Communications in Combinatorics and Optimization
Vol. 3 No. 2, 2018 pp.151-172
DOI: 10.22049/CCO.2018.26132.1079

CCO
Commun. Comb. Optim.

Some results on a supergraph of the comaximal ideal graph of a commutative ring

S. Visweswaran and J. Parejiya

Department of Mathematics, Saurashtra University, Rajkot, India 360 005
s_visweswaran2006@yahoo.co.in

Received: 22 January 2018; Accepted: 6 August 2018
Published Online: 8 August 2018

Communicated by ***Stephan Wagner***

Abstract: The rings considered in this article are commutative with identity which admit at least two maximal ideals. We denote the set of all maximal ideals of a ring R by $Max(R)$ and we denote the Jacobson radical of R by $J(R)$. Let R be a ring such that $|Max(R)| \geq 2$. Let $\mathbb{I}(R)$ denote the set of all proper ideals of R. In this article, we associate an undirected graph denoted by $\mathbb{INC}(R)$ with a subcollection of ideals of R whose vertex set is $\{I \in \mathbb{I}(R) | I \not\subseteq J(R)\}$ and two distinct vertices I_1, I_2 are adjacent in $\mathbb{INC}(R)$ if and only if $I_1 \not\subseteq I_2$ and $I_2 \not\subseteq I_1$ (that is, I_1 and I_2 are not comparable under the inclusion relation). The aim of this article is to investigate the interplay between the graph-theoretic properties of $\mathbb{INC}(R)$ and the ring-theoretic properties of R.

Keywords: Chained ring, diameter of a graph, bipartite graph, split graph, complemented graph

AMS Subject classification: 13A15, 05C25

1. Introduction

The rings considered in this article are commutative with identity which admit at least two maximal ideals. Let R be a ring. We denote the set of all maximal ideals of R by $Max(R)$ and we denote the Jacobson radical of R by $J(R)$. As in [8], we denote the collection of all proper ideals of R by $\mathbb{I}(R)$. For a set A, we denote the cardinality of A by the notation $|A|$. This article is motivated by the interesting theorems proved by M. Ye and T. Wu in [17]. Let R be a ring with $|Max(R)| \geq 2$. Inspired by the research work done on the comaximal graph of a ring in [10, 12–16] and the research work done on the annihilating-ideal graph of a ring in [8, 9]. M. Ye and T. Wu in [17], introduced and investigated an undirected graph associated with R whose vertex set equals $\{I \in \mathbb{I}(R)|I \not\subseteq J(R)\}$ and distinct vertices I_1 and I_2 are joined by an edge if and only if $I_1 + I_2 = R$. M. Ye and T. Wu called the graph introduced and studied by them in [17] as the *comaximal ideal graph* of R and denoted it by the notation $\mathscr{C}(R)$. For a ring R, we denote the set of all units of R by $U(R)$ and the set of all nonunits of R by $NU(R)$.

This article is also motivated by the inspiring theorems proved on cozero-divisor graph of a commutative ring by M. Afkhami and K. Khashyarmanesh in [1–3]. Let R be a ring. Recall from [1] that the *cozero-divisor graph* of R denoted by $\Gamma'(R)$ is an undirected graph whose vertex set is $NU(R)\backslash\{0\}$ and distinct vertices a, b are joined by an edge if and only if $a \notin Rb$ and $b \notin Ra$. That is, a, b are joined by an edge if and only if Ra and Rb are not comparable under the inclusion relation.

Let R be a ring with $|Max(R)| \geq 2$. Motivated by the research work on the comaximal ideal graph of a commutative ring in [17] and by the research work on the cozero-divisor graph of a ring in [1–3], in this article, we introduce an undirected graph structure associated with R, denoted by $\mathbb{INC}(R)$ whose vertex set equals $\{I \in \mathbb{I}(R)|I \not\subseteq J(R)\}$ and distinct vertices I_1, I_2 are joined by an edge if and only if $I_1 \not\subseteq I_2$ and $I_2 \not\subseteq I_1$. That is, I_1 and I_2 are joined by an edge if and only if I_1 and I_2 are not comparable under the inclusion relation. The aim of this article is to study the interplay between the graph-theoretic properties of $\mathbb{INC}(R)$ and the ring-theoretic properties of R.

The graphs considered in this article are undirected and simple. Let $G = (V, E)$ be a graph. We denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. If H is a subgraph of G, then we say that G is a *supergraph* of H. A subgraph H of G is said to be a *spanning subgraph* of G if $V(H) = V(G)$. Let R be a ring with $|Max(R)| \geq 2$. Observe that $V(\mathscr{C}(R)) = V(\mathbb{INC}(R)) = \{I \in \mathbb{I}(R) \mid I \not\subseteq J(R)\}$. Let $I_1, I_2 \in V(\mathscr{C}(R))$ be such that $I_1 \neq I_2$. If there is an edge of $\mathscr{C}(R)$ joining I_1 and I_2, then $I_1 + I_2 = R$. Since $I_1, I_2 \in \mathbb{I}(R)$, it follows that $I_1 \not\subseteq I_2$ and $I_2 \not\subseteq I_1$. Hence, there is an edge of $\mathbb{INC}(R)$ joining I_1 and I_2. Therefore, $\mathscr{C}(R)$ is a spanning subgraph of $\mathbb{INC}(R)$. In this article, we study the influence of some graph parameters of $\mathbb{INC}(R)$ on the structure of the ring R.

It is useful to recall the following definitions from graph theory before we describe the results that are proved in this article on $\mathbb{INC}(R)$, where R is a ring with $|Max(R)| \geq 2$. Let $G = (V, E)$ be a graph. Let $a, b \in V$ with $a \neq b$. Recall that the *distance* between a and b, denoted by $d(a, b)$ is defined as the length of a shortest path in in G if there exists such a path in G; otherwise, we define $d(a, b) = \infty$. We define $d(a, a) = 0$. The *diameter* of G, denoted by $diam(G)$ is defined as $diam(G) = sup\{d(a, b)|a, b \in V\}$ [6]. A graph $G = (V, E)$ is said to be *connected* if for any distinct $a, b \in V$, there exists a path in G between a and b. Let $G = (V, E)$ be a connected graph. Let $a \in V$. Then the *eccentricity* of a, denoted by $e(a)$ is defined as $e(a) = sup\{d(a, b)|b \in V\}$. The *radius* of G, denoted by $r(G)$ is defined as $r(G) = min\{e(a)|a \in V\}$ [6]. Let $G = (V, E)$ be a graph. Recall from [[6], page 159] that the *girth* of G, denoted by $girth(G)$ is defined as the length of a shortest cycle in G if G admits at least one cycle. If G does not admit any cycle, then we set $girth(G) = \infty$. A simple graph $G = (V, E)$ is said to be *complete* if every pair of distinct vertices of G are adjacent

in G [[6], Definition 1.1.11]. Recall from [[6], Definition 1.2.2] that a *clique* of G is a complete subgraph of G. A subset S of G is said to be an *independent set* if no two members of S are adjacent in G.
A graph $G = (V, E)$ is said to be *bipartite* if V can be partitioned into nonempty subsets V_1 and V_2 such that each edge of G has one end in V_1 and the other in V_2. A bipartite graph with vertex partition V_1 and V_2 is said to be *complete* if each element of V_1 is adjacent to every element of V_2. A complete bipartite graph with vertex partition V_1 and V_2 is called a *star* if either $|V_1| = 1$ or $|V_2| = 1$ [[6], Definition 1.1.12]. Let I be an ideal of a ring R. As in [15], we denote $\{\mathfrak{m} \in Max(R) | \mathfrak{m} \supseteq I\}$ by $M(I)$. The Krull dimension of a ring R is simply referred to as the dimension of R and is denoted by the notation $dim R$. A ring which has only one maximal ideal is referred to as a *quasilocal* ring. A ring which has only a finite number of maximal ideals is referred to as a *semiquasilocal* ring. A Noetherian quasilocal (respectively, semiquasilocal) ring is referred to as a *local* (respectively, *semilocal*) ring. Recall from [[11], page 184] that a ring R is said to be a *chained ring* if the set of ideals of R is linearly ordered by inclusion. Whenever a set A is a subset of a set B and $A \neq B$, then we denote it symbolically by the notation $A \subset B$. For $n \in \mathbb{N}$ with $n \geq 2$, we denote the ring of integers modulo n by $\mathbb{Z}_n$.
Let R be a ring such that $|Max(R)| \geq 2$. In Section 2 of this article, some basic properties of $\mathbb{INC}(R)$ are proved. It is proved in Lemma 1 that $\mathbb{INC}(R)$ is connected and $diam(\mathbb{INC}(R)) \leq 2$. If $|Max(R)| \geq 3$, then it is shown that $diam(\mathbb{INC}(R)) = r(\mathbb{INC}(R)) = 2$ (see, Lemmas 1 and 2). Let R be a ring with $Max(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$. We denote $\{I \in V(\mathbb{INC}(R)) \mid M(I) = \{\mathfrak{m}_i\}\}$ by V_i for each $i \in \{1, 2\}$. It is proved in Lemma 3 that $diam(\mathbb{INC}(R)) = r(\mathbb{INC}(R)) = 2$ if $|V_i| \geq 2$ for each $i \in \{1, 2\}$. It is shown in Proposition 1 that $\mathbb{INC}(R)$ is complete if and only if $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$. It is observed in Lemma 4 that if $|Max(R)| \geq 3$, then $\mathscr{C}(R) \neq \mathbb{INC}(R)$. The rest of Section 2 is devoted to characterizing rings R with $|Max(R)| = 2$ such that $\mathscr{C}(R) = \mathbb{INC}(R)$. If $|Max(R)| = 2$, then we know from (3) $\Rightarrow$ (1) of [[17], Theorem 4.5] that $\mathscr{C}(R)$ is a complete bipartite graph. Hence, we focus on characterizing rings R such that $\mathbb{INC}(R)$ is a bipartite graph. If $\mathbb{INC}(R)$ is bipartite, then it is verified in Proposition 2 that $\mathbb{INC}(R) = \mathscr{C}(R)$ is a complete bipartite graph. Let $R = R_1 \times R_2$, where $(R_i, \mathfrak{m}_i)$ is a quasilocal ring for each $i \in \{1, 2\}$. It is shown in Proposition 3 that $\mathbb{INC}(R)$ is a bipartite graph if and only if R_i is a chained ring for each $i \in \{1, 2\}$. For a ring R with $|Max(R)| \geq 2$, it is proved in Proposition 4 that $\mathbb{INC}(R)$ is a star graph if and only if $R \cong R_1 \times R_2$ as rings, where R_i is a chained ring for each $i \in \{1, 2\}$ with R_i is a field for at least one $i \in \{1, 2\}$. If $dim R = 0$, then it is shown in Proposition 5 that $\mathbb{INC}(R)$ is a bipartite graph if and only if $R \cong R_1 \times R_2$ as rings, where R_i is a zero-dimensional chained ring for each $i \in \{1, 2\}$. In Example 1 (respectively, in Example 2), an example is provided to illustrate that $(i) \Rightarrow (ii)$ of Proposition 5 can fail to hold if the hypothesis $dim R = 0$ is omitted in Proposition 5. For a ring R with $|Max(R)| \geq 2$, it is verified in Proposition 6 that $girth(\mathbb{INC}(R)) \in \{3, 4, \infty\}$. Moreover, it is proved in Proposition 6 that $girth(\mathbb{INC}(R)) = \infty$ if and only if $R \cong R_1 \times R_2$ as rings, where R_1 is a field and R_2 is a chained ring.

