollary 4.** For any graph \( G \) with at most one pendant vertex, \( a^1(G) \geq g \), where \( g \) is the girth of \( G \).

**Theorem 5.** For a graph \( G \) having at most one pendant vertex, with \( \Delta(G) \leq 3 \) and girth \( g \), \( a^1(G) = g \).

**Proof.** Let \( G \) be a graph having at most one pendant vertex with \( \Delta(G) \leq 3 \) and girth \( g \). From Corollary 4, \( a^1(G) \leq g \). Let \( v_1, v_2, \ldots, v_g \) be the vertices of the cycle of length \( g \) in \( G \). Consider \( S = \{v_1, v_2, \ldots, v_g\} \). Then for each \( v_j \in S \), \( \deg_G(v_j) \leq 3 \) implies that \( |N[v_j] \cap S| = 3 \) and \( |N[v_j] - S| \leq 1 \). Hence \( S \) is a defensive alliance in \( G \), and for any \( v_k \in S \), the set \( S' = S - \{v_k\} \) is also a defensive alliance (since \( |N[v_k] \cap S| = 3, |N[v_k] - S| \leq 1 \) if \( v_k \) is not adjacent to \( v_j \) in \( \langle S \rangle \), and \( |N[v_k] \cap S| = 3 \text{ if } v_k = d \)).
2, |N[v_k] − S| ≤ 2 if v_k is adjacent to v_j in \langle S\rangle). Therefore, S is a 1-strong alliance in G and hence \( a^1(G) = |S| = g \).

\[ \text{Corollary 5.} \quad \text{For any cubic graph G with girth g, } a^1(G) = g. \]

One of the very famous cubic graphs is the Petersen graph, which is a graph of order 10, size 15, and girth 5. Therefore, for the Petersen graph, the 1-strong alliance number is 5. The generalized Petersen graph is denoted by \( GP(n, k) \) where \( n \geq 3 \) is a graph with vertex set \( \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\} \) and edge set \( \{u_iu_{i+1}, u_iv_i, v_{i+k} : i = 1, 2, \ldots, n\} \), where subscripts are taken modulo \( n \) and \( k < n/2 \). The vertices \( \{u_1, u_2, \ldots, u_n\} \) are called the inner polygon vertices and the vertices \( \{v_1, v_2, \ldots, v_n\} \) are called the outer polygon vertices. In particular, if \( k = 1 \), then the generalized Petersen graph \( GP(n, 1) \) which is nothing but the cartesian product \( C_n \square P_2 \). The graph \( GP(n, 1) \) is a cubic graph of girth 4. Therefore \( a^1(GP(n, 1)) = 4 \).

\[ \text{Theorem 6.} \quad \text{For integers } n \geq 5 \text{ and } 2 \leq k < \frac{n}{2}, \]

\[ a^1(GP(n, k)) = \begin{cases} \min \left\{ k + 3, \frac{n}{k} \right\}, & \text{if } k \text{ divides } n \\ \min \left\{ k + 3, \left\lfloor \frac{n}{k} \right\rfloor + 3 \right\}, & \text{otherwise} \end{cases} \]

\[ \text{Proof.} \quad \text{The graph } GP(n, k) \text{ is cubic and hence by Theorem 5, } a^1(GP(n, k)) = g, \]

the girth of the graph. Suppose \( k \) divides \( n \), then there will be only three types of chordless cycles in \( GP(n, k) \) of lengths \( n, k + 3 \), and \( \frac{n}{k} \). Since \( 2 \leq k < \frac{n}{2} \), we get \( n \geq k + 3 \) and hence girth of \( GP(n, k) \) is \( \min \left\{ k + 3, \frac{n}{k} \right\} \). Suppose if \( k \) does not divide \( n \), then there will be only three types of chordless cycles in \( GP(n, k) \) of lengths \( n, k + 3 \), and \( \left\lfloor \frac{n}{k} \right\rfloor + 3 \). Since \( n \geq k + 3 \), the girth of \( GP(n, k) \) must be \( \min \left\{ k + 3, \left\lfloor \frac{n}{k} \right\rfloor + 3 \right\} \).

