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Complexity and approximation ratio of semitotal domination in graphs

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Abstract: A set $S \subseteq V(G)$ is a semitotal dominating set of a graph G if it is a dominating set of G and every vertex in S is within distance 2 of another vertex of S. The semitotal domination number $\gamma_{t2}(G)$ is the minimum cardinality of a semitotal dominating set of G. We show that the semitotal domination problem is APX-complete for bounded-degree graphs, and the semitotal domination problem in any graph of maximum degree Δ can be approximated with an approximation ratio of $2 + \ln(\Delta - 1)$.

Keywords: semitotal domination, APX-complete, NP-complete

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1. Terminology and introduction

In this paper, we shall only consider graphs without multiple edges or loops or isolated vertices. For a graph G, $S \subseteq V(G)$, $v \in V(G)$, the *open neighborhood* of v in S is denoted by $N_S(v)$ (or simply $N(v)$), i.e. $N_S(v) = \{u : uv \in E(G), u \in S\}$.

Domination and its variations in graphs have attracted considerable attention [3, 5, 6]. A set $S \subseteq V(G)$ is a dominating set of a graph G if every vertex in $V(G) \setminus S$ is adjacent to a vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. A set $S \subseteq V(G)$ is a semitotal dominating set of a graph G if it is a dominating set of G and every vertex in S is within distance 2 of another vertex of S. The semitotal domination number $\gamma_{t2}(G)$ is the minimum cardinality of a semitotal dominating set of G.

The semitotal domination problem consists of finding the semitotal domination number of a graph G. It has been proved to be NP-complete and was claimed that there is a linear-time algorithm for trees [4]. Henning studied the semitotal domination in cubic claw-free graphs and proposed a conjecture, and soon the conjecture was confirmed in [8]. In this paper, we continue studying the complexity of the semitotal domination problem and extend these studies by investigating the approximation hardness of the semitotal domination problem in graphs.

2. NP-completeness of semitotal domination

Goddard et al. proved that the semitotal domination problem is NP-complete for general graphs, where the semitotal domination problem is stated as follows.

Semitotal Domination Problem	
Input:	A graph G, and an integer k.
Question:	Is there a semitotal dominating set of G with cardinality at most k ?

We show that it is NP-complete for bipartite, or chordal, or planar graphs via reduction from the dominating set problem.

Theorem 1. *The semitotal domination problem is NP-complete for bipartite, or chordal, or planar graphs.*

Proof. It is clear the semitotal domination problem is in NP, since it is easy to verify a yes instance of the semitotal domination problem in polynomial time. Now let us show how to transform any instance (G, k) of DOM into an instance (G', k') of the semitotal domination problem so that G has a dominating set of order k if and only if G' has a semitotal dominating set of order k'.

Let G be an arbitrary graph, we construct a graph G' as follows. For each vertex $v \in V(G)$, we build a star $K^v_{1,4} = \{w_{v,1}, w_{v,2}, w_{v,3}, w_{v,4}\}$ centered at $w_{v,1}$, and add the star $K^v_{1,4}$ and connect $w_{v,1}$ to v. That is to say, $V(G') = V(G) \cup \{w_{v,1}, w_{v,2}, w_{v,3}, w_{v,4} : v \in V(G)\}$, and $E(G') = E(G) \cup \{w_{v,1}w_{v,2},\ w_{v,1}w_{v,3},\ w_{v,1}w_{v,4},\ w_{v,1}v : v \in V(G)\}$.

Suppose that G has a dominating set of D, then we have $D' = D \cup \{w_{v,1} : v \in V(G)\}$ is a semitotal dominating set of G'. It can be seen that $|D'| = |D| + |V(G)|$.

Conversely, suppose that G' has a semitotal dominating set D'. Then we can obtain a semitotal dominating set D'' such that $|D''| \leq |D'|$ and $D'' \supseteq \{w_{v,1} : v \in V(G)\}$ and $D'' \cap \{w_{v,2}, w_{v,3}, w_{v,4} : v \in V(G)\} = \emptyset$. Now, we claim that $D = D'' \setminus \{w_{v,1} : v \in V(G)\}$ is a dominating set of G. It can be seen that $|D| = |D''| - |V(G)|$.