Let $G = (V, E)$ be a graph. Recall that G is a *split graph* if V is the disjoint union of two nonempty subsets K and S such that the subgraph of G induced on K is complete and S is an independent set of G. In [12], M. I. Jinnah and S.C. Mathew classified rings R such that the comaximal graph of R is a split graph. In Section 3 of this article, we try to characterize rings R with $|Max(R)| \geq 2$ such that $\mathbb{INC}(R)$ is a split graph. It is proved in Proposition 7 that if $\mathbb{INC}(R)$ is a split graph, then $|Max(R)| = 2$. Let R be a ring such that $|Max(R)| = 2$. It is shown in Theorem 2 that $\mathbb{INC}(R)$ is a split graph if and only if $R \cong R_1 \times R_2$ as rings, where R_i is a quasilocal ring for each $i \in \{1, 2\}$ with R_i is a field for at least one $i \in \{1, 2\}$ and if R_i is not a field for some $i \in \{1, 2\}$, then either R_i is a chained ring or $\mathbb{I}(R_i) = W_1 \cup W_2$ satisfying the property that $|W_k| \geq 2$ for each $k \in \{1, 2\}$ such that W_1 is a chain under the inclusion relation and no two distinct members of W_2 are comparable under the inclusion relation. Some examples are given in Example 4 to illustrate Theorem 2.
Let $G = (V, E)$ be a graph. Recall from [4] that two distinct vertices u, v of G are said to be *orthogonal*, written $u \perp v$ if u and v are adjacent in G and there is no vertex of G which is adjacent to both u and v in G; that is, the edge $u - v$ is not the edge of any triangle in G. Let $u \in V$. A vertex v of G is said to be a *complement* of u if $u \perp v$ [4]. Moreover, recall from [4] that G is *complemented* if each vertex of G admits a complement in G. Furthermore, G is said to be *uniquely complemented* if G is complemented and whenever the vertices u, v, w of G are such that $u \perp v$ and $u \perp w$, then a vertex x of G is adjacent to v in G if and only if x is adjacent to w in G. Let R be a ring which is not an integral domain. The authors of [4] determined in Section 3 of [4] rings R such that $\Gamma(R)$ is complemented or uniquely complemented, where $\Gamma(R)$ is the zero-divisor graph of R. Let R be a ring with $|Max(R)| \geq 2$. In [[15], Proposition 3.11], it was shown that the subgraph of the comaximal graph of R induced on $NU(R) \backslash J(R)$ is complemented if and only if $dim(\frac{R}{J(R)}) = 0$. In Section 4 of this article, we try to characterize rings R with $|Max(R)| \geq 2$ such that $\mathbb{INC}(R)$ is complemented. It is proved in Lemma 10 that if $\mathbb{INC}(R)$ is complemented, then $|Max(R)| \leq 3$. Let R be a ring with $|Max(R)| = 2$. Let $Max(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$. Let $V_1 = \{I \in \mathbb{I}(R) \mid M(I) = \{\mathfrak{m}_1\}\}$ and let $V_2 = \{J \in \mathbb{I}(R) \mid M(J) = \{\mathfrak{m}_2\}\}$. If $|V_i| = 1$ for each $i \in \{1, 2\}$, (it is noted in Remark 3 that this can happen if and only if $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$) then it is clear that $\mathbb{INC}(R)$ is a complete graph on two vertices and hence, $\mathbb{INC}(R)$ is complemented. Suppose that $|V_1| = 1$ and $|V_2| \geq 2$. Then it is shown in Proposition 9 that $\mathbb{INC}(R)$ is complemented if and only if $R \cong R_1 \times R_2$ as rings, where R_1 is a chained ring which is not a field and R_2 is a field. Suppose that $|V_i| \geq 2$ for each $i \in \{1, 2\}$. It is proved in Proposition 10 that $\mathbb{INC}(R)$ is complemented if and only if $\mathbb{INC}(R)$ is a complete bipartite graph. We are not able to characterize rings R with $|Max(R)| = 2$ such that $|V_i| \geq 2$ for each $i \in \{1, 2\}$ and $\mathbb{INC}(R)$ is complemented. However, if $dim R = 0$, it is shown in Proposition 11 that R has the above mentioned properties if and only if $R \cong R_1 \times R_2$ as rings, where R_i is a chained ring which is not a field for each $i \in \{1, 2\}$. In Remark 4, an example is mentioned to illustrate that the hypothesis $dim R = 0$ cannot be omitted in Proposition 11. Let R be a ring such that $|Max(R)| = 3$. It is proved in Theorem 3 that $\mathbb{INC}(R)$ is complemented if and only if $R \cong F_1 \times F_2 \times F_3$

as rings, where F_i is a field for each $i \in \{1, 2, 3\}$.

2. Basic properties of $\mathbb{INC}(R)$

In this section we investigate some basic properties of $\mathbb{INC}(R)$.

Lemma 1. Let R be a ring such that $|Max(R)| \geq 2$. Then $\mathbb{INC}(R)$ is connected and $diam(\mathbb{INC}(R)) \leq 2$. If $|Max(R)| \geq 3$, then $diam(\mathbb{INC}(R)) = 2$.

Proof. Let $I_1, I_2 \in V(\mathbb{INC}(R))$ be such that $I_1 \neq I_2$. Suppose that I_1 and I_2 are not adjacent in $\mathbb{INC}(R)$. Then either $I_1 \subset I_2$ or $I_2 \subset I_1$. Without loss of generality, we can assume that $I_1 \subset I_2$. Since $I_1 \in V(\mathbb{INC}(R))$, there exists $\mathfrak{m} \in Max(R)$ such that $I_1 \not\subseteq \mathfrak{m}$. As $I_1 \subset I_2$, it follows that $I_2 \not\subseteq \mathfrak{m}$. Thus $I_i + \mathfrak{m} = R$ for each $i \in \{1, 2\}$ and so, $I_1 - \mathfrak{m} - I_2$ is a path in $\mathscr{C}(R)$ and hence, it is a path in $\mathbb{INC}(R)$. This proves that $\mathbb{INC}(R)$ is connected and $diam(\mathbb{INC}(R)) \leq 2$.
Suppose that $|Max(R)| \geq 3$. Let $\{\mathfrak{m}_i \mid i \in \{1, 2, 3\}\} \subseteq Max(R)$. Note that $\mathfrak{m}_1, \mathfrak{m}_1 \cap \mathfrak{m}_2 \in V(\mathbb{INC}(R))$ and as $\mathfrak{m}_1 \cap \mathfrak{m}_2 \subset \mathfrak{m}_1$, it follows that $\mathfrak{m}_1$ and $\mathfrak{m}_1 \cap \mathfrak{m}_2$ are not adjacent in $\mathbb{INC}(R)$. Hence, we obtain that $diam(\mathbb{INC}(R)) \geq 2$ and so, $diam(\mathbb{INC}(R)) = 2$. □

Lemma 2. Let R be a ring such that $|Max(R)| \geq 3$. Then $r(\mathbb{INC}(R)) = 2$.

Proof. We know from Lemma 1 that $\mathbb{INC}(R)$ is connected and $diam(\mathbb{INC}(R)) = 2$. Let $I \in V(\mathbb{INC}(R))$. Then $I \subseteq \mathfrak{m}$ for some $\mathfrak{m} \in Max(R)$. We consider the following cases.

Case 1. $I = \mathfrak{m}$.
Let $\mathfrak{m}' \in Max(R)$ be such that $\mathfrak{m}' \neq \mathfrak{m}$. As $|Max(R)| \geq 3$, $\mathfrak{m} \cap \mathfrak{m}' \in V(\mathbb{INC}(R))$. From $\mathfrak{m} \cap \mathfrak{m}' \subset \mathfrak{m}$, we obtain that $\mathfrak{m}$ and $\mathfrak{m} \cap \mathfrak{m}'$ are not adjacent in $\mathbb{INC}(R)$. Hence, $d(\mathfrak{m}, \mathfrak{m} \cap \mathfrak{m}') \geq 2$ in $\mathbb{INC}(R)$. This shows that $e(\mathfrak{m}) \geq 2$ in $\mathbb{INC}(R)$.

Case 2. $I \subset \mathfrak{m}$.
Now, I and $\mathfrak{m}$ are not adjacent in $\mathbb{INC}(R)$ and so, $d(I, \mathfrak{m}) \geq 2$ in $\mathbb{INC}(R)$. Hence, $e(I) \geq 2$ in $\mathbb{INC}(R)$.
This proves that $e(I) \geq 2$ in $\mathbb{INC}(R)$ for any $I \in V(\mathbb{INC}(R))$ and from $diam(\mathbb{INC}(R)) = 2$, it follows that $e(I) = 2$ for each $I \in \mathbb{INC}(R)$. Therefore, $r(\mathbb{INC}(R)) = 2$. □

Lemma 3. Let R be a ring such that $|Max(R)| = 2$. Let $\{\mathfrak{m}_1, \mathfrak{m}_2\}$ denote the set of all maximal ideals of R. Let $V_1 = \{I \in V(\mathbb{INC}(R)) \mid M(I) = \{\mathfrak{m}_1\}\}$ and let $V_2 = \{J \in V(\mathbb{INC}(R)) \mid M(J) = \{\mathfrak{m}_2\}\}$. If $|V_i| \geq 2$ for each $i \in \{1, 2\}$, then $diam(\mathbb{INC}(R)) = r(\mathbb{INC}(R)) = 2$.

Proof. We know from Lemma 1 that $\mathbb{INC}(R)$ is connected and $diam(\mathbb{INC}(R)) \leq 2$. Suppose that $|V_i| \geq 2$ for each $i \in \{1, 2\}$. Note that $\mathfrak{m}_i \in V_i$ for each $i \in \{1, 2\}$.

Observe that there exist $I_1 \in V_1$ such that $I_1 \neq \mathfrak{m}_1$ and $I_2 \in V_2$ such that $I_2 \neq \mathfrak{m}_2$. It is clear that $\mathfrak{m}_1$ and any $I \in V_1 \backslash \{\mathfrak{m}_1\}$ are not adjacent in $\mathbb{INC}(R)$. Therefore, $e(A) \geq 2$ in $\mathbb{INC}(R)$ for any $A \in V_1$. If J is any element of V_2 with $J \neq \mathfrak{m}_2$, then J and $\mathfrak{m}_2$ are not adjacent in $\mathbb{INC}(R)$. Hence, $e(B) \geq 2$ in $\mathbb{INC}(R)$ for any $B \in V_2$. As $V(\mathbb{INC}(R)) = V_1 \cup V_2$ and $diam(\mathbb{INC}(R)) \leq 2$, we obtain that $e(A) = 2$ for any $A \in V(\mathbb{INC}(R))$. Therefore, we obtain that $diam(\mathbb{INC}(R)) = r(\mathbb{INC}(R)) = 2$. □

Let R be a ring such that $|Max(R)| \geq 2$. In Proposition 1, we characterize such rings R whose $\mathbb{INC}$ graph is complete.

Proposition 1. Let R be a ring such that $|Max(R)| \geq 2$. The following statements are equivalent:
(i) $\mathbb{INC}(R)$ is complete.
(ii) $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$.

Proof. $(i) \Rightarrow (ii)$ We are assuming that $\mathbb{INC}(R)$ is complete. Hence, we obtain from Lemma 1 that $|Max(R)| = 2$. Let $\{\mathfrak{m}_1, \mathfrak{m}_2\}$ denote the set of all maximal ideals of R. Let V_1, V_2 be as in the statement of Lemma 3. Note that $\mathfrak{m}_i \in V_i$ for each $i \in \{1, 2\}$. Let $i \in \{1, 2\}$. Let $I \in V_i$. We claim that $I = \mathfrak{m}_i$. Suppose that $I \neq \mathfrak{m}_i$. Then $I \subset \mathfrak{m}_i$ and so, I and $\mathfrak{m}_i$ are not adjacent in $\mathbb{INC}(R)$. This is in contradiction to the assumption that $\mathbb{INC}(R)$ is complete. Therefore, $I = \mathfrak{m}_i$ and this shows that $|V_i| = 1$ for each $i \in \{1, 2\}$. Note that $V_i = \{\mathfrak{m}_i\}$ for each $i \in \{1, 2\}$. Let $a \in \mathfrak{m}_1 \backslash \mathfrak{m}_2$. Then $Ra \in V_1$ and so, $Ra = \mathfrak{m}_1$. Let $b \in \mathfrak{m}_2 \backslash \mathfrak{m}_1$. Then $Rb \in V_2$ and so, $\mathfrak{m}_2 = Rb$. Now, for each $i \in \{1, 2\}$, $\mathfrak{m}_i^2 \in V_i$ and hence, $\mathfrak{m}_i = \mathfrak{m}_i^2$. Therefore, $\mathfrak{m}_1 = Ra = Ra^2$ and $\mathfrak{m}_2 = Rb = Rb^2$. Thus we get that $Rab = Ra^2b^2$. This implies that $ab = ra^2b^2$ for some $r \in R$ and hence, $ab(1 - rab) = 0$. Since $ab \in \mathfrak{m}_1 \cap \mathfrak{m}_2 = J(R)$, we obtain that $1 - rab \in U(R)$ and so, $ab = 0$. Hence, $\mathfrak{m}_1\mathfrak{m}_2 = Rab = (0)$. As $\mathfrak{m}_1 + \mathfrak{m}_2 = R$, it follows from [5, Proposition 1.10(i)] that $\mathfrak{m}_1 \cap \mathfrak{m}_2 = \mathfrak{m}_1\mathfrak{m}_2$. Therefore, $\mathfrak{m}_1 \cap \mathfrak{m}_2 = (0)$. Now, it follows from the Chinese remainder theorem [[5], Proposition 1.10(ii) and (iii)] that $R \cong \frac{R}{\mathfrak{m}_1} \times \frac{R}{\mathfrak{m}_2}$, as desired.
$(ii) \Rightarrow (i)$ We are assuming that $R \cong F_1 \times F_2$ as rings, where F_1 and F_2 are fields. Let us denote the ring $F_1 \times F_2$ by T. Note that $V(\mathbb{INC}(T)) = \{(0) \times F_2, F_1 \times (0)\}$ and $(0) \times F_2$ and $F_1 \times (0)$ are adjacent in $\mathscr{C}(T)$ and so, they are adjacent in $\mathbb{INC}(T)$. This proves that $\mathbb{INC}(T)$ is complete and therefore, we obtain that $\mathbb{INC}(R)$ is complete. □

Let R be a ring such that $|Max(R)| \geq 2$. We want to determine such rings R which satisfies $\mathscr{C}(R) = \mathbb{INC}(R)$.