\[ \text{Remark 2.} \quad \text{Let } G = (V, E) \text{ be any graph with } \delta(G) \geq 2. \text{ If } G \text{ has a cycle } C_k \text{ (} k \geq 3 \text{) such that degree of each vertex in } C_k \text{ is at most } 3, \text{ then similar to the proof of Theorem 5, we see that } S = V(C_k) \text{ is a 1-strong defensive alliance and hence } a^1(G) \leq k. \]

\[ \text{Theorem 7.} \quad \text{For any graph } G \text{ of order } n \geq 4 \text{ and } \delta(G) \geq 3, \]

\[ a^1(G) \leq n - \left\lfloor \frac{\delta(G) - 1}{2} \right\rfloor \]

\[ \text{Proof.} \quad \text{Let } G \text{ be a graph of order } n \geq 4 \text{ and } v \text{ be a vertex of smallest degree in } G. \text{ Let } v_1, v_2, \ldots, v_{\delta(G)} \text{ be the vertices which are adjacent to } v \text{ in } G. \text{ Consider the set } S = V - \{v, v_1, v_2, \ldots, v_{\left\lfloor \frac{\delta(G) - 1}{2} \right\rfloor - 1}\}. \text{ Let } x \text{ be any arbitrary vertex in } S \text{ and } \bar{S} = V - S. \text{ Then } |N[x] \cap \bar{S}| \leq \left\lfloor \frac{\delta(G) - 1}{2} \right\rfloor < \left\lfloor \frac{\delta(G)}{2} \right\rfloor \text{ and } |N[x] \cap S| = |N[x]| - |N[x] \cap \bar{S}| \geq (1 + \delta(G)) - \left\lfloor \frac{\delta(G) - 1}{2} \right\rfloor \geq 1 + \left\lfloor \frac{\delta(G)}{2} \right\rfloor. \text{ So } S \text{ is a defensive alliance in } G. \text{ Also, } S' = S - \{x\} \text{ is a defensive alliance in } G. \text{ In fact, for any } x' \in S', |N[x'] \cap \bar{S}'| \leq \left\lfloor \frac{\delta(G) - 1}{2} \right\rfloor + 1 \text{ and } |N[x'] \cap S'| = |N[x']| - |N[x'] \cap \bar{S}'| \geq (1 + \delta(G)) - \left( \left\lfloor \frac{\delta(G) - 1}{2} \right\rfloor + 1 \right) \geq \delta(G) - \left\lfloor \frac{\delta(G) - 1}{2} \right\rfloor \geq \frac{\delta(G)}{2} \geq |S| \geq |S'|. \]

\]
The upper bound in Theorem 7 is tight for the graph $G = P_3 + K_2$ (by Theorem 15). Now we characterize the graphs of order $n$ having 1-strong alliance number $n$. A nonseparable graph is connected, nontrivial, and has no cutvertex. A block of a graph is its maximal nonseparable subgraph. If a graph $G$ is nonseparable, then $G$ itself is a block. If we take the blocks of $G$ as the family $F$ of sets, then the intersection graph of $F$ is the block graph of $G$, denoted by $B(G)$. The blocks of $G$ correspond to the vertices of $B(G)$ and two of these vertices in $B(G)$ are adjacent whenever the corresponding blocks in $G$ contain a common cutvertex.

For an integer $n \geq 3$, let $\Gamma_n$ be the class of graphs such that $G \in \Gamma_n$ if and only if $G$ is a graph of order $n$ with every block of $G$ is a cycle and block graph of $G$ is a tree. In 1963, Frank Harary [6] had obtained a characterization of block graphs in the form of the following theorem:

**Theorem 8 (F. Harary [6]).** A graph $H$ is a block graph of some graph if and only if every block of $H$ is complete.

**Remark 3.** The block graph of a graph $G$ is a tree if and only if each cutvertex of $G$ lies on exactly two blocks of $G$.