Since G is bipartite (resp. chordal, planar), G' is also bipartite (resp. chordal, planar). Note that the dominating set problem is NP-complete for bipartite, or chordal, or planar graphs, so the semitotal dominating set problem is also NP-complete for such graphs. $\square$

3. APX-completeness of semitotal domination

The notation of L-reduction can be found in [1, 2, 7]. Given two NP optimization problems G_1 and G_2 and a polynomial time transformation h from instances of G_1 to instances of G_2, h is said to be an L-reduction if there are positive constants α and β such that for every instance x of G_1, we have

(1) $opt_{G_1}(h(x)) \leq \alpha \cdot opt_{G_2}(x)$;

(2) for every feasible solution y of $h(x)$ with objective value $m_G(h(x), y) = c_2$ we can in polynomial time find a solution y of x with $m_{G_1}(x, y) = c_1$ such that $|opt_{G_1}(x) - c_1| \leq \beta \cdot |opt_{G_2}(h(x)) - c_2|$.

To show that a problem $\mathcal{P} \in APX$ is APX-complete, we need to show that there is an L-reduction from some APX-complete problem to $\mathcal{P}$. The following problem was proved to be APX-complete (see [2]):

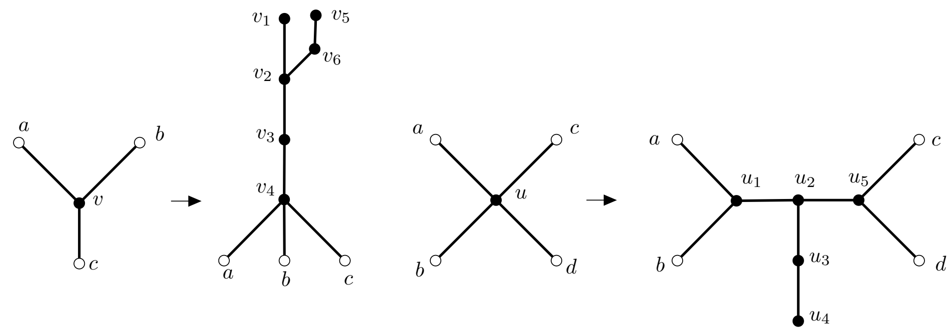
MIN DOM SET-B
Input: A graph $G = (V, E)$ with degree at most B.
Solution: A dominating set S of G.
Measure: Cardinality of the dominating set S.

Now we consider the following problem with $B \in \{3, 4\}$:

SEMITOTAL DOM-B
Input: A graph $G = (V, E)$ with degree at most B.
Solution: A semitotal dominating set S of G.
Measure: Cardinality of the semitotal dominating set S.

Theorem 2.
i) SEMITOTAL DOM-4 is APX-complete;
ii) SEMITOTAL DOM-3 is APX-complete.

Proof. i) It is clear that SEMITOTAL DOM-4 $\in APX$, and so we just have to show SEMITOTAL DOM-4 is APX-hard. We will construct an L-reduction f from MIN DOM-3 for cubic graphs to SEMITOTAL DOM-4 for graphs with maximum degree 4. Given a cubic graph G, we construct a graph G' with maximum degree 4 in the following way. For each vertex v with $N(v) = \{a, b, c\}$, we split the vertex v and transform to the gadget depicted in Fig. 1 (b).



Let D is a dominating set of G, we construct a vertex set TD of G' in the following way:

- If $v \in D$, then v_2, v_4, v_6 are put to TD.

- If $v \notin D$, then we have that one of $\{a, b, c\}$ is in D and thus v_2, v_6 are put to TD.