Lemma 4. Let R be a ring such that $|Max(R)| \geq 3$. Then $\mathscr{C}(R) \neq \mathbb{INC}(R)$.

Proof. Let $\{\mathfrak{m}_i \mid i \in \{1, 2, 3\}\} \subseteq Max(R)$. Let $I_1 = \mathfrak{m}_1 \cap \mathfrak{m}_2$ and let $I_2 = \mathfrak{m}_1 \cap \mathfrak{m}_3$. Note that $I_1, I_2 \in V(\mathscr{C}(R)) = V(\mathbb{INC}(R))$, $I_1 \neq I_2$, and as $I_1 + I_2 \subseteq \mathfrak{m}_1$, it follows that I_1 and I_2 are not adjacent in $\mathscr{C}(R)$. It is clear that $I_1 \not\subseteq I_2$ and $I_2 \not\subseteq I_1$. Hence, I_1 and I_2 are adjacent in $\mathbb{INC}(R)$. This proves that $\mathscr{C}(R) \neq \mathbb{INC}(R)$. □

Remark 1. Let R be a ring such that $|Max(R)| \geq 2$. If $\mathscr{C}(R) = \mathbb{INC}(R)$, then we know from Lemma 4 that $|Max(R)| = 2$. If $|Max(R)| = 2$, then it follows from (3) $\Rightarrow$ (2) of [[17], Theorem 4.5] that $\mathscr{C}(R)$ is a bipartite graph. Thus if $\mathscr{C}(R) = \mathbb{INC}(R)$, then $\mathbb{INC}(R)$ is necessarily a bipartite graph.

Motivated by the results proved on $\mathscr{C}(R)$ in Section 4 of [17], we next try to characterize rings R with $|Max(R)| \geq 2$ such that $\mathbb{INC}(R)$ is bipartite.

Lemma 5. Let R be a ring such that $|Max(R)| \geq 2$. If $\mathbb{INC}(R)$ is bipartite, then $|Max(R)| = 2$.

Proof. It is already noted in the introduction that $\mathscr{C}(R)$ is a spanning subgraph of $\mathbb{INC}(R)$. Thus if $\mathbb{INC}(R)$ is bipartite, then $\mathscr{C}(R)$ is also a bipartite graph. Hence, we obtain from (2) $\Rightarrow$ (3) of [[17], Theorem 4.5] that $|Max(R)| = 2$. □

We use Observation 1 in the proof of Proposition 2. As this observation is easy to prove, we omit its proof.

Observation 1. *Let H be a spanning subgraph of a graph $G = (V, E)$. Suppose that H is a complete bipartite graph. If G is a bipartite graph, then $H = G$.*

Proposition 2. Let R be a ring with $|Max(R)| \geq 2$. The following statements are equivalent:
(i) $\mathbb{INC}(R)$ is a bipartite graph.
(ii) $\mathbb{INC}(R) = \mathscr{C}(R)$ is a complete bipartite graph.
(iii) $\mathbb{INC}(R)$ is a complete bipartite graph.

Proof. $(i) \Rightarrow (ii)$ Assume that $\mathbb{INC}(R)$ is a bipartite graph. Then we know from Lemma 5 that $|Max(R)| = 2$. Hence, we obtain from (3) $\Rightarrow$ (1) of [[17], Theorem 4.5] that $\mathscr{C}(R)$ is a complete bipartite graph. As $\mathscr{C}(R)$ is a spanning subgraph of $\mathbb{INC}(R)$, we obtain from Observation 1 that $\mathbb{INC}(R) = \mathscr{C}(R)$. Therefore, $\mathbb{INC}(R) = \mathscr{C}(R)$ is a complete bipartite graph.

$(ii) \Rightarrow (iii)$ This is clear.

$(iii) \Rightarrow (i)$ This is clear. □

Proposition 3. Let $(R_i, \mathfrak{m}_i)$ be a quasilocal ring for each $i \in \{1, 2\}$ and let $R = R_1 \times R_2$. The following statements are equivalent:
(i) $\mathbb{INC}(R)$ is a bipartite graph.
(ii) $\mathbb{INC}(R) = \mathscr{C}(R)$ is a complete bipartite graph.
(iii) R_i is a chained ring for each $i \in \{1, 2\}$.

Proof. Note that $|Max(R)| = 2$ and $Max(R) = \{\mathfrak{M}_1 = \mathfrak{m}_1 \times R_2, \mathfrak{M}_2 = R_1 \times \mathfrak{m}_2\}$. Observe that $\mathscr{C}(R)$ is a complete bipartite graph with vertex partition V_1

and V_2, where $V_1 = \{I \times R_2 \mid I \text{ is a proper ideal of } R_1\}$ and $V_2 = \{R_1 \times J \mid J \text{ is a proper ideal of } R_2\}$.

$(i) \Rightarrow (ii)$ This follows from $(i) \Rightarrow (ii)$ of Proposition 2.

$(ii) \Rightarrow (iii)$ Let I_1, I_2 be distinct proper ideals of R_1. Now, $A_i = I_i \times R_2 \in V_1$ for each $i \in \{1, 2\}$ and $A_1 \neq A_2$. Hence, A_1 and A_2 are not adjacent in $\mathbb{INC}(R)$. Therefore, either $A_1 = I_1 \times R_2 \subset A_2 = I_2 \times R_2$ or $A_2 \subset A_1$. This implies that either $I_1 \subset I_2$ or $I_2 \subset I_1$. This shows that R_1 is a chained ring. Similarly, using the fact that no two distinct elements of V_2 are adjacent in $\mathbb{INC}(R)$, it can be shown that R_2 is a chained ring.

$(iii) \Rightarrow (i)$ Assume that R_i is a chained ring for each $i \in \{1, 2\}$. Note that if A_1, A_2 are any two distinct members of V_1, then $A_i = I_i \times R_2$ for some proper ideal I_i of R_1 for each $i \in \{1, 2\}$. It is clear that $I_1 \neq I_2$. Since R_1 is a chained ring, it follows that either $I_1 \subset I_2$ or $I_2 \subset I_1$ and so, either $A_1 \subset A_2$ or $A_2 \subset A_1$. Hence, A_1 and A_2 are not adjacent in $\mathbb{INC}(R)$. Similarly, using the hypothesis that R_2 is a chained ring, it can be shown that no distinct members of V_2 are adjacent in $\mathbb{INC}(R)$. Let $A \in V_1$ and $B \in V_2$. Then A and B are adjacent in $\mathscr{C}(R)$ and so, they are adjacent in $\mathbb{INC}(R)$. Hence, it follows that $\mathbb{INC}(R)$ is a complete bipartite graph with vertex partition V_1 and V_2. □

Corollary 1. Let $(R_i, \mathfrak{m}_i)$ be a quasilocal ring for each $i \in \{1, 2\}$. Let $R = R_1 \times R_2$. Then the following statements are equivalent:
(i) $\mathbb{INC}(R)$ is a star graph.
(ii) $\mathscr{C}(R) = \mathbb{INC}(R)$ is a star graph.
(iii) R_i is a chained ring for each $i \in \{1, 2\}$ with R_i is a field for at least one $i \in \{1, 2\}$.

Proof. Note that $\mathscr{C}(R)$ is a complete bipartite graph with vertex partition V_1 and V_2, where $V_1 = \{I \times R_2 \mid I \text{ is a proper ideal of } R_1\}$ and $V_2 = \{R_1 \times J \mid J \text{ is a proper ideal of } R_2\}$.

$(i) \Rightarrow (ii)$ Since any star graph is a bipartite graph, it follows from $(i) \Rightarrow (ii)$ of Proposition 3 that $\mathscr{C}(R) = \mathbb{INC}(R)$ is a star graph.

$(ii) \Rightarrow (iii)$ We know from $(ii) \Rightarrow (iii)$ of Proposition 3 that R_i is a chained ring for each $i \in \{1, 2\}$. Since $\mathscr{C}(R)$ is a star graph, it follows that $|V_i| = 1$ for at least one $i \in \{1, 2\}$. Without loss of generality, we can assume that $|V_1| = 1$. Then we obtain that (0) is the only proper ideal of R_1 and so, R_1 is a field.

$(iii) \Rightarrow (i)$ It is shown in $(iii) \Rightarrow (i)$ of Proposition 3 that $\mathbb{INC}(R)$ is a complete bipartite graph with vertex partition V_1 and V_2. Without loss of generality, we can assume that R_1 is a field. Hence, $|V_1| = 1$ and so, $\mathbb{INC}(R)$ is a star graph. □

Let R be a ring such that $|Max(R)| \geq 2$. In Proposition 4, we characterize such rings R whose $\mathbb{INC}$ graph is a star graph.

Proposition 4. Let R be a ring such that $|Max(R)| \geq 2$. Then the following statements are equivalent:

(i) $\mathbb{INC}(R)$ is a star graph.
(ii) $R \cong R_1 \times R_2$ as rings, where R_i is a chained ring for each $i \in \{1, 2\}$ with R_i is a field for at least one $i \in \{1, 2\}$.

Proof. $(i) \Rightarrow (ii)$ We know from the proof of $(i) \Rightarrow (ii)$ of Proposition 2 that $|Max(R)| = 2$ and $\mathscr{C}(R) = \mathbb{INC}(R)$. Let $\{\mathfrak{m}_1, \mathfrak{m}_2\}$ denote the set of all maximal ideals of R and let V_1, V_2 be as in the statement of Lemma 3. As $\mathscr{C}(R)$ is a star graph with vertex partition V_1 and V_2, we can assume without loss of generality that $|V_1| = 1$. Therefore, $V_1 = \{\mathfrak{m}_1\}$. Let $a \in \mathfrak{m}_1 \backslash \mathfrak{m}_2$. Then $Ra, Ra^2 \in V_1$ and so, $\mathfrak{m}_1 = Ra = Ra^2$. Let $r \in R$ be such that $a = ra^2$. Then $e = ra$ is a nontrivial idempotent element of R. Hence, the mapping $f : R \to Re \times R(1 - e)$ defined by $f(x) = (xe, x(1 - e))$ is an isomorphism of rings. Let us denote the ring Re by R_1 and $R(1 - e)$ by R_2. Since $|Max(R)| = 2$, it follows that R_1 and R_2 are quasilocal rings. Let us denote the ring $R_1 \times R_2$ by T. As $\mathbb{INC}(T)$ is a star graph, it follows from $(i) \Rightarrow (iii)$ of Corollary 1 that R_i is a chained ring for each $i \in \{1, 2\}$ with R_i is a field for at least one $i \in \{1, 2\}$.

$(ii) \Rightarrow (i)$ Let us denote the ring $R_1 \times R_2$ by T. We know from $(iii) \Rightarrow (i)$ of Corollary 1 that $\mathbb{INC}(T)$ is a star graph. It follows from $R \cong T$ as rings that $\mathbb{INC}(R)$ is a star graph. □

Let R be a ring and let $\mathfrak{m} \in Max(R)$. Let $f : R \to R_{\mathfrak{m}}$ denote the ring homomorphism given by $f(r) = \frac{r}{1}$. For any ideal I of R, $f^{-1}(I_{\mathfrak{m}})$ is called the *saturation of* I *with respect to the multiplicatively closed set* $R\backslash\mathfrak{m}$ and is denoted by the notation $S_{\mathfrak{m}}(I)$. It is well-known that for any ideal I of R, $I = \cap_{\mathfrak{m} \in Max(R)} S_{\mathfrak{m}}(I)$. Let R be a ring with $|Max(R)| \geq 2$. Suppose that $dim R = 0$. In Proposition 5, we characterize such rings R whose $\mathbb{INC}$ graph is a bipartite graph.

Remark 2. Let R be a semiquasilocal ring with $|Max(R)| = n \geq 2$. Suppose that $dim R = 0$. Then $R \cong R_1 \times R_2 \times \cdots \times R_n$ as rings, where $(R_i, \mathfrak{n}_i)$ is a quasilocal ring for each $i \in \{1, 2, \ldots, n\}$.