**Theorem 9.** Let $G$ be a graph of order $n \geq 3$. Then $a^1(G) = n$ if and only if $G \in \Gamma_n$.

**Proof.** Let $G \in \Gamma_n$ and $C_{n_1}, C_{n_2}, \ldots, C_{n_k}$ be the blocks of $G$. We prove the result by induction on $k$, the number of blocks. For $k = 1$, $G \cong C_n$ and by Theorem 4, $a^1(G) = n$. For $k = 2$, let $V(C_{n_1}) = \{v_1, v_2, \ldots, v_{n_1}\}$ and $V(C_{n_2}) = \{u_1, u_2, \ldots, u_{n_2}\}$ such that $v_1 = u_1$ is the common cutvertex. Then $\deg(v_1) = \deg(u_1) = 4$ and $\deg(v_i) = \deg(u_j) = 2$ for all $i, j$ with $2 \leq i \leq n_1$ and $2 \leq j \leq n_2$. Let $S$ be a 1-strong defensive alliance in $G$. Then without loss of generality, we may assume $S \cap V(C_{n_1}) \neq \emptyset$. Now by Theorem 4, $v_i \in S$ for each $i$, $1 \leq i \leq n_1$. In particular, $v_1 = u_1 \in S$. Since $\deg(u_1) = 4$ and $S$ is a 1-strong defensive alliance in $G$, at least one of the vertices in $V(C_{n_2})$ which is adjacent to $u_1$ must be in $S$. Again by Theorem 4, $u_j \in S$ for each $j$, $1 \leq j \leq n_2$. Therefore, $S$ contains every vertex of $G$ and hence $a^1(G) = n$ in this case.

Assume that the result holds for any $k \geq 2$. Consider a graph $G \in \Gamma_n$ having $k + 1$ blocks $C_{n_1}, C_{n_2}, \ldots, C_{n_{k+1}}$. Let $G'$ be the graph obtained from $G$ by deleting vertices and edges which are only in $C_{n_{k+1}}$. Then by inductive assumption, $a^1(G') = n - n_{k+1} + 1$ and no proper subset of $V(G')$ is a 1-strong defensive alliance in $G'$, as well as in $G$. Also by Theorem 4, no proper subset of $V(C_{n_{k+1}})$ is a 1-strong defensive alliance in $G$. Let $S$ be a 1-strong defensive alliance in $G$. Then $S \cap V(G') \neq \emptyset$ or $S \cap V(C_{n_{k+1}}) \neq \emptyset$. If $S \cap V(C_{n_{k+1}}) \neq \emptyset$, then by Theorem 4, $V(C_{n_k}) \subseteq S$, which implies that a common cutvertex, say $w$, in $S$. If $S \cap V(G') \neq \emptyset$, then by inductive assumption, $a^1(G') = n - n_{k+1} + 1$ and hence $V(G') \subseteq S$. Again which implies that
\( w \in S \). Thus in either case the cut-vertex common to \( G' \) and \( C_{k+1} \) must be in \( S \), which implies \( S \) contains every vertex of \( G \). Hence \( a^1(G) = n \). Thus by principle of induction, the result holds for any graph \( G \in \Gamma_n \).

Conversely, let \( G = (V, E) \) be a graph of order \( n \geq 3 \) with \( a^1(G) = n \). Then by Proposition 5, \( G \) has no pendant vertex. Hence degree of every vertex in \( G \) is at least 2. If \( G \cong C_n \), then there is nothing to prove. Suppose \( G \not\cong C_n \), then \( G \) has a vertex of degree 3.

**Claim 1**: Every edge of \( G \) lies in some cycle of \( G \).

If possible, let \( e \) be an edge in \( G \) which is not in any cycle of \( G \). Then \( e \) is a bridge and \( G - e \) has exactly two components. Let \( G_1 \) and \( G_2 \) be the components of \( G - e \). Then both \( G_1 \) and \( G_2 \) are connected nontrivial graphs (because \( G \) has no pendant vertex). The set \( S = V(G_1) \) is 1-strong defensive alliance in \( G_1 \), as well as in \( G \). But \( |V(G_1)| \leq n - 2 \) implies \( a^1(G) \leq n - 2 \), which is a contradiction to the fact that \( a^1(G) = n \). Hence the claim 1.