It can be checked that TD is a semitotal dominating set of G' and $|TD| = |D| + 2|V(G)|$. In particular, we have $|TD^*| \leq |D^*| + 2|V(G)|$, where TD^* is a minimum semitotal dominating set of G' and D^* is a minimum dominating set of G. It is well known that $\gamma(G) \geq \frac{|V(G)|}{\Delta+1}$, so we have $|V(G)| \leq 5|D^*|$. Therefore, we have $|TD^*| \leq |D^*| + 2|V(G)| \leq 11|D^*|$.
Let TD' be a semitotal dominating set of G'. We construct a vertex set D' of G as follows. Let $T(v) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ for any $v \in V(G)$ and $s(v) = |T(v) \cap TD'|$. If $s(v) = 3$, we put v to D'. We claim that D' is a dominating set of G. $|D'| \leq |TD'| - 2|V(G)|$. In particular, we have $|D^*| \leq |TD^*| - 2|V(G)|$ and thus $|D^*| = |TD^*| - 2|V(G)|$. In addition, $|D| - |D^*| \leq |TD'| - |TD^*|$. As a result, f is an L-reduction with $\alpha = 11$ and $\beta = 1$.
ii) It is clear that SEMITOTAL DOM-3 $\in APX$, and so we just have to show SEMITOTAL DOM-3 is APX-hard. Since we have shown that SEMITOTAL DOM-4 is APX-complete, we now consider the following L-reduction g from SEMITOTAL DOM-4 to SEMITOTAL DOM-3.
Given a graph G with maximum degree 4, we construct a graph G' with maximum degree 3 in the following way. For each vertex u with degree 4 and $N(u) = \{a, b, c, d\}$, we split the vertex u and transform to the gadget depicted in Fig. 1 (a). Let TD^1 be a semitotal dominating set of G, we construct a vertex set TD^2 of G' in the following way:

- If $u \in TD^1$ and $d(u) \leq 3$, then u is put to TD^2.

- If $u \in TD^1$ and $d(u) = 4$, then u_1, u_3, u_5 are put to TD^2.

- If $u \notin TD^1$ and $d(u) = 4$, then u_2, u_3 are put to TD^2.

It can be checked that TD^2 is a semitotal dominating set of G' and $|TD^2| = |TD^1| + 2k$, where k is the number of vertices with degree 4 in G. In particular, $|TD^{2*}| =$

$|TD^{1*}| + 2k$, where TD^{1*} is a minimum semitotal dominating set of G and TD^{2*} is a minimum semitotal dominating set of G'. Since $\Delta(G) \le 4$, we have $5|TD^1| \ge |V(G)| \ge k$, and so $|TD^2| \le |TD^1| + 2k \le 11|TD^1|$.
Let TD^2 be a semitotal dominating set of G'. We construct a vertex set TD^1 of G as follows. For any $u \in V(G)$ with $d(u) = 4$, let $T(u) = \{u_1, u_2, u_3, u_4, u_5\}$ and $s(u) = |T(u) \cap TD^2|$. For any $u \in V(G)$ with $d(u) \le 3$, let $T(u) = \{u\}$ and $s(u) = |T(u) \cap TD^2|$. If $s(u) = 3$ or $s(u) = 1$, we put u to TD^1. We claim that TD^1 is a semitotal dominating set of G. $|TD^1| \le |TD^2| - 2k$, where k is the number of vertices with degree 4 in G. In particular, we have $|TD^{1*}| \le |TD^{2*}| - 2k$ and thus $|TD^{1*}| = |TD^{2*}| - 2k$ In addition, $|TD^1| - |TD^{1*}| \le |TD^2| - |TD^{2*}|$. As a result, g is an L-reduction with $\alpha = 11$ and $\beta = 1$. $\square$

4. Approximation ratio of semitotal domination

Given a graph G, let $v \in V(G)$ and $\mathcal{A}$ be a family of subset of $V(G)$, we define $\mathcal{F}(\mathcal{A}, v) = \{S : S \cap N[v] \ne \emptyset, S \in \mathcal{A}\}$ and $f(\mathcal{A}, v) = |\mathcal{F}(\mathcal{A}, v)|$.