Proof. Let $\{\mathfrak{m}_i \mid i \in \{1, 2, \ldots, n\}\}$ denote the set of all maximal ideals of R. Let $i \in \{1, 2, \ldots, n\}$. Since $dim R = 0$, it follows that $\sqrt{(0)_{\mathfrak{m}_i}} = (\mathfrak{m}_i)_{\mathfrak{m}_i}$ and so, $\sqrt{S_{\mathfrak{m}_i}((0))} = \mathfrak{m}_i$. Hence, we obtain from [[5], Proposition 4.2] that $S_{\mathfrak{m}_i}((0))$ is a $\mathfrak{m}_i$-primary ideal of R. Let us denote the ideal $S_{\mathfrak{m}_i}((0))$ by $\mathfrak{q}_i$ for each $i \in \{1, 2, \ldots, n\}$. Observe that $(0) = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_n$. From $\mathfrak{m}_i + \mathfrak{m}_j = R$ for all distinct $i, j \in \{1, 2, \ldots, n\}$, we obtain from [[5], Proposition 1.16] that $\mathfrak{q}_i + \mathfrak{q}_j = R$. It now follows from the Chinese remainder theorem [[5], Proposition 1.10(ii) and (iii)] that $R \cong \frac{R}{\mathfrak{q}_1} \times \frac{R}{\mathfrak{q}_2} \times \cdots \times \frac{R}{\mathfrak{q}_n}$. For each i with $1 \leq i \leq n$, let us denote the ring $\frac{R}{\mathfrak{q}_i}$ by R_i. Note that R_i is quasilocal with $\mathfrak{n}_i = \frac{\mathfrak{m}_i}{\mathfrak{q}_i}$ as its unique maximal ideal and $R \cong R_1 \times R_2 \times \cdots \times R_n$ as rings. □

Proposition 5. Let R be a ring such that $|Max(R)| \geq 2$ and let $dim R = 0$. Then the following statements are equivalent:
(i) $\mathbb{INC}(R)$ is a bipartite graph.

(ii) $R \cong R_1 \times R_2$ as rings, where $(R_i, \mathfrak{n}_i)$ is a chained ring with $dim R_i = 0$ for each $i \in \{1, 2\}$.

Proof. $(i) \Rightarrow (ii)$ Assume that $\mathbb{INC}(R)$ is a bipartite graph. Then we know from Lemma 5 that $|Max(R)| = 2$. Since $dim R = 0$, we know from Remark 2 that $R \cong R_1 \times R_2$ as rings, where $(R_i, \mathfrak{n}_i)$ is a quasilocal ring for each $i \in \{1, 2\}$. Let us denote the ring $R_1 \times R_2$ by T. Since $R \cong T$ as rings, we get that $\mathbb{INC}(T)$ is a bipartite graph. Hence, we obtain from $(i) \Rightarrow (iii)$ of Proposition 3 that R_i is a chained ring for each $i \in \{1, 2\}$. Since $dim R = 0$, it is clear that $dim R_i = 0$ for each $i \in \{1, 2\}$.
$(ii) \Rightarrow (i)$ This follows from $(iii) \Rightarrow (i)$ of Proposition 3. □

We provide an example in Example 1 to illustrate that $(i) \Rightarrow (ii)$ of Proposition 5 can fail to hold if the hypothesis $dim R = 0$ is omitted.

Lemma 6. Let R be a principal ideal domain with $|Max(R)| \geq 2$. The following statements are equivalent:
(i) $\mathscr{C}(R) = \mathbb{INC}(R)$.
(ii) $|Max(R)| = 2$.

Proof. $(i) \Rightarrow (ii)$ Assume that $\mathscr{C}(R) = \mathbb{INC}(R)$. Then we obtain from Lemma 4 that $|Max(R)| = 2$. (For this part of the proof, we do not need the assumption that R is a principal ideal domain.)
$(ii) \Rightarrow (i)$ Assume that R is a principal ideal domain with $|Max(R)| = 2$. Let $\{\mathfrak{m}_1 = Rp, \mathfrak{m}_2 = Rq\}$ denote the set of all maximal ideals of R. It is already noted in the introduction that for any ring T with $|Max(T)| \geq 2$, $\mathscr{C}(T)$ is a spanning subgraph of $\mathbb{INC}(T)$. Observe that $V(\mathbb{INC}(R)) = V_1 \cup V_2$, where $V_1 = \{I \in \mathbb{I}(R) \mid I \subseteq Rp$ but $I \nsubseteq Rq\}$ and $V_2 = \{J \in \mathbb{I}(R) \mid J \subseteq Rq$ but $J \nsubseteq Rp\}$. Let $I_1, I_2 \in V(\mathbb{INC}(R))$ be such that I_1 and I_2 are adjacent in $\mathbb{INC}(R)$. We assert that $I_1 + I_2 = R$. Suppose that $I_1 + I_2 \neq R$. Then either $I_1 + I_2 \subseteq Rp$ or $I_1 + I_2 \subseteq Rq$. Without loss of generality, we can assume that $I_1 + I_2 \subseteq Rp$. Note that $I_i \in V_1$ for each $i \in \{1, 2\}$. Hence, $I_1 = Rp^n$ and $I_2 = Rp^m$ for some $n, m \in \mathbb{N}$. Therefore, I_1 and I_2 are comparable under the inclusion relation and so, I_1 and I_2 are not adjacent in $\mathbb{INC}(R)$. This is in contradiction to the assumption that I_1 and I_2 are adjacent in $\mathbb{INC}(R)$. Therefore, $I_1 + I_2 = R$ and so, I_1 and I_2 are adjacent in $\mathscr{C}(R)$. This proves that $\mathscr{C}(R) = \mathbb{INC}(R)$. □

Example 1 mentioned below was mentioned in [[17], Example 4.10] to illustrate that [[17], Proposition 4.7] can fail to hold if the hypothesis R satisfies d.c.c on principal ideals is omitted.

Example 1. Let p, q be distinct prime numbers. Let $R = S^{-1}\mathbb{Z}$, where $S = \mathbb{Z} \backslash (p\mathbb{Z} \cup q\mathbb{Z})$. Then $\mathbb{INC}(R)$ is a complete bipartite graph but the statement (ii) of Proposition 5 does not hold.

Proof. Note that R is a principal ideal domain with $|Max(R)| = 2$ and $\{Rp, Rq\}$ is the set of all maximal ideals of R. Hence, we obtain from $(ii) \Rightarrow (i)$ of Lemma 6 that $\mathscr{C}(R) = \mathbb{INC}(R)$. We know from $(3) \Rightarrow (1)$ of [[17], Theorem 4.5] that $\mathscr{C}(R)$ is a complete bipartite graph. Therefore, $\mathbb{INC}(R)$ is a complete bipartite graph. As R is an integral domain, R has no nontrivial idempotent element. Hence, the statement (ii) of Proposition 5 does not hold. □

In Example 2, we provide another example to illustrate that $(i) \Rightarrow (ii)$ of Proposition 5 can fail to hold if the hypothesis $dim R = 0$ is omitted.

Example 2. Let $V = \mathbb{Q}[[X]]$ be the power series ring in one variable X over $\mathbb{Q}$. Let us denote VX by $\mathfrak{m}$. Let $T = R + \mathfrak{m}$, where R is as in Example 1. Then $\mathbb{INC}(T)$ is a complete bipartite graph but the statement (ii) of Proposition 5 does not hold.

Proof. Observe that $V = \mathbb{Q} + \mathfrak{m}$ is a discrete valuation ring. We know from [[7], Theorem 2.1(c)] that each ideal of T compares with $\mathfrak{m}$ under inclusion. As $Max(R) = \{Rp, Rq\}$, it follows from [[7], Theorem 2.1(d)] that $Max(T) = \{\mathfrak{m}_1 = Rp + \mathfrak{m}, \mathfrak{m}_2 = Rq + \mathfrak{m}\}$. Let $V_1 = \{I \in \mathbb{I}(T) \mid M(I) = \{\mathfrak{m}_1\}\}$ and let $V_2 = \{J \in \mathbb{I}(T) \mid M(J) = \{\mathfrak{m}_2\}\}$. Observe that $V(\mathbb{INC}(T)) = V_1 \cup V_2$. Let $I_1, I_2 \in V_1$ be such that $I_1 \neq I_2$. Let $i \in \{1, 2\}$. It is not hard to verify that $I_i = A_i + \mathfrak{m}$ for some $A_i \in \mathbb{I}(R) \backslash \{(0)\}$ such that $M(A_i) = \{Rp\}$. It is clear that $A_1 \neq A_2$. Observe that there exist distinct $n, m \in \mathbb{N}$ such that $I_1 = Rp^n + \mathfrak{m}$ and $I_2 = Rp^m + \mathfrak{m}$. Hence, I_1 and I_2 are comparable under the inclusion relation and so, they are not adjacent in $\mathbb{INC}(T)$. Similarly, if J_1, J_2 are any two distinct members of V_2, then $J_i = B_i + \mathfrak{m}$ for some distinct $B_1, B_2 \in \mathbb{I}(R) \backslash \{(0)\}$ such that $M(B_i) = \{Rq\}$ for each $i \in \{1, 2\}$. Hence, there exist distinct $k, t \in \mathbb{N}$ such that $J_1 = Rq^k + \mathfrak{m}$ and $J_2 = Rq^t + \mathfrak{m}$. Therefore, it follows that J_1 and J_2 are comparable under the inclusion relation and so, they are not adjacent in $\mathbb{INC}(T)$. If $I \in V_1$ and $J \in V_2$, then $I + J = T$ and so, they are adjacent in $\mathbb{INC}(T)$. This shows that $\mathbb{INC}(T)$ is a complete bipartite graph. We know from [[7], Theorem 2.1(f)] that $dim T = dim V + dim R = 1 + 1 = 2$. Indeed, it follows from [[7], Theorem 2.1 $(c), (d), (e)$] and the fact that (0) and $\mathfrak{m}$ are the only prime ideals of V that $\{(0), \mathfrak{m}, Rp + \mathfrak{m}, Rq + \mathfrak{m}\}$ is the set of all prime ideals of T. Hence, $(0) \subset \mathfrak{m}$, $(0) \subset \mathfrak{m} \subset Rp + \mathfrak{m}$, and $(0) \subset \mathfrak{m} \subset Rq + \mathfrak{m}$ are the only chains of prime ideals of T of positive length and so, $dim T = 2$. Since T is an integral domain, we obtain that T has no nontrivial idempotent. Hence, the statement (ii) of Proposition 5 does not hold. □

In Proposition 6, we determine $girth(\mathbb{INC}(R))$, where R is a ring with $|Max(R)| \geq 2$ and moreover, we characterize such rings R which satisfies $girth(\mathbb{INC}(R)) = \infty$.

Proposition 6. Let R be a ring with $|Max(R)| \geq 2$. Then $girth(\mathbb{INC}(R)) \in \{3, 4, \infty\}$. Moreover, $girth(\mathbb{INC}(R)) = \infty$ if and only if $R \cong R_1 \times R_2$ as rings, where R_1 is a field and R_2 is a chained ring.

Proof. Suppose that $|Max(R)| \geq 3$. Let $\{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3\} \subseteq Max(R)$. Note that $\mathfrak{m}_1 - \mathfrak{m}_2 - \mathfrak{m}_3 - \mathfrak{m}_1$ is a cycle of length three in $\mathscr{C}(R)$ and hence, a cycle of length three in $\mathbb{INC}(R)$. Therefore, $girth(\mathbb{INC}(R)) = 3$.
Suppose that $|Max(R)| = 2$. Let $\{\mathfrak{m}_1, \mathfrak{m}_2\}$ denote the set of all maximal ideals of R. Let V_1, V_2 be as in the statement of Lemma 3. We consider the following cases.

Case 1. $|V_i| \geq 2$ for each $i \in \{1, 2\}$.
Let $I \in V_1 \backslash \{\mathfrak{m}_1\}$ and let $J \in V_2 \backslash \{\mathfrak{m}_2\}$. Observe that $I + \mathfrak{m}_2 = I + J = R$ and so, $\mathfrak{m}_1 - \mathfrak{m}_2 - I - J - \mathfrak{m}_1$ is a cycle of length four in $\mathscr{C}(R)$ and hence, a cycle of length four in $\mathbb{INC}(R)$. Therefore, $girth(\mathbb{INC}(R)) \leq 4$.