**Claim 2** Every edge of \( G \) lies on exactly one cycle.

If possible, let us assume that there is an edge common to two cycles in \( G \). Since \( G \) has no pendant vertex, if every vertex of \( G \) is of degree at least 3, then by Theorem 7, \( a^1(G) \leq n - 1 \), again a contradiction to the fact that \( a^1(G) = n \). Thus there must be a vertex of degree 2 in \( G \). Let \( v \) be the vertex of degree 2 in \( G \) and \( w_1, w_2 \) be the vertices which are adjacent to \( v \) in \( G \). Then by the above Claim 1, it follows that \( w_1, v, \) and \( w_2 \) are on some cycle of \( G \). Thus there is a path in \( G \) between \( w_1 \) and \( w_2 \) not containing \( v \). Let \( P : w_1 = v_1 - v_2 - \cdots - v_l = w_2 \) be a \( w_1 - w_2 \) path in \( G \) not containing \( v \). Let \( v_j \) be the first vertex of degree more than 2 in \( P \) (while tracing the path from \( w_1 \) to \( w_2 \)) for some \( j, 1 \leq j \leq l \) (we note that such a vertex \( v_j \) exists in \( P \) because \( G \) is connected and \( G \not\cong C_n \)). Let \( v_k \) be the first vertex of degree more than 2 in \( P \) while tracing \( P \) from \( w_2 \) to \( w_1 \) \((v_k \) may be equal to \( v_j \)).

Let \( S = V - \{v, v_1, \ldots, v_{j-1}, v_{k+1}, v_{k+2}, \ldots, v_l\} \). Then we observe that for any vertex \( x \in S - \{v_j, v_k\} \), \( N[x] - S = \phi \) and \( |N[v_j] - S| = |N[v_k] - S| = 1 \). Since both \( v_j \) and \( v_k \) are of degree at least 3, it follows that the set \( S \) is a 1-strong defensive alliance in \( G \). But \( |S| \leq n - 1 \) implies \( a^1(G) \leq n - 1 \), again a contradiction to the fact that \( a^1(G) = n \). Therefore, every edge of \( G \) lies in exactly one cycle of \( G \). Hence the Claim 2.

**Claim 3**: Every cut vertex of \( G \) lies on exactly two blocks of \( G \).

If not, let us assume that \( v \) is a cutvertex of \( G \) common to the block \( B_1, B_2, \ldots, B_k \), where \( k \geq 3 \). Then by the Claim 1 and Claim 2, each block is a cycle, hence \( |V(B_i)| \geq 3 \) for each \( i, \ 1 \leq i \leq k \). So, the set \( S = V - (V(B_1) - \{v\}) \) is a 1-strong defensive alliance in \( G \) with \( |S| \leq n - 2 \), a contradiction to the fact that \( a^1(G) = n \). Hence the Claim 3.

Now by the above three claims and by Remark 3, it follows \( G \in \Gamma_n \). This completes the proof of the theorem. \( \square \)
4. 1-Strong Alliance Number of Standard Graphs

**Theorem 10.** For an integer \( n \geq 3 \), \( a^1(K_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1 \) and \( a^1(C_n) = n \).

*Proof.* In \( K_n \), any subset \( S \) containing \( \left\lfloor \frac{n}{2} \right\rfloor + 1 \) vertices of \( K_n \) is a 1-defensive alliance. Hence \( a^1(K_n) \leq |S| = \left\lfloor \frac{n}{2} \right\rfloor + 1 \). Therefore, by Theorem 1, \( a^1(K_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1 \). Further by Theorem 9, \( a^1(C_n) = n \). \( \square \)