Algorithm 1: GreedySemiTotalDom(G);
Output: D;
begin
$\mathcal{A} \leftarrow \{\{v_1\}, \{v_2\}, \cdots, \{v_n\}\}$;
$\mathcal{B} \leftarrow \emptyset$;
$D \leftarrow \emptyset$;
$i \leftarrow 0$;
$\mathcal{A}_i \leftarrow \mathcal{A}$;
while($\mathcal{A} \ne \emptyset$)
begin
  find a vertex $v \in V(G) \setminus D$ which maximizes $f(\mathcal{A}, v)$;
  $\mathcal{T} \leftarrow \{S | S \in \mathcal{A}, S \cap N[v] \ne \emptyset\}$;
  $\mathcal{A} \leftarrow \mathcal{A} \setminus \mathcal{T}$;
  if ($\forall b \in \mathcal{B}$, $b \cap N[v] = \emptyset$)
    $\mathcal{A} \leftarrow \mathcal{A} \cup \{N[v]\}$;
  endif
  $\mathcal{B} \leftarrow \mathcal{B} \cup \{N[v]\}$;
  $D \leftarrow D \cup \{v\}$;
  $i \leftarrow i + 1$;
  $\mathcal{A}_i \leftarrow \mathcal{A}$;
end
$g \leftarrow i$;
end.

Remark: Although it seems that $\mathcal{A}_i$ is not used in Algorithm 1, it will be used in the analysis of the approximation ratio of Algorithm 1 for finding the semitotal domination number of a graph in Theorem 3.

Theorem 3. *The SEMITOTAL DOM in any graph $G = (V, E)$ of maximum degree $\Delta(\geq 2)$ can be approximated with an approximation ratio of $2 + \ln(\Delta - 1)$.*

Proof. We proceed with some claims.

Claim 1. $|\mathcal{A}_{i+1}| \leq |\mathcal{A}_i| - 1$ and Algorithm 1 can terminate within finite steps.

Proof. If $\exists b \in \mathcal{B}$ such that $b \cap N[v] \neq \emptyset$, then we have $T \neq \emptyset$ and thus $|\mathcal{A}_{i+1}| \leq |\mathcal{A}| - |\mathcal{T}| \leq |\mathcal{A}_i| - 1$.
If $\forall b \in \mathcal{B}$, $b \cap N[v] = \emptyset$, then $\forall v' \in N[v]$ we have $v' \in \mathcal{A}_i$. Since $|N[v]| \geq 2$, we have $|\mathcal{T}| \geq 2$ and thus $|\mathcal{A}_{i+1}| \leq |\mathcal{A}_i| - |\mathcal{T}| + 1 \leq |\mathcal{A}_i| - 1$. Therefore, we have Algorithm 1 can terminate within finite steps. □

Claim 2. D is a semitotal dominating set of G.

Proof. Firstly, we have $\forall v \in V(G)$, there exist $v' \in D$ such that $v \in N[v']$. Otherwise, $\{v\} \notin \mathcal{T}$ for any iteration of Algorithm 1 (see lines 9 and 10), since $\{v\} \in A_0$ we have $\{v\} \in A_g$, a contradiction.
Secondly, we have $\forall v' \in D$, $\exists v \neq v'$ such that $v \in D$ and $v' \in N[v]$. Otherwise, we assume the vertex v' is selected at the i-th iteration of Algorithm 1, then $N[v'] \in \mathcal{A}_{i+1}$. Since $A_g = \emptyset$, we have there exists a vertex v such that $N[v] \cap N[v'] \neq \emptyset$, a contradiction. □

Claim 3. Algorithm 1 has approximation ratio of $2 + \ln(\Delta - 1)$.