Case 2. $|V_i| = 1$ for at least one $i \in \{1, 2\}$.
In such a case, it follows as in the proof of $(i) \Rightarrow (ii)$ of Proposition 4 that there exist quasilocal rings R_1 and R_2 such that at least one between R_1 and R_2 is a field and $R \cong R_1 \times R_2$ as rings. Without loss of generality, we can assume that R_1 is a field. Let us denote the ring $R_1 \times R_2$ by T. Note that $\mathbb{INC}(T)$ contains a cycle if and only if there are at least two distinct nonzero proper ideals J_1 and J_2 of R_2 such that J_1 and J_2 are not comparable under the inclusion relation. Hence, $(0) \times R_2 - R_1 \times J_1 - R_1 \times J_2 - (0) \times R_2$ is a cycle of length three in $\mathbb{INC}(T)$. From $R \cong T$ as rings, it follows that $girth(\mathbb{INC}(R)) = girth(\mathbb{INC}(T)) = 3$. Observe that $\mathbb{INC}(T)$ (equivalently, $\mathbb{INC}(R)$) does not contain any cycle if and only if the set of ideals of R_2 is linearly ordered by inclusion, that is, R_2 is a chained ring.
It is clear from the above discussion that $girth(\mathbb{INC}(R)) \in \{3, 4, \infty\}$ and $girth(\mathbb{INC}(R)) = \infty$ if and only if $R \cong R_1 \times R_2$ as rings, where R_1 is a field and R_2 is a chained ring. Now, it is clear that *girth* of any star graph equals ∞. It follows from $(ii) \Rightarrow (i)$ of Proposition 4 that if $girth(\mathbb{INC}(R)) = \infty$, then $\mathbb{INC}(R)$ is a star graph. □

In Example 3, we provide some examples to illustrate Proposition 6.

Example 3. (i) Let $T = \mathbb{Z}_2[X, Y]$ be the polynomial ring in two variables X, Y over $\mathbb{Z}_2$. Let $\mathfrak{m} = TX + TY$. Let $R = F \times \frac{T}{\mathfrak{m}^2}$, where F is a field. Then $girth(\mathbb{INC}(R)) = 3$.
(ii) Let p, q be distinct prime numbers and let $R = S^{-1}\mathbb{Z}$, where $S = \mathbb{Z} \backslash (p\mathbb{Z} \cup q\mathbb{Z})$. Then $girth(\mathbb{INC}(R)) = 4$.
(iii) Let p be a prime number and let $R = F \times \mathbb{Z}_{p\mathbb{Z}}$, where F is a field. Then $girth(\mathbb{INC}(R)) = \infty$.

Proof. (i) Note that $\frac{T}{\mathfrak{m}^2}$ is a local ring with $\frac{\mathfrak{m}}{\mathfrak{m}^2}$ as its unique maximal ideal. It is clear that $|Max(R)| = 2$ and $Max(R) = \{(0) \times \frac{T}{\mathfrak{m}^2}, F \times \frac{\mathfrak{m}}{\mathfrak{m}^2}\}$. It is convenient to denote $\frac{T}{\mathfrak{m}^2}$ by T_1, $X + \mathfrak{m}^2$ by x, and $Y + \mathfrak{m}^2$ by y. Note that $(0) \times T_1 - F \times T_1 x - F \times T_1 y - (0) \times T_1$ is a cycle of length three in $\mathbb{INC}(R)$ and so, $girth(\mathbb{INC}(R)) = 3$.
(ii) We know from Example 1 that $\mathbb{INC}(R)$ is a complete bipartite graph with vertex partition V_1 and V_2, where $V_1 = \{Rp^n \mid n \in \mathbb{N}\}$ and $V_2 = \{Rq^n \mid n \in \mathbb{N}\}$. As $|V_i| \geq 2$ for each $i \in \{1, 2\}$, we obtain that $girth(\mathbb{INC}(R)) = 4$.

(iii) We know from [[5], Example (1), page 94] that $\mathbb{Z}_{p\mathbb{Z}}$ is a discrete valuation ring and so, it is a chained ring. It follows from the moreover part of Proposition 6 that $girth(\mathbb{INC}(R)) = \infty$. □

3. When is $\mathbb{INC}(R)$ a split graph?

The aim of this section is to characterize rings R with $|Max(R)| \geq 2$ such that $\mathbb{INC}(R)$ is a split graph. Let R be a ring such that $|Max(R)| \geq 2$. Note that $\mathbb{INC}(R)$ is a split graph if and only if there exist nonempty subsets K, S of $V(\mathbb{INC}(R))$ such that $V(\mathbb{INC}(R)) = K \cup S$, $K \cap S = \emptyset$, satisfying the property that the subgraph of $\mathbb{INC}(R)$ induced on K is a clique and S is an independent set of $\mathbb{INC}(R)$. Throughout this section, whenever we consider rings R with $\mathbb{INC}(R)$ is a split graph, we use K and S with the above mentioned properties.
Let R be a ring with $|Max(R)| \geq 2$. In Proposition 7, we determine a necessary condition on $|Max(R)|$ in order that $\mathbb{INC}(R)$ is a split graph. In Theorem 2, we characterize such rings R whose $\mathbb{INC}$ graph is a split graph.

Lemma 7. Let R be a ring with $|Max(R)| \geq 3$. If $\mathbb{INC}(R)$ is a split graph with $V(\mathbb{INC}(R)) = K \cup S$, then $Max(R) = K$.

Proof. As distinct maximal ideals of R are not comparable under the inclusion relation, it follows that distinct maximal ideals of R are adjacent in $\mathbb{INC}(R)$. Since S is an independent set of $\mathbb{INC}(R)$, we obtain that $|S \cap Max(R)| \leq 1$. By hypothesis, $|Max(R)| \geq 3$. Hence, there exist distinct $\mathfrak{m}_1, \mathfrak{m}_2 \in Max(R)$ such that $\mathfrak{m}_i \in K$ for each $i \in \{1, 2\}$. It follows from $|Max(R)| \geq 3$ that $\mathfrak{m}_1 \cap \mathfrak{m}_2 \in V(\mathbb{INC}(R)) = K \cup S$. Since $\mathfrak{m}_1 \cap \mathfrak{m}_2 \subset \mathfrak{m}_1$, we get that $\mathfrak{m}_1$ and $\mathfrak{m}_1 \cap \mathfrak{m}_2$ are not adjacent in $\mathbb{INC}(R)$. As $\mathfrak{m}_1 \in K$, it follows that $\mathfrak{m}_1 \cap \mathfrak{m}_2 \in S$. Let $\mathfrak{m} \in Max(R)\backslash\{\mathfrak{m}_1, \mathfrak{m}_2\}$. It is clear that $\mathfrak{m} \not\subseteq \mathfrak{m}_1 \cap \mathfrak{m}_2$ and from [[5], Proposition 1.11(ii)], it follows that $\mathfrak{m}_1 \cap \mathfrak{m}_2 \not\subseteq \mathfrak{m}$. This shows that $\mathfrak{m}$ and $\mathfrak{m}_1 \cap \mathfrak{m}_2$ are adjacent in $\mathbb{INC}(R)$. Since $\mathfrak{m}_1 \cap \mathfrak{m}_2 \in S$, we obtain that $\mathfrak{m} \in K$. This proves that $Max(R) \subseteq K$. Let $I \in K$. Note that there exists $\mathfrak{m} \in Max(R)$ such that $I \subseteq \mathfrak{m}$. If $I \neq \mathfrak{m}$, then I and $\mathfrak{m}$ are not adjacent in $\mathbb{INC}(R)$. This is in contradiction to the fact that the subgraph of $\mathbb{INC}(R)$ induced on K is complete. Therefore, $I = \mathfrak{m} \in Max(R)$ and so, we obtain that $Max(R) = K$. □

Proposition 7. Let R be a ring such that $|Max(R)| \geq 2$. If $\mathbb{INC}(R)$ is a split graph, then $|Max(R)| = 2$.

Proof. Let $V(\mathbb{INC}(R)) = K \cup S$. Suppose that $|Max(R)| \geq 3$. Then we know from Lemma 7 that $Max(R) = K$. Let $\{\mathfrak{m}_i \mid i \in \{1, 2, 3\}\} \subseteq Max(R)$. Now, $\mathfrak{m}_i \in K$ for each $i \in \{1, 2, 3\}$. Let us denote $\mathfrak{m}_1 \cap \mathfrak{m}_2$ by A and $\mathfrak{m}_2 \cap \mathfrak{m}_3$ by B. From the assumption that $|Max(R)| \geq 3$, it is clear that $A, B \in V(\mathbb{INC}(R))$. As $A \subset \mathfrak{m}_1$, it follows that A and $\mathfrak{m}_1$ are not adjacent in $\mathbb{INC}(R)$ and so, from $\mathfrak{m}_1 \in K$, we get that $A \in S$. Similarly, as $B \subset \mathfrak{m}_2$ and $\mathfrak{m}_2 \in K$, it follows that $B \in S$. Now, it follows from [[5], Proposition 1.11(ii)] that $A \not\subseteq \mathfrak{m}_3$ and $B \not\subseteq \mathfrak{m}_1$. Therefore, we obtain that $A \not\subseteq B$ and $B \not\subseteq A$. Hence, A and B are adjacent in $\mathbb{INC}(R)$. This is impossible since $A, B \in S$. Therefore, $|Max(R)| \leq 2$ and so, $|Max(R)| = 2$. □

Lemma 8. Let R be a ring with $|Max(R)| = 2$. Let $\{\mathfrak{m}_1, \mathfrak{m}_2\}$ denote the set of all maximal ideals of R. Let V_1, V_2 be as in the statement of Lemma 3. If $\mathbb{INC}(R)$ is a split graph, then $|V_i| = 1$ for at least one $i \in \{1, 2\}$.

Proof. Let $V(\mathbb{INC}(R)) = K \cup S$. Since $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are adjacent in $\mathbb{INC}(R)$, it follows that $|S \cap Max(R)| \leq 1$. We consider the following cases.

Case 1. $Max(R) \subseteq K$.
Suppose that $|V_i| \geq 2$ for each $i \in \{1, 2\}$. Let $I \in V_1 \backslash \{\mathfrak{m}_1\}$ and let $J \in V_2 \backslash \{\mathfrak{m}_2\}$. Since $I \subset \mathfrak{m}_1$, it follows that I and $\mathfrak{m}_1$ are not adjacent in $\mathbb{INC}(R)$ and so, $I \in S$. Similarly, since $J \subset \mathfrak{m}_2$, it follows that $J \in S$. As $I + J = R$, we obtain that I and J are adjacent in $\mathscr{C}(R)$ and so, they are adjacent in $\mathbb{INC}(R)$. This is impossible, since $I, J \in S$. Therefore, $|V_i| = 1$ for at least one $i \in \{1, 2\}$.

Case 2. $|S \cap Max(R)| = 1$.
Without loss of generality, we can assume that $\mathfrak{m}_1 \in S$. Then $\mathfrak{m}_2 \in K$. We claim that $|V_2| = 1$. Suppose that $|V_2| \geq 2$. Let $J \in V_2 \backslash \{\mathfrak{m}_2\}$. Since $J \subset \mathfrak{m}_2$ and $\mathfrak{m}_2 \in K$, we obtain that $J \notin K$ and so, $J \in S$. As $J + \mathfrak{m}_1 = R$, it follows that J and $\mathfrak{m}_1$ are adjacent in $\mathbb{INC}(R)$. This is impossible, since $J, \mathfrak{m}_1 \in S$. Therefore, $|V_2| = 1$.
This proves that $|V_i| = 1$ for at least one $i \in \{1, 2\}$. □

Proposition 8. Let $(R_i, \mathfrak{m}_i)$ be a quasilocal ring for each $i \in \{1, 2\}$ and let $R = R_1 \times R_2$. The following statements are equivalent:
(i) $\mathbb{INC}(R)$ is a split graph.
(ii) R_i is a field for at least one $i \in \{1, 2\}$ and if R_i is not a field for some $i \in \{1, 2\}$, then either R_i is a chained ring or $\mathbb{I}(R_i) = W_1 \cup W_2$, where $|W_k| \geq 2$ for each $k \in \{1, 2\}$ with the property that $W_1 \cap W_2 = \emptyset$, W_1 is a chain under the inclusion relation, and no two distinct members of W_2 are comparable under the inclusion relation.

Proof. $(i) \Rightarrow (ii)$ We are assuming that $\mathbb{INC}(R)$ is a split graph. Let $V(\mathbb{INC}(R)) = K \cup S$. Note that $V(\mathbb{INC}(R)) = V_1 \cup V_2$, where $V_1 = \{I \times R_2 \mid I \text{ is a proper ideal of } R_1\}$ and $V_2 = \{R_1 \times J \mid J \text{ is a proper ideal of } R_2\}$. It follows from Lemma 8 that $|V_i| = 1$ for at least one $i \in \{1, 2\}$. Without loss of generality, we can assume that $|V_1| = 1$. Hence, we obtain that R_1 is a field. We can assume that R_2 is not a field. Now, $V(\mathbb{INC}(R)) = V_1 \cup V_2 = K \cup S$. We consider the following cases.