**Theorem 11.** For integers \( m, n \geq 2 \), \( a^1(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 2 \).

*Proof.* Let \( V_1 \) and \( V_2 \) be the vertex partition of \( K_{m,n} \) with \( |V_1| = m \), \( |V_2| = n \) and every vertex in \( V_1 \) is adjacent to every vertex in \( V_2 \). Let \( S \) be a 1-strong defensive alliance in \( K_{m,n} \) with minimum cardinality. For any vertex \( v \in S \), we have either \( v \in V_1 \) or \( v \in V_2 \). Suppose \( v \in V_1 \) (in case of \( v \in V_2 \) we have a similar argument), then among the \( n \) vertices of \( V_2 \), adjacent to \( v \), at least \( \left\lfloor \frac{n}{2} \right\rfloor + 1 \) vertices must be in \( S \). Similarly, at least \( \left\lfloor \frac{m}{2} \right\rfloor + 1 \) vertices of \( V_1 \) must belong to \( S \). Therefore, \( a^1(K_{m,n}) \geq \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 2 \). The set \( S \) containing any \( \left\lfloor \frac{m}{2} \right\rfloor + 1 \) vertices of \( V_1 \) and any \( \left\lfloor \frac{n}{2} \right\rfloor + 1 \) vertices of \( V_2 \) is a 1-strong defensive alliance and hence \( a^1(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 2 \). \( \square \)

**Theorem 12.** For an integer \( n \geq 3 \), \( a^1(W_{1,n}) = \left\lfloor \frac{n+1}{2} \right\rfloor + 1 \).

*Proof.* Let \( v \) be the central vertex and \( v_1, v_2, \ldots, v_n \) be the rim vertices of the wheel \( W_{1,n} \). Let \( S \) be a 1-strong defensive alliance in \( W_{1,n} \). If \( v \in S \), then by Remark 1, we have \( a^1(W_{1,n}) \geq \left\lceil \frac{\deg(v)+1}{2} \right\rceil + 1 = \left\lfloor \frac{n+1}{2} \right\rfloor + 1 \). Else if \( v \notin S \), then clearly \( S \) contains a rim vertex, say \( u \). Let \( w_1 \) and \( w_2 \) be the rim vertices which are adjacent to \( u \). Since \( S \) is a defensive alliance and \( v \notin S \), \( w_1 \) or \( w_2 \) must be in \( S \). We claim that both \( w_1 \) and \( w_2 \) must be in \( S \). If not without loss of generality, we may assume that \( w_1 \in S \) and \( w_2 \notin S \). Then for the set \( S' = S - \{w_1\} \), clearly \( |N[u] \cap S'| = 1 \) and \( |N[u] - S'| = 3 \). Hence \( S' \) is not a defensive alliance. Which implies that \( S \) is not a 1-strong defensive alliance, a contradiction. Thus the claim follows. Continuing the same argument for neighboring rim vertices, we see that \( S \) contains all rim vertices. Thus \( |S| = n \) in this case. Therefore, \( |a^1(W_{1,n})| \geq \min \left\{ \left\lfloor \frac{n+1}{2} \right\rfloor + 1, n \right\} = \left\lfloor \frac{n+1}{2} \right\rfloor + 1 \).

On the other hand, it is easy to verify that the set \( S = \{v, v_1, v_2, \ldots, v_{\left\lfloor \frac{n+1}{2} \right\rfloor}\} \) is a 1-strong defensive alliance in \( W_{1,n} \). Therefore, \( a^1(W_{1,n}) = \left\lfloor \frac{n+1}{2} \right\rfloor + 1 \). \( \square \)

**Theorem 13.** For an integer \( n \geq 4 \), \( a^1(K_n - e) = \left\lfloor \frac{n}{2} \right\rfloor + 1 \).

*Proof.* For \( n = 4 \), the result follows by Proposition 4. For \( n \geq 5 \), let \( V(K_n - e) = \{v_1, v_2, \ldots, v_n\} \) and \( S \) be a 1-strong defensive alliance in \( K_n - e \). Without loss of generality, we assume that \( v_1 \) and \( v_2 \) are nonadjacent in \( K_n - e \). Then \( \deg(v_1) = \deg(v_2) = n - 2 \) and \( \deg(v_i) = n - 1 \) for \( 3 \leq i \leq n \). Since \( K_n - e \) has no pendant vertex, by Corollary 1, \( |S| \geq 3 \) and hence there exists a vertex \( v_k \in S \) for some
Let $G_1$ and $G_2$ be any two graphs having disjoint vertex set $V_1$ and $V_2$ and edge sets $E_1$ and $E_2$, respectively. Then the join of $G_1$ and $G_2$ is denoted by $G_1 + G_2$ and is the graph whose vertex set is $V_1 \cup V_2$ and consists of all the edges of $G_1$, $G_2$ and all the edges joining every vertex of $V_1$ with every vertex of $V_2$. In particular, $K_{m,n} = K_m + K_n$ and $W_{1,n} = C_n + K_1$. The graph $P_8 + K_3$ is shown in Figure 2.