Proof. Let m be the semitotal domination number of G and $U = \{u_1, u_2, \cdots, u_m\}$ be a semitotal dominating set of G with $|U| = m$. If $|\mathcal{A}_0| \leq 2m$, we have Claim 3 holds. Now we only need to consider the case $|\mathcal{A}_0| > 2m$.
Note that g equals $|D|$ which is the output of Algorithm 1, we will show that $g \leq m(2 + \ln(\Delta - 1))$.
Firstly, we show that $|\mathcal{A}_i| - |\mathcal{A}_{i+1}| \geq \frac{|\mathcal{A}_i|}{m} - 1$ if $|\mathcal{A}_i| \geq 2m$. Since $U = \{u_1, u_2, \cdots, u_m\}$ is a semitotal dominating set of G, we have there exist a j such that

$$f(\mathcal{A}_i, u_j) \geq \frac{|\mathcal{A}_i|}{m} > 1.$$

Since $f(\mathcal{A}_i, u_j)$ is an integer, we have $\frac{|\mathcal{A}_i|}{m} \geq 2$ and $u_j \notin D_i$. Now we have

$$\begin{aligned} m|\mathcal{A}_i| - m|\mathcal{A}_{i+1}| &\geq |\mathcal{A}_i| - m, \\ (m-1)|\mathcal{A}_i| &\geq m|\mathcal{A}_{i+1}| - m. \end{aligned}$$

Consequently,

$$\begin{aligned}|\mathcal{A}_{i+1}| &\leq (1-\frac{1}{m})|\mathcal{A}_i|+1,\\ &\leq (1-\frac{1}{m})(1-\frac{1}{m}|\mathcal{A}_{i-1}|+1)+1,\\ &\leq 1+\sum_{j=1}^{i}(1-\frac{1}{m})^j+(1-\frac{1}{m})^{i+1}|\mathcal{A}_0|.\end{aligned}$$

Since $|\mathcal{A}_0| = n$, we have

$$|\mathcal{A}_{i+1}| \leq m\left(1-(1-\frac{1}{m})^{i+1}\right)+(1-\frac{1}{m})^{i+1}n. \tag{1}$$

Since $|\mathcal{A}_0| > 2m$, we have there exists an integer i such that $|\mathcal{A}_i| \geq 2m$ and $|\mathcal{A}_{i+1}| < 2m$. By inequality (1), we have

$$2m \leq |\mathcal{A}_i| \leq m\left(1-(1-\frac{1}{m})^{i}\right)+(1-\frac{1}{m})^{i}n, \tag{2}$$

and thus

$$2 \leq \left(1-(1-\frac{1}{m})^{i}\right)+(1-\frac{1}{m})^{i}\frac{n}{m}. \tag{3}$$

Since $\frac{n}{m} \leq \Delta$ and $(1-\frac{1}{m})^i \leq e^{-\frac{i}{m}}$, we have

$$i \leq m(\ln(\Delta-1)). \tag{4}$$

Since $|\mathcal{A}_{i+1}| < 2m$, we have $g-(i+1) < 2m$. Since $g-i$ is an integer, we have $g \leq i+2m \leq m(2+\ln(\Delta-1))$. □

□

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References

[1] G. Ausiello, P. Crescenzi, G. Gambosi, V. Kann, A. Marchetti-Spaccamela, and M. Protasi, *Complexity and approximation*, Springer, Berlin, 1999.

[2] L. Chen, W. Zeng, and C. Lu, *Np-completeness and apx-completeness of restrained domination in graphs*, Theoret. Comput. Sci. **448** (2012), 1–8.
[3] M.R. Garey and D.S. Johnson, *Computers and intractability: A guide to the theory of npcompleteness (series of books in the mathematical sciences), ed*, Computers and Intractability (1979), 340.
[4] W. Goddard, M.A. Henning, and C.A. McPillan, *Semitotal domination in graphs*, Util. Math. **94** (2014), 67–81.
[5] T.W. Haynes, S. Hedetniemi, and P. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.
[6] O. [Ore, *Theory of graphs*, vol. 38, American Mathematical Society, 1962.
[7] C.H. Papadimitriou and M. Yannakakis, *Optimization, approximation, and complexity classes*, J. Comput. System Sci. **43** (1991), no. 3, 425–440.
[8] E. Zhu, Z. Shao, and J. Xu, *Semitotal domination in claw-free cubic graphs*, Graphs Combin. **33** (2017), no. 5, 1119–1130.