Case 1. $Max(R) \subseteq K$.
Note that $(0) \times R_2$ and $R_1 \times \mathfrak{m}_2 \in K$. Let J_1, J_2 be any two distinct proper ideals of R_2. We claim that J_1 and J_2 are comparable under the inclusion relation. This is clear if either $J_1 = \mathfrak{m}_2$ or $J_2 = \mathfrak{m}_2$. Hence, we can assume that $J_i \neq \mathfrak{m}_2$ for each $i \in \{1, 2\}$. As $R_1 \times \mathfrak{m}_2 \in K$, we obtain that $R_1 \times J_i \in S$ for each $i \in \{1, 2\}$. Since S is an independent set of $\mathbb{INC}(R)$, we obtain that $R_1 \times J_1$ and $R_1 \times J_2$ are not adjacent in

$\mathbb{INC}(R)$. Hence, J_1 and J_2 are comparable under the inclusion relation. This proves that R_2 is a chained ring.

Case 2. $|Max(R) \cap S| = 1$.
If $(0) \times R_2 \in S$, then $R_1 \times \mathfrak{m}_2, R_1 \times (0) \in K$. This is impossible since $R_1 \times \mathfrak{m}_2$ and $R_1 \times (0)$ are not adjacent in $\mathbb{INC}(R)$. Therefore, $(0) \times R_2 \notin S$ and so, $R_1 \times \mathfrak{m}_2 \in S$. We can assume that R_2 is not a chained ring. Let $W_1 = \{J \in \mathbb{I}(R_2) \mid R_1 \times J \in S\}$ and let $W_2 = \{J \in \mathbb{I}(R_2) \mid R_1 \times J \in K\}$. Note that $R_1 \times \mathfrak{m}_2 \in S$ and so, $W_1 \neq \emptyset$. Since R_2 is not a chained ring by assumption, there exist proper ideals J_1, J_2 of R_2 such that they are not comparable under the inclusion relation. Let $a \in J_1 \backslash J_2$ and let $b \in J_2 \backslash J_1$. Let $A = R_2 a, B = R_2 b$, and $C = R_2(a + b)$. It is clear that $A \not\subseteq B$ and $B \not\subseteq A$. As $C \not\subseteq J_1$ and $C \not\subseteq J_2$, we obtain that $C \not\subseteq A$ and $C \not\subseteq B$. We claim that $A \not\subseteq C$ and $B \not\subseteq C$. For if $A \subseteq C$, then $a = y(a+b)$ for some $y \in R_2$. Suppose that $y \in \mathfrak{m}_2$. Then $1 - y \in U(R_2)$ and from $a(1 - y) = yb \in J_2$, we get that $a = (1 - y)^{-1} yb \in J_2$. This is impossible. If $y \in U(R_2)$, then from $a = y(a+b)$, it follows that $a+b = y^{-1} a \in J_1$. This is impossible. Therefore, $A \not\subseteq C$. Similarly, it can be shown that $B \not\subseteq C$. Hence, $R_1 \times A - R_1 \times B - R_1 \times C - R_1 \times A$ is a cycle of length 3 in $\mathbb{INC}(R)$. As S is an independent set of $\mathbb{INC}(R)$, it follows that at least two among $R_1 \times A, R_1 \times B, R_1 \times C$ must be in K. Hence, at least two among A, B, C must be in W_2 and so, $|W_2| \geq 2$. Observe that $R_1 \times (0)$ must be in S. Thus $R_1 \times \mathfrak{m}_2, R_1 \times (0) \in S$ and so, $|W_1| \geq 2$. It is clear that $W_1 \cup W_2 \subseteq \mathbb{I}(R_2)$. Let $J \in \mathbb{I}(R_2)$. Then $R_1 \times J \in V_2 \subseteq K \cup S$. If $R_1 \times J \in S$, then $J \in W_1$ and if $R_1 \times J \in K$, then $J \in W_2$. This proves that $\mathbb{I}(R_2) = W_1 \cup W_2$. It follows from $K \cap S = \emptyset$ that $W_1 \cap W_2 = \emptyset$.

$(ii) \Rightarrow (i)$ If both R_1 and R_2 are fields, then $\mathbb{INC}(R)$ is a complete graph on two vertices and so, $\mathbb{INC}(R)$ is a split graph. We can assume that R_1 is a field and R_2 is not a field. If R_2 is a chained ring, then we know from $(ii) \Rightarrow (i)$ of Proposition 4 that $\mathbb{INC}(R)$ is a star graph and so, $\mathbb{INC}(R)$ is a split graph. Suppose that $\mathbb{I}(R_2) = W_1 \cup W_2$, where $|W_i| \geq 2$ for each $i \in \{1, 2\}$ satisfying the property that $W_1 \cap W_2 = \emptyset$, W_1 is a chain under the inclusion relation, and no two distinct members of W_2 are comparable under the inclusion relation. Let $K = \{(0) \times R_2, R_1 \times I \mid I \in W_2\}$ and let $S = \{R_1 \times I \mid I \in W_1\}$. It is clear that $V(\mathbb{INC}(R)) = K \cup S$, $K \neq \emptyset, S \neq \emptyset, K \cap S = \emptyset$, the subgraph of $\mathbb{INC}(R)$ induced on K is a clique, and S is an independent set of $\mathbb{INC}(R)$. Therefore, $\mathbb{INC}(R)$ is a split graph. □

Theorem 2. Let R be a ring with $|Max(R)| \geq 2$. The following statements are equivalent:
(i) $\mathbb{INC}(R)$ is a split graph.
(ii) $R \cong R_1 \times R_2$ as rings, where R_1 and R_2 are quasilocal rings which satisfy the conditions mentioned in the statement (ii) of Proposition 8.

Proof. $(i) \Rightarrow (ii)$ We are assuming that $\mathbb{INC}(R)$ is a split graph. Let $V(\mathbb{INC}(R)) = K \cup S$. We know from Proposition 7 that $|Max(R)| = 2$. Let $\{\mathfrak{m}_1, \mathfrak{m}_2\}$ denote the set of all maximal ideals of R. Let V_1, V_2 be as in the statement of Lemma 3. It follows from Lemma 8 that $|V_i| = 1$ for at least one $i \in \{1, 2\}$. Without loss of generality, we can assume that $|V_1| = 1$. Now, it can be shown as in the proof of $(i) \Rightarrow (ii)$ of Proposition 4 that there exist nonzero rings R_1 and R_2 such that $R \cong R_1 \times R_2$ as rings. As $|Max(R)| = 2$, it is clear that R_1 and R_2 are quasilocal rings. Let us denote the ring $R_1 \times R_2$ by T. Since $\mathbb{INC}(T)$ is a split graph, we obtain from $(i) \Rightarrow (ii)$ of Proposition 8 that the rings R_1, R_2 satisfy the conditions mentioned in the statement (ii) of Proposition 8.

$(ii) \Rightarrow (i)$ Assume that $R \cong R_1 \times R_2$ as rings, where R_1 and R_2 are quasilocal rings and they satisfy the conditions mentioned in the statement (ii) of Proposition 8. Let us denote the ring $R_1 \times R_2$ by T. We know from $(ii) \Rightarrow (i)$ of Proposition 8 that $\mathbb{INC}(T)$ is a split graph. Since $R \cong T$ as rings, we obtain that $\mathbb{INC}(R)$ is a split graph. □

We provide some examples in Example 4 to illustrate Theorem 2.

Example 4. (i) Let F be a field and let $T = \mathbb{Z}_{p\mathbb{Z}}$, where p is a prime number. Let $R = F \times T$. Then $\mathbb{INC}(R)$ is a split graph.
(ii) Let $T = \mathbb{Z}_2[X, Y]$ be the polynomial ring in two variables X, Y over $\mathbb{Z}_2$ and let $\mathfrak{m} = TX + TY$. Let $R = F \times \frac{T}{\mathfrak{m}^2}$, where F is a field. Then $\mathbb{INC}(R)$ is a split graph.
(iii) Let $A = \mathbb{Z}_2[X, Y, Z]$ be the polynomial ring in three variables X, Y, Z over $\mathbb{Z}_2$ and let $\mathfrak{m} = AX + AY + AZ$. Let $R = F \times \frac{A}{\mathfrak{m}^2}$, where F is a field. Then $\mathbb{INC}(R)$ is not a split graph.

Proof. (i) We know from [[5], Example (1), page 94] that $T = \mathbb{Z}_{p\mathbb{Z}}$ is a discrete valuation ring and so, T is a chained ring. Hence, we obtain from $(ii) \Rightarrow (i)$ of Proposition 8 that $\mathbb{INC}(R)$ is a split graph.

(ii) It is convenient to denote $X + \mathfrak{m}^2$ by x and $Y + \mathfrak{m}^2$ by y. It is clear that $\frac{T}{\mathfrak{m}^2}$ is a local ring with unique maximal ideal $\frac{\mathfrak{m}}{\mathfrak{m}^2}$. Observe that $\mathbb{I}(\frac{T}{\mathfrak{m}^2}) = W_1 \cup W_2$, where $W_1 = \{(0 + \mathfrak{m}^2), \frac{\mathfrak{m}}{\mathfrak{m}^2}\}$ and $W_2 = \{\frac{T}{\mathfrak{m}^2}x, \frac{T}{\mathfrak{m}^2}y, \frac{T}{\mathfrak{m}^2}(x + y)\}$. It is clear that W_1 is a chain under the inclusion relation and no two distinct members of W_2 are comparable under the inclusion relation. Hence, we obtain from $(ii) \Rightarrow (i)$ of Proposition 8 that $\mathbb{INC}(R)$ is a split graph.

(iii) It is convenient to denote $X + \mathfrak{m}^2$ by x, $Y + \mathfrak{m}^2$ by y, and $Z + \mathfrak{m}^2$ by z. It is clear that $\frac{A}{\mathfrak{m}^2}$ is a local ring with $\frac{\mathfrak{m}}{\mathfrak{m}^2}$ as its unique maximal ideal. It is convenient to denote $\frac{A}{\mathfrak{m}^2}$ by A_1 and $\frac{\mathfrak{m}}{\mathfrak{m}^2}$ by $\mathfrak{m}_1$. Observe that $\mathbb{I}(A_1) = \{(0 + \mathfrak{m}^2), A_1x, A_1y, A_1z, A_1(x + y), A_1(y + z), A_1(z + x), A_1(x + y + z), A_1x + A_1y, A_1y + A_1z, A_1z + A_1x, A_1x + A_1(y + z), A_1y + A_1(x + z), A_1z + A_1(x + y), \mathfrak{m}_1\}$. Note that A_1 is not a field and is not a chained ring. Let W_1, W_2 be subsets of $\mathbb{I}(A_1)$ such that W_1 is a chain under the inclusion relation and no two distinct members of W_2 are comparable under the inclusion relation. We claim that $\mathbb{I}(A_1) \neq W_1 \cup W_2$. Suppose that $\mathbb{I}(A_1) = W_1 \cup W_2$. If $A_1x \in W_1$, then $A_1(x+y+z), A_1y+A_1(x+z)$ must be in W_2. This is impossible since $A_1(x+y+z) \subset A_1y+A_1(x+z)$. Hence, $A_1x \notin W_1$. If $A_1x \in W_2$, then both A_1x+A_1y and A_1x+A_1z must be in W_1. This is impossible since A_1x+A_1y and A_1x+A_1z are not comparable under the inclusion relation. Therefore, $\mathbb{I}(A_1) \neq W_1 \cup W_2$. Hence, it follows from $(i) \Rightarrow (ii)$ of Proposition 8 that $\mathbb{INC}(R)$ is not a split graph. □

4. When is $\mathbb{INC}(R)$ complemented?

Let R be a ring with $|Max(R)| \geq 2$. In this section, we try to characterize such rings R whose $\mathbb{INC}$ graph is complemented.

Lemma 9. Let R be a ring such that $|Max(R)| \geq 2$. Let $I \in V(\mathbb{INC}(R))$. If J is a vertex in $\mathbb{INC}(R)$ such that $I \perp J$ in $\mathbb{INC}(R)$, then $IJ \subseteq J(R)$.

Proof. Now, by assumption $I \perp J$ in $\mathbb{INC}(R)$. Hence, I and J are adjacent in $\mathbb{INC}(R)$ and there is no $A \in V(\mathbb{INC}(R))$ which is adjacent to both I and J in $\mathbb{INC}(R)$. Let $\mathfrak{m} \in Max(R)$. We claim that $IJ \subseteq \mathfrak{m}$. This is clear if $\mathfrak{m} \in \{I, J\}$. Hence, we can assume that $\mathfrak{m} \notin \{I, J\}$. Since $I \perp J$, either $\mathfrak{m}$ is not adjacent to I or $\mathfrak{m}$ is not adjacent to J in $\mathbb{INC}(R)$. Hence, either $I \subset \mathfrak{m}$ or $J \subset \mathfrak{m}$. Therefore, $IJ \subset \mathfrak{m}$. This is true for any $\mathfrak{m} \in Max(R)$ and so, $IJ \subseteq J(R)$. □

Lemma 10. Let R be a ring such that $|Max(R)| \geq 2$. If $\mathbb{INC}(R)$ is complemented, then $|Max(R)| \leq 3$.