Figure 2. The graph $P_8 + K_3$.

Theorem 14. For any positive integer $n$, $a^1(P_n + K_1) = \left\lceil \frac{n+1}{2} \right\rceil + 1$.

Proof. Let $G = P_n + K_1$. Let $v$ be the vertex of $K_1$ and $v_1, v_2, \ldots, v_n$ be the vertices of $P_n$ with $v_i$ adjacent to $v_{i+1}$, for $1 \leq i \leq n - 1$. Let $S$ be a 1-strong defensive alliance in $P_n + K_1$. For $n = 1, 2$, the result follows by Corollary 2 and Theorem 10. When $n = 3$, the result follows by Theorem 13.

Let $n \geq 4$ and $S$ be a 1-strong defensive alliance of $G$.

Claim: $v \in S$.

Let us suppose to contrary that $v \notin S$. Since $\delta(G) = 2$, by Lemma 1, we have $|S| \geq 3$ and hence $v_i \in S$ for some $i$, $2 \leq i \leq n - 1$. But then, $\{v_{i-1}, v_{i+1}\} \subseteq S$, otherwise $|N[v_i] \cap S'| = 1 < 3 = |N[v_i] - S'|$ where $S' = S - \{v_{i-1}, v_{i+1}\}$, a contradiction to the fact that $S$ is a 1-strong defensive alliance. Thus, $S = \{v_1, v_2, \ldots, v_n\}$. Now for the set $S'' = S - \{v_2\}$, we get $|N[v_1] \cap S''| = 1 < 2 = |N[v_1] - S''|$, again a contradiction. Hence the claim follows. Thus, by Remark 1, $a^1(G) \geq \left\lceil \frac{\deg(v)+1}{2} \right\rceil + 1 = \left\lceil \frac{n+1}{2} \right\rceil + 1$.

On the other hand, the set $S = \{v, v_1, v_2, \ldots, v_{\left\lceil \frac{n+1}{2} \right\rceil}\}$ is a 1-strong defensive alliance in $G$. Hence $a^1(P_n + K_1) = \left\lceil \frac{n+1}{2} \right\rceil + 1$. \qed
Theorem 15. For any positive integers \( n \) and \( m \),

\[
a^1(P_n + \overline{K}_m) = \begin{cases} 
\lceil \frac{n+1}{2} \rceil + 1 & \text{if } m = 1 \\
2 & \text{if } m \geq 2 \text{ and } n = 1 \\
\lceil \frac{m}{2} \rceil + 2 & \text{if } m \geq 2 \text{ and } n = 2 \\
\lceil \frac{n+1}{2} \rceil + \lceil \frac{m+1}{2} \rceil & \text{if } m \geq 2, n \geq 3 
\end{cases}
\]

Proof. Let \( G = P_n + \overline{K}_m \). The case \( m = 1 \) follows by Theorem 14 and the case \( m \geq 2, n = 1 \) follows by Corollary 2. If \( m = 2 \) and \( n = 2 \), then \( P_2 + \overline{K}_2 \cong K_4 - e \) and the result follows by Theorem 13. We now suppose that \( m \geq 2 \) and \( n \geq 2 \). Let \( u_1, u_2, \ldots, u_m \) be the vertices of \( \overline{K}_m \) and \( v_1, v_2, \ldots, v_n \) be the vertices of \( P_n \) with \( v_i \) adjacent to \( v_{i+1} \), for \( 1 \leq i \leq n-1 \). Let \( S \) be a 1-strong defensive alliance in \( G \). Then, by Theorem 4, \( v_i \in S \) for at least one \( i \), \( 1 \leq i \leq n \) (otherwise \( S \) is disconnected).

