

Complexity and approximation ratio of semitotal domination in graphs

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Abstract: A set $S \subseteq V(G)$ is a semitotal dominating set of a graph G if it is a dominating set of G and every vertex in S is within distance 2 of another vertex of S . The semitotal domination number $\gamma_{t2}(G)$ is the minimum cardinality of a semitotal dominating set of G . We show that the semitotal domination problem is APX-complete for bounded-degree graphs, and the semitotal domination problem in any graph of maximum degree Δ can be approximated with an approximation ratio of $2 + \ln(\Delta - 1)$.

Keywords: semitotal domination, APX-complete, NP-complete

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1. Terminology and introduction

In this paper, we shall only consider graphs without multiple edges or loops or isolated vertices. For a graph G , $S \subseteq V(G)$, $v \in V(G)$, the *open neighborhood* of v in S is denoted by $N_S(v)$ (or simply $N(v)$), i.e. $N_S(v) = \{u : uv \in E(G), u \in S\}$.

Domination and its variations in graphs have attracted considerable attention [3, 5, 6]. A set $S \subseteq V(G)$ is a dominating set of a graph G if every vertex in $V(G) \setminus S$ is adjacent to a vertex in S . The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G . A set $S \subseteq V(G)$ is a semitotal dominating set of a graph G if it is a dominating set of G and every vertex in S is within distance 2 of another vertex of S . The semitotal domination number $\gamma_{t2}(G)$ is the minimum cardinality of a semitotal dominating set of G .

The semitotal domination problem consists of finding the semitotal domination number of a graph G . It has been proved to be NP-complete and was claimed that there

is a linear-time algorithm for trees [4]. Henning studied the semitotal domination in cubic claw-free graphs and proposed a conjecture, and soon the conjecture was confirmed in [8]. In this paper, we continue studying the complexity of the semitotal domination problem and extend these studies by investigating the approximation hardness of the semitotal domination problem in graphs.

2. NP-completeness of semitotal domination

Goddard et al. proved that the semitotal domination problem is NP-complete for general graphs, where the semitotal domination problem is stated as follows.

SEMITOTAL DOMINATION PROBLEM

Input: A graph G , and an integer k .

Question: Is there a semitotal dominating set of G with cardinality at most k ?

We show that it is NP-complete for bipartite, or chordal, or planar graphs via reduction from the dominating set problem.

Theorem 1. *The semitotal domination problem is NP-complete for bipartite, or chordal, or planar graphs.*

Proof. It is clear the semitotal domination problem is in NP, since it is easy to verify a yes instance of the semitotal domination problem in polynomial time. Now let us show how to transform any instance (G, k) of *DOM* into an instance (G', k') of the semitotal domination problem so that G has a dominating set of order k if and only if G' has a semitotal dominating set of order k' .

Let G be an arbitrary graph, we construct a graph G' as follows. For each vertex $v \in V(G)$, we build a star $K_{1,4}^v = \{w_{v,1}, w_{v,2}, w_{v,3}, w_{v,4}\}$ centered at $w_{v,1}$, and add the star $K_{1,4}^v$ and connect $w_{v,1}$ to v . That is to say, $V(G') = V(G) \cup \{w_{v,1}, w_{v,2}, w_{v,3}, w_{v,4} : v \in V(G)\}$, and $E(G') = E(G) \cup \{w_{v,1}w_{v,2}, w_{v,1}w_{v,3}, w_{v,1}w_{v,4}, w_{v,1}v : v \in V(G)\}$.

Suppose that G has a dominating set of D , then we have $D' = D \cup \{w_{v,1} : v \in V(G)\}$ is a semitotal dominating set of G' . It can be seen that $|D'| = |D| + |V(G)|$.

Conversely, suppose that G' has a semitotal dominating set D' . Then we can obtain a semitotal dominating set D'' such that $|D''| \leq |D'|$ and $D'' \supseteq \{w_{v,1} : v \in V(G)\}$ and $D'' \cap \{w_{v,2}, w_{v,3}, w_{v,4} : v \in V(G)\} = \emptyset$. Now, we claim that $D = D'' \setminus \{w_{v,1} : v \in V(G)\}$ is a dominating set of G . It can be seen that $|D| = |D''| - |V(G)|$.

Since G is bipartite (resp. chordal, planar), G' is also bipartite (resp. chordal, planar). Note that the dominating set problem is NP-complete for bipartite, or chordal, or planar graphs, so the semitotal dominating set problem is also NP-complete for such graphs. \square

3. APX-completeness of semitotal domination

The notation of L -reduction can be found in [1, 2, 7]. Given two NP optimization problems G_1 and G_2 and a polynomial time transformation h from instances of G_1 to instances of G_2 , h is said to be an L -reduction if there are positive constants α and β such that for every instance x of G_1 , we have

- (1) $opt_{G_1}(h(x)) \leq \alpha \cdot opt_{G_2}(x)$;
- (2) for every feasible solution y of $h(x)$ with objective value $m_G(h(x), y) = c_2$ we can in polynomial time find a solution y of x with $m_{G_1}(x, y) = c_1$ such that $|opt_{G_1}(x) - c_1| \leq \beta \cdot |opt_{G_2}(h(x)) - c_2|$.

To show that a problem $\mathcal{P} \in APX$ is APX-complete, we need to show that there is an L -reduction from some APX-complete problem to \mathcal{P} . The following problem was proved to be APX-complete (see [2]):

MIN DOM SET-B

Input: A graph $G = (V, E)$ with degree at most B .

Solution: A dominating set S of G .

Measure: Cardinality of the dominating set S .

Now we consider the following problem with $B \in \{3, 4\}$:

SEMITOTAL DOM-B

Input: A graph $G = (V, E)$ with degree at most B .