Proof. Assume that $\mathbb{INC}(R)$ is complemented. Suppose that $|Max(R)| \geq 4$. Let $\{\mathfrak{m}_i \mid i \in \{1, 2, 3, 4\}\} \subseteq Max(R)$. Note that $\mathfrak{m}_1 \cap \mathfrak{m}_2 \in V(\mathbb{INC}(R))$. Let us denote $\mathfrak{m}_1 \cap \mathfrak{m}_2$ by I. Since $\mathbb{INC}(R)$ is complemented, there exists $J \in V(\mathbb{INC}(R))$ such that $I \perp J$ in $\mathbb{INC}(R)$. We know from Lemma 9 that $IJ \subseteq J(R)$. It follows from [[5], Proposition 1.11(ii)] that $I \not\subseteq \mathfrak{m}$ for any $\mathfrak{m} \in Max(R) \backslash \{\mathfrak{m}_1, \mathfrak{m}_2\}$. From $IJ \subseteq J(R)$, we obtain that $J \subseteq \mathfrak{m}_3 \cap \mathfrak{m}_4$. Since I and J are adjacent in $\mathbb{INC}(R)$, we get that $J \not\subseteq I$. Hence, either $J \not\subseteq \mathfrak{m}_1$ or $J \not\subseteq \mathfrak{m}_2$. Without loss of generality, we can assume that $J \not\subseteq \mathfrak{m}_1$. Consider the ideal $A = \mathfrak{m}_1 \cap \mathfrak{m}_3$. It is clear that $A \in V(\mathbb{INC}(R))$ and $I = \mathfrak{m}_1 \cap \mathfrak{m}_2 \not\subseteq A = \mathfrak{m}_1 \cap \mathfrak{m}_3$ and $A \not\subseteq I$. Since $A \not\subseteq \mathfrak{m}_4$, we obtain that $A \not\subseteq J$. From $J \not\subseteq \mathfrak{m}_1$, it follows that $J \not\subseteq A$. Hence, we get that A is adjacent to both I and J in $\mathbb{INC}(R)$. This is in contradiction to the assumption that $I \perp J$ in $\mathbb{INC}(R)$. Therefore, $|Max(R)| \leq 3$. □

Let R be a ring such that $|Max(R)| = 2$. We try to characterize such rings R whose $\mathbb{INC}$ graph is complemented.

Remark 3. Let R be a ring such that $|Max(R)| = 2$. Let $Max(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$. Let V_1, V_2 be as in the statement of Lemma 3. If $|V_i| = 1$ for each $i \in \{1, 2\}$, then it is verified in the proof of $(i) \Rightarrow (ii)$ of Proposition 1 that $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$ and in such a case, it is observed in $(ii) \Rightarrow (i)$ of Proposition 1 that $\mathbb{INC}(R)$ is a complete graph on two vertices. Hence, $\mathbb{INC}(R)$ is complemented. Thus in characterizing rings R with $|Max(R)| = 2$ whose $\mathbb{INC}$ graph is complemented, we assume that $|V_i| \geq 2$ for at least one $i \in \{1, 2\}$.

Lemma 11. Let R be a ring such that $|Max(R)| = 2$. Let $Max(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$. Let V_1, V_2 be as in the statement of Lemma 3. Suppose that $|V_i| \geq 2$ for some $i \in \{1, 2\}$. If $\mathbb{INC}(R)$ is complemented, then any $I_1, I_2 \in V_i$ are comparable under the inclusion relation.

Proof. Suppose that $|V_1| \geq 2$. Let $I_1 \in V_1$. We are assuming that $\mathbb{INC}(R)$ is complemented. Hence, there exists $J_1 \in V(\mathbb{INC}(R))$ such that $I_1 \perp J_1$ in $\mathbb{INC}(R)$. We know from Lemma 9 that $I_1 J_1 \subseteq J(R) = \mathfrak{m}_1 \cap \mathfrak{m}_2$. From $I_1 \not\subseteq \mathfrak{m}_2$, we obtain that $J_1 \subseteq \mathfrak{m}_2$. Hence, $M(J_1) = \{\mathfrak{m}_2\}$ and so, $J_1 \in V_2$. Let $I_2 \in V_1$ be such that $I_2 \neq I_1$. Since $I_2 + J_1 = R$, I_2 and J_1 are adjacent in $\mathscr{C}(R)$ and hence, they are adjacent in $\mathbb{INC}(R)$. As $I_1 \perp J_1$ in $\mathbb{INC}(R)$, I_2 and I_1 cannot be adjacent in $\mathbb{INC}(R)$. Therefore, I_1 and I_2 are comparable under the inclusion relation. Similarly, if $|V_2| \geq 2$, it can be shown that any two members of V_2 are comparable under the inclusion relation. □

Proposition 9. Let R be a ring such that $|Max(R)| = 2$. Let $Max(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$. Let V_1, V_2 be as in the statement of Lemma 3. Suppose that $|V_1| = 1$ and $|V_2| \geq 2$. The following statements are equivalent:
(i) $\mathbb{INC}(R)$ is complemented.
(ii) $R \cong R_1 \times R_2$ as rings, where R_1 is a chained ring which is not a field and R_2 is a field.

Proof. $(i) \Rightarrow (ii)$ Let $a \in \mathfrak{m}_1 \backslash \mathfrak{m}_2$. It follows from $|V_1| = 1$ that $\mathfrak{m}_1 = Ra = \mathfrak{m}_1^2 = Ra^2$. Hence, there exists a nontrivial idempotent $e \in \mathfrak{m}_1$ such that $\mathfrak{m}_1 = Re$. Note that the mapping $f : R \to Re \times R(1-e)$ defined by $f(r) = (re, r(1-e))$ is an isomorphism of rings. Let us denote the ring Re by $R_1, R(1-e)$ by R_2, and $R_1 \times R_2$ by T. Observe that $f(\mathfrak{m}_1) = R_1 \times (0)$ and as $f(\mathfrak{m}_1) \in Max(T)$, it follows that R_2 is a field. Since $R \cong T$ as rings, we obtain that $|Max(T)| = 2$ and so, R_1 is quasilocal. Let us denote the unique maximal ideal of R_1 by $\mathfrak{n}_1$. It is clear that $f(\mathfrak{m}_2) = \mathfrak{n}_1 \times R_2$. Note that under the isomorphism f, V_1 is mapped onto $W_1 = \{R_1 \times (0)\}$ and V_2 is mapped onto $W_2 = \{I \times R_2 \mid I \in \mathbb{I}(R_1)\}$. We are assuming that $\mathbb{INC}(R)$ is complemented. Therefore, $\mathbb{INC}(T)$ is complemented. From $|W_2| \geq 2$, it follows from Lemma 11 that any two members of W_2 are comparable under the inclusion relation. Hence, if $I_1, I_2 \in \mathbb{I}(R_1)$, then I_1 and I_2 are comparable under the inclusion relation. Therefore, we obtain that R_1 is a chained ring and it follows from $|W_2| \geq 2$ that R_1 is not a field.

$(ii) \Rightarrow (i)$ Assume that $R \cong R_1 \times R_2$ as rings, where R_1 is a chained ring which is not a field and R_2 is a field. It follows from $(ii) \Rightarrow (i)$ of Proposition 4 that $\mathbb{INC}(R)$ is a star graph and so, $\mathbb{INC}(R)$ is complemented. □

Proposition 10. Let R be a ring such that $|Max(R)| = 2$. Let $\{\mathfrak{m}_1, \mathfrak{m}_2\}$ denote the set of all maximal ideals of R. Let V_1, V_2 be as in the statement of Lemma 3. Suppose that $|V_i| \geq 2$ for each $i \in \{1, 2\}$. Then the following statements are equivalent:
(i) $\mathbb{INC}(R)$ is complemented.
(ii) Any two members of V_i are comparable under the inclusion relation for each $i \in \{1, 2\}$.
(iii) $\mathbb{INC}(R) = \mathscr{C}(R)$ is a complete bipartite graph.

Proof. $(i) \Rightarrow (ii)$ We are assuming that $\mathbb{INC}(R)$ is complemented. By hypothesis, $|V_i| \geq 2$ for each $i \in \{1, 2\}$. Hence, we obtain from Lemma 11 that any two members of V_i are comparable under the inclusion relation for each $i \in \{1, 2\}$.

$(ii) \Rightarrow (iii)$ Note that $V(\mathbb{INC}(R)) = V_1 \cup V_2$. Observe that $V_1 \cap V_2 = \emptyset$. It follows from (ii) that if $I_1 - I_2$ is an edge of $\mathbb{INC}(R)$, then both I_1, I_2 cannot be in the same V_i for any $i \in \{1,2\}$. If $I \in V_1$ and $J \in V_2$, then $I + J = R$ and so, I and J are adjacent in $\mathbb{INC}(R)$. Therefore, $\mathbb{INC}(R) = \mathscr{C}(R)$ is a complete bipartite graph.

$(iii) \Rightarrow (i)$ This is clear. □

Let R be a ring with $|Max(R)| = 2$ satisfying the hypothesis of Proposition 10. We are not able to characterize such rings R which satisfies the statement (ii) of Proposition 10. However, we mention one instance where the statement (ii) of Proposition 10 is satisfied. Let R_1, R_2 be chained rings which are not fields and let $R = R_1 \times R_2$. Let $i \in \{1,2\}$ and let $\mathfrak{m}_i$ denote the unique maximal ideal of R_i. Note that in this case, $V_1 = \{I \times R_2 \mid I \in \mathbb{I}(R_1)\}$ and $V_2 = \{R_1 \times J \mid J \in \mathbb{I}(R_2)\}$. Since R_1 and R_2 are not fields, we obtain that $|V_i| \geq 2$ for each $i \in \{1,2\}$. As R_i is a chained ring for each $i \in \{1,2\}$, we obtain that R satisfies the statement (ii) of Proposition 10. Therefore, $\mathbb{INC}(R)$ is complemented. In Proposition 11, we characterize zero-dimensional rings R with $|Max(R)| = 2$ such that $\mathbb{INC}(R)$ is complemented.

Proposition 11. Let R be a ring with $|Max(R)| = 2$. Let $dim R = 0$. Let $Max(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$ and let V_1, V_2 be as in the statement of Lemma 3. Suppose that $|V_i| \geq 2$ for each $i \in \{1,2\}$. Then the following statements are equivalent:
(i) $\mathbb{INC}(R)$ is complemented.
(ii) $R \cong R_1 \times R_2$ as rings, where R_i is a chained ring which is not a field for each $i \in \{1,2\}$.

Proof. $(i) \Rightarrow (ii)$ Since $dim R = 0$ and $|Max(R)| = 2$, we obtain from Remark 2 that $R \cong R_1 \times R_2$ as rings, where $(R_i, \mathfrak{n}_i)$ is a quasilocal ring for each $i \in \{1,2\}$. Let us denote the ring $R_1 \times R_2$ by T. Note that under the isomorphism from R onto T, V_1 is mapped onto $W_1 = \{I \times R_2 \mid I \in \mathbb{I}(R_1)\}$ and V_2 is mapped onto $W_2 = \{R_1 \times J \mid J \in \mathbb{I}(R_2)\}$. Since $R \cong T$ as rings, we obtain that $\mathbb{INC}(T)$ is complemented. By hypothesis, $|V_i| \geq 2$ for each $i \in \{1,2\}$ and so, $|W_i| \geq 2$ for each $i \in \{1,2\}$. Therefore, R_i is not a field for each $i \in \{1,2\}$. Let $i \in \{1,2\}$. We know from Lemma 11 that any two members of W_i are comparable under the inclusion relation and so, any two proper ideals of R_i are comparable under the inclusion relation. Therefore, R_i is a chained ring.

$(ii) \Rightarrow (i)$ Let us denote the ring $R_1 \times R_2$ by T. It follows from $(iii) \Rightarrow (ii)$ of Proposition 3 that $\mathbb{INC}(T)$ is a complete bipartite graph. Therefore, $\mathbb{INC}(T)$ is complemented and so, $\mathbb{INC}(R)$ is complemented. □

Remark 4. In this Remark, we mention an example to illustrate that $(i) \Rightarrow (ii)$ of Proposition 11 can fail to hold if the hypothesis $dim R = 0$ is omitted in Proposition 11. Let p, q be distinct prime numbers and let $R = S^{-1}\mathbb{Z}$, where $S = \mathbb{Z} \backslash (p\mathbb{Z} \cup q\mathbb{Z})$. Note that R is a principal ideal domain and $Max(R) = \{pR, qR\}$. It is verified in Example 1 that $\mathscr{C}(R) = \mathbb{INC}(R)$ is a complete bipartite graph. Hence, $\mathbb{INC}(R)$ is complemented. Since R is an integral domain, 0 and 1 are the only idempotent elements of R. Therefore, (ii) of Proposition 11 does not hold.