We now show that \( u_k \in S \) for some \( k, 1 \leq k \leq m \). In fact, if \( S \cap \{u_1, u_2, \ldots, u_m\} = \emptyset \), then certainly \( m \leq 3 \) (since \( \deg_G(v_i) \geq m + 2 \)) and \( v_j \in S \), where \( j = i + 1 \) if \( i < n \) or \( j = i - 1 \) if \( i = n \). Now, for the set \( S' = S - \{v_j\} \), we have \( |N[v_i] - S'| \geq 3 \) and \( |N[v_i] \cap S'| \leq 2 \), a contradiction to the fact that \( S \) is a 1-strong defensive alliance in \( G \). For this vertex \( u_k \in S \), \( \deg_G(u_k) = n \), \( N(u_k) = V(P_n) \), and hence by Remark 1, the set \( S \) should contain at least \( \lceil \frac{n+1}{2} \rceil \) vertices of \( P_n \).

Case 1: \( n = 2 \).

In this case, \( \{v_1, v_2\} \subseteq S \), \( \deg_G(v_1) = \deg_G(v_2) = m + 1 \) and hence by Remark 1, the set \( S \) should contain at least \( \lceil \frac{m}{2} \rceil + 1 \) vertices of \( G \), of which at least \( \lceil \frac{m}{2} \rceil \) vertices in \( \overline{K}_m \). Therefore, \( |S| = |V(P_n) \cap S| + |V(\overline{K}_m) \cap S| \geq 2 + \lceil \frac{m}{2} \rceil \).

On the other hand, let \( S = \{v_1, v_2, u_1, u_2, \ldots, u_{\lceil \frac{m}{2} \rceil}\} \), then \( |S| = 2 + \lceil \frac{m}{2} \rceil \) and for any vertex \( x \in S \), the subset \( S - \{x\} \) is a defensive alliance. Hence \( S \) is a 1-strong defensive alliance in \( G \). Therefore, \( a^1(P_n + \overline{K}_m) = 2 + \lceil \frac{m}{2} \rceil \).

Case 2: \( n \geq 3 \).

In this case, \( \deg_G(v_i) \geq m + 1 \) for all \( v_i \), \( 1 \leq i \leq n \). If \( \deg_G(v_i) = m + 1 \) for all \( v_i \in S \), then clearly \( S = \{v_1, v_n\} \) and \( n = 3 \) (since \( S \) has at least \( \lceil \frac{n+1}{2} \rceil \) vertices of \( P_n \)). The vertex \( v_2 \) is an attacker for \( v_1 \), therefore, similar than above, by Remark 1, \( S \) should contain at least \( \lceil \frac{m}{2} \rceil + 1 \) vertices of \( \overline{K}_m \) adjacent to \( v_1 \) (since \( N(v_1) \cap S \cap V(P_n) = \emptyset \)). Else if \( \deg_G(v_i) = m + 2 \) for at least one vertex \( v_i \in S \), then it is easy to see that \( S \) has a vertex \( v_j \) such that either \( v_{j-1} \notin S \) or \( v_{j+1} \notin S \) (\( v_j \) may be one of the end vertices). So by Remark 1, \( |S| \geq \lceil \frac{\deg(v_j)+1}{2} \rceil = \lceil \frac{m+1}{2} \rceil + 1 \). This implies that \( S \) should contain at least \( \lceil \frac{m+1}{2} \rceil \) neighbouring vertices of \( v_j \) in \( \overline{K}_m \) (since \( N(v_j) \cap V(P_n) = \emptyset \)).

Thus, in any case, \( |S| = |V(P_n) \cap S| + |V(\overline{K}_m) \cap S| \leq \lceil \frac{n+1}{2} \rceil + \lceil \frac{m+1}{2} \rceil \).

On the other hand, let \( S = \{u_1, u_2, \ldots, u_{\lceil \frac{m+1}{2} \rceil}, v_1, v_2, \ldots, v_{\lceil \frac{m+1}{2} \rceil}\} \), then \( |S| = \lceil \frac{n+1}{2} \rceil + \lceil \frac{m+1}{2} \rceil \) and for any vertex \( x \in S \), the subset \( S - \{x\} \) is a defensive alliance. Hence \( S \) is a 1-strong defensive alliance in \( G \). Therefore, \( a^1(P_n + \overline{K}_m) = \lceil \frac{n+1}{2} \rceil + \lceil \frac{m+1}{2} \rceil \).

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