Let R be a ring with $|Max(R)| = 3$. In Theorem 3, we characterize such rings R whose $\mathbb{INC}$ graph is complemented.

Lemma 12. Let R be a ring such that $|Max(R)| = 3$. Let $\{\mathfrak{m}_i | i \in \{1,2,3\}\}$ denote the set of all maximal ideals of R. If $\mathbb{INC}(R)$ is complemented, then $\mathfrak{m}_i = \mathfrak{m}_i^2$ for each $i \in \{1,2,3\}$.

Proof. We are assuming that $\mathbb{INC}(R)$ is complemented and $Max(R) = \{\mathfrak{m}_i \mid i \in \{1,2,3\}\}$. We claim that $\mathfrak{m}_i = \mathfrak{m}_i^2$ for each $i \in \{1,2,3\}$. Since $\mathbb{INC}(R)$ is complemented, there exists $J \in V(\mathbb{INC}(R))$ such that $\mathfrak{m}_1^2 \perp J$ in $\mathbb{INC}(R)$. It now follows from Lemma 9 that $\mathfrak{m}_1^2 J \subseteq J(R) = \cap_{i=1}^3 \mathfrak{m}_i$. This implies that $J \subseteq \mathfrak{m}_2 \cap \mathfrak{m}_3$. Let us denote the ideal $\mathfrak{m}_1\mathfrak{m}_3$ by A. It is clear that $A \in V(\mathbb{INC}(R))$. Observe that $A \not\subseteq \mathfrak{m}_2$, whereas $J \subseteq \mathfrak{m}_2$ and so, $A \not\subseteq J$. Since $J \not\subseteq J(R)$, it follows that $J \not\subseteq \mathfrak{m}_1$. As $A \subseteq \mathfrak{m}_1$, we obtain that $J \not\subseteq A$. Hence, A and J are adjacent in $\mathbb{INC}(R)$. Since $\mathfrak{m}_1^2 \not\subseteq \mathfrak{m}_3$, whereas $A \subseteq \mathfrak{m}_3$, we obtain that $\mathfrak{m}_1^2 \not\subseteq A$. As $\mathfrak{m}_1^2 \perp J$ in $\mathbb{INC}(R)$, it follows that $\mathfrak{m}_1^2$ and A cannot be adjacent in $\mathbb{INC}(R)$. Therefore, $A = \mathfrak{m}_1\mathfrak{m}_3 \subseteq \mathfrak{m}_1^2$. We know from [[5], Proposition 4.2] that $\mathfrak{m}_1^2$ is a $\mathfrak{m}_1$-primary ideal of R. As $\mathfrak{m}_3 \not\subseteq \mathfrak{m}_1 = \sqrt{\mathfrak{m}_1^2}$, we get that $\mathfrak{m}_1 \subseteq \mathfrak{m}_1^2$ and so, $\mathfrak{m}_1 = \mathfrak{m}_1^2$. Similarly, it can be shown that $\mathfrak{m}_2 = \mathfrak{m}_2^2$ and $\mathfrak{m}_3 = \mathfrak{m}_3^2$. □

Lemma 13. Let R be a ring such that $|Max(R)| = 3$. Let $\{\mathfrak{m}_i \mid i \in \{1,2,3\}\}$ denote the set of all maximal ideals of R. If $\mathbb{INC}(R)$ is complemented, then $R_{\mathfrak{m}_i}$ is a field for each $i \in \{1,2,3\}$.

Proof. We are assuming that $\mathbb{INC}(R)$ is complemented. We first verify that $R_{\mathfrak{m}_1}$ is a field. Since $\mathfrak{m}_1 \not\subseteq \mathfrak{m}_2 \cup \mathfrak{m}_3$, there exists $a \in \mathfrak{m}_1 \backslash (\mathfrak{m}_2 \cup \mathfrak{m}_3)$. Note that $Ra \in V(\mathbb{INC}(R))$. As $\mathbb{INC}(R)$ is complemented, there exists $J \in V(\mathbb{INC}(R))$ such that $Ra \perp J$ in $\mathbb{INC}(R)$. We know from Lemma 9 that $(Ra)J \subseteq J(R) = \cap_{i=1}^3 \mathfrak{m}_i$. Hence, we obtain that $J \subseteq \mathfrak{m}_2 \cap \mathfrak{m}_3$. Let us denote the ideal $\mathfrak{m}_1\mathfrak{m}_2$ by A. It is clear that $A \in V(\mathbb{INC}(R))$. As $J \subseteq \mathfrak{m}_3$ and $A \not\subseteq \mathfrak{m}_3$, we obtain that $A \not\subseteq J$. Since $J \not\subseteq J(R)$, it follows that $J \not\subseteq \mathfrak{m}_1$ and so, $J \not\subseteq \mathfrak{m}_1\mathfrak{m}_2 = A$. Hence, A and J are not comparable under the inclusion relation and therefore, A and J are adjacent in $\mathbb{INC}(R)$. As $a \notin \mathfrak{m}_2$, it follows that $Ra \not\subseteq A$. Since $Ra \perp J$ in $\mathbb{INC}(R)$, we obtain that Ra and A cannot be adjacent in $\mathbb{INC}(R)$. Therefore, $A = \mathfrak{m}_1\mathfrak{m}_2 \subseteq Ra$. This implies that $(\mathfrak{m}_1\mathfrak{m}_2)_{\mathfrak{m}_1} \subseteq (Ra)_{\mathfrak{m}_1} \subseteq (\mathfrak{m}_1)_{\mathfrak{m}_1}$. From $(\mathfrak{m}_2)_{\mathfrak{m}_1} = R_{\mathfrak{m}_1}$, we get that $(\mathfrak{m}_1)_{\mathfrak{m}_1} = R_{\mathfrak{m}_1}(\frac{a}{1})$. We know from Lemma 12 that $\mathfrak{m}_1 = \mathfrak{m}_1^2$. Hence, we obtain that $R_{\mathfrak{m}_1}(\frac{a}{1}) = (\mathfrak{m}_1)_{\mathfrak{m}_1} = (\mathfrak{m}_1^2)_{\mathfrak{m}_1} = R_{\mathfrak{m}_1}(\frac{a^2}{1})$. Hence, $\frac{a}{1} = \frac{r}{s}\frac{a^2}{1}$ for some $r \in R$ and $s \in R\backslash\mathfrak{m}_1$. Therefore, $\frac{a}{1}(\frac{1}{1} - \frac{ra}{s}) = \frac{0}{1}$. Since $R_{\mathfrak{m}_1}$ is quasilocal with $(\mathfrak{m}_1)_{\mathfrak{m}_1}$ as its unique maximal ideal, it follows that $\frac{1}{1} - \frac{ra}{s}$ is a unit in $R_{\mathfrak{m}_1}$, and so, we obtain that $\frac{a}{1} = \frac{0}{1}$. Therefore, $(\mathfrak{m}_1)_{\mathfrak{m}_1} = (\frac{0}{1})$. This proves that $R_{\mathfrak{m}_1}$ is a field. Similarly, it can be shown that $R_{\mathfrak{m}_i}$ is a field for each $i \in \{2,3\}$. □

Lemma 14. Let $R = F_1 \times F_2 \times F_3$, where F_i is a field for each $i \in \{1,2,3\}$. Then $\mathbb{INC}(R)$ is complemented.

Proof. Note that $Max(R) = \{\mathfrak{m}_1 = (0) \times F_2 \times F_3, \mathfrak{m}_2 = F_1 \times (0) \times F_3, \mathfrak{m}_3 = F_1 \times F_2 \times (0)\}$. It is clear that $J(R) = (0) \times (0) \times (0)$ and $V(\mathbb{INC}(R)) = \{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \mathfrak{m}_1 \cap \mathfrak{m}_2, \mathfrak{m}_2 \cap \mathfrak{m}_3, \mathfrak{m}_1 \cap \mathfrak{m}_3\}$. It is clear that $\mathfrak{m}_1 \perp (\mathfrak{m}_2 \cap \mathfrak{m}_3), \mathfrak{m}_2 \perp (\mathfrak{m}_1 \cap \mathfrak{m}_3)$, and $\mathfrak{m}_3 \perp (\mathfrak{m}_1 \cap \mathfrak{m}_2)$ in $\mathbb{INC}(R)$. This proves that $\mathbb{INC}(R)$ is complemented. □

Theorem 3. Let R be a ring such that $|Max(R)| = 3$. The following statements are equivalent:
(i) $\mathbb{INC}(R)$ is complemented.
(ii) $R \cong F_1 \times F_2 \times F_3$ as rings, where F_i is a field for each $i \in \{1, 2, 3\}$.

Proof. $(i) \Rightarrow (ii)$ Assume that $\mathbb{INC}(R)$ is complemented. Let $\{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3\}$ denote the set of all maximal ideals of R. We know from Lemma 13 that $R_{\mathfrak{m}_i}$ is a field for each $i \in \{1, 2, 3\}$. Hence, $(J(R))_{\mathfrak{m}_i} = (\mathfrak{m}_i)_{\mathfrak{m}_i} = (\frac{0}{1})$ for each $i \in \{1, 2, 3\}$. Therefore, we obtain from $(iii) \Rightarrow (i)$ of [[5], Proposition 3.8] that $J(R) = (0)$. Thus $\cap_{i=1}^{3} \mathfrak{m}_i = (0)$. As distinct maximal ideals of a ring are comaximal, it follows from the Chinese remainder theorem [[5], Proposition 1.10(ii) and (iii)] that $R \cong \frac{R}{\mathfrak{m}_1} \times \frac{R}{\mathfrak{m}_2} \times \frac{R}{\mathfrak{m}_3}$.
$(ii) \Rightarrow (i)$ Let us denote the ring $F_1 \times F_2 \times F_3$ by T. We know from Lemma 14 that $\mathbb{INC}(T)$ is complemented. Since $R \cong T$ as rings, we obtain that $\mathbb{INC}(R)$ is complemented. □

References

[1] M. Afkhami and K. Khashyarmanesh, *The cozero-divisor graph of a commutative ring.*, Southeast Asian Bull. Math. **35** (2011), no. 5, 753–762.

[2] ______, *On the cozero-divisor graphs and comaximal graphs of commutative rings*, J. Algebra Appl. **12** (2013), no. 3, Article ID: 1250173, 9 pages.

[3] ______, *On the cozero-divisor graphs of commutative rings*, Applied Mathematics **4** (2013), no. 7, 979–985.

[4] D.F. Anderson, R. Levy, and J. Shapiro, *Zero-divisor graphs, von neumann regular rings, and boolean algebras*, J. Pure Appl. Algebra **180** (2003), no. 3, 221–241.

[5] M.F. Atiyah and I.G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley Publishing Company, Reading Massachusetts, 1969.

[6] R. Balakrishnan and K. Ranganathan, *A Textbook of Graph Theory*, Universitext Springer, 2000.

[7] E. Bastida and R. Gilmer, *Overrings and divisorial ideals of rings of the form $D + M$.*, Michigan Math. J. **20** (1973), no. 1, 79–95.

[8] M. Behboodi and Z. Rakeei, *The annihilating-ideal graph of commutative rings I*, J. Algebra Appl. **10** (2011), no. 4, 727–739.

[9] ______, *The annihilating-ideal graph of commutative rings II*, J. Algebra Appl. **10** (2011), no. 4, 741–753.

[10] S.M. Bhatwadekar and P.K. Sharma, *A note on graphical representation of rings*, J. Algebra **176** (1995), no. 1, 124–127.

[11] R. Gilmer, *Multiplicative Ideal Theory*, Marcel-Dekker, New York, 1972.

[12] M.I. Jinnah and S.C. Mathew, *When is the comaximal graph split?*, Comm. Algebra **40** (2012), no. 7, 2400–2404.

[13] H.R. Maimani, M. Salimi, A. Sattari, and S. Yassemi, *Comaximal graph of commutative rings*, J. Algebra **319** (2008), no. 4, 1801–1808.

[14] S.M. Moconja and Z.Z. Petrović, *On the structure of comaximal graphs of commutative rings with identity*, Bull. Aust. Math. Soc. **83** (2011), no. 1, 11–21.

[15] K. Samei, *On the comaximal graph of a commutative ring*, Canad. Math. Bull. **57** (2014), no. 2, 413–423.

[16] H.-Ju Wang, *Graphs associated to co-maximal ideals of commutative rings*, J. Algebra **320** (2008), no. 7, 2917–2933.

[17] M. Ye and T. Wu, *Co-maximal ideal graphs of commutative rings*, J. Algebra Appl. **11** (2012), no. 6, Article ID: 1250114, 14 pages.