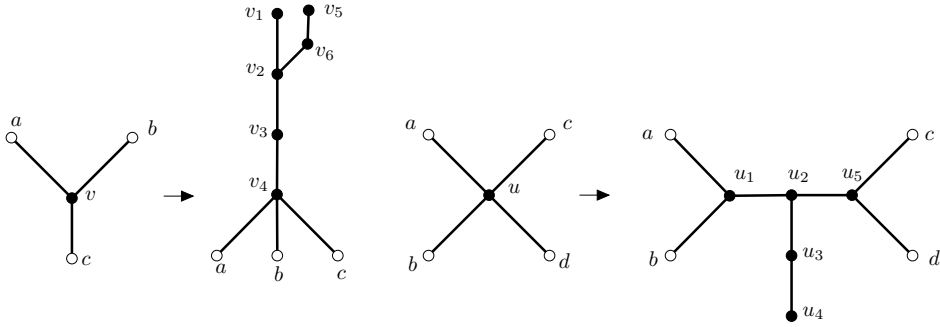
Solution: A semitotal dominating set S of G .

Measure: Cardinality of the semitotal dominating set S .

Theorem 2.

- i) SEMITOTAL DOM-4 is APX-complete;
- ii) SEMITOTAL DOM-3 is APX-complete.

Proof. i) It is clear that SEMITOTAL DOM-4 $\in APX$, and so we just have to show SEMITOTAL DOM-4 is APX-hard. We will construct an L -reduction f from MIN DOM-3 for cubic graphs to SEMITOTAL DOM-4 for graphs with maximum degree 4. Given a cubic graph G , we construct a graph G' with maximum degree 4 in the following way. For each vertex v with $N(v) = \{a, b, c\}$, we split the vertex v and transform to the gadget depicted in Fig. 1 (b).



Let D is a dominating set of G , we construct a vertex set TD of G' in the following way:

- If $v \in D$, then v_2, v_4, v_6 are put to TD .
- If $v \notin D$, then we have that one of $\{a, b, c\}$ is in D and thus v_2, v_6 are put to TD .

It can be checked that TD is a semitotal dominating set of G' and $|TD| = |D| + 2|V(G)|$. In particular, we have $|TD^*| \leq |D^*| + 2|V(G)|$, where TD^* is a minimum semitotal dominating set of G' and D^* is a minimum dominating set of G . It is well known that $\gamma(G) \geq \frac{|V(G)|}{\Delta+1}$, so we have $|V(G)| \leq 5|D^*|$. Therefore, we have $|TD^*| \leq |D^*| + 2|V(G)| \leq 11|D^*|$.

Let TD' be a semitotal dominating set of G' . We construct a vertex set D' of G as follows. Let $T(v) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ for any $v \in V(G)$ and $s(v) = |T(v) \cap TD'|$. If $s(v) = 3$, we put v to D' . We claim that D' is a dominating set of G . $|D'| \leq |TD'| - 2|V(G)|$. In particular, we have $|D^*| \leq |TD^*| - 2|V(G)|$ and thus $|D^*| = |TD^*| - 2|V(G)|$. In addition, $|D| - |D^*| \leq |TD'| - |TD^*|$. As a result, f is an L-reduction with $\alpha = 11$ and $\beta = 1$.

ii) It is clear that SEMITOTAL DOM-3 \in APX, and so we just have to show SEMITOTAL DOM-3 is APX-hard. Since we have shown that SEMITOTAL DOM-4 is APX-complete, we now consider the following L-reduction g from SEMITOTAL DOM-4 to SEMITOTAL DOM-3.

Given a graph G with maximum degree 4, we construct a graph G' with maximum degree 3 in the following way. For each vertex u with degree 4 and $N(u) = \{a, b, c, d\}$, we split the vertex u and transform to the gadget depicted in Fig. 1 (a). Let TD^1 be a semitotal dominating set of G , we construct a vertex set TD^2 of G' in the following way:

- If $u \in TD^1$ and $d(u) \leq 3$, then u is put to TD^2 .
- If $u \in TD^1$ and $d(u) = 4$, then u_1, u_3, u_5 are put to TD^2 .
- If $u \notin TD^1$ and $d(u) = 4$, then u_2, u_3 are put to TD^2 .

It can be checked that TD^2 is a semitotal dominating set of G' and $|TD^2| = |TD^1| + 2k$, where k is the number of vertices with degree 4 in G . In particular, $|TD^{2*}| =$

$|TD^{1*}| + 2k$, where TD^{1*} is a minimum semitotal dominating set of G and TD^{2*} is a minimum semitotal dominating set of G' . Since $\Delta(G) \leq 4$, we have $5|TD^1| \geq |V(G)| \geq k$, and so $|TD^2| \leq |TD^1| + 2k \leq 11|TD^1|$.

Let TD^2 be a semitotal dominating set of G' . We construct a vertex set TD^1 of G as follows. For any $u \in V(G)$ with $d(u) = 4$, let $T(u) = \{u_1, u_2, u_3, u_4, u_5\}$ and $s(u) = |T(u) \cap TD^2|$. For any $u \in V(G)$ with $d(u) \leq 3$, let $T(u) = \{u\}$ and $s(u) = |T(u) \cap TD^2|$. If $s(u) = 3$ or $s(u) = 1$, we put u to TD^1 . We claim that TD^1 is a semitotal dominating set of G . $|TD^1| \leq |TD^2| - 2k$, where k is the number of vertices with degree 4 in G . In particular, we have $|TD^{1*}| \leq |TD^{2*}| - 2k$ and thus $|TD^{1*}| = |TD^{2*}| - 2k$. In addition, $|TD^1| - |TD^{1*}| \leq |TD^2| - |TD^{2*}|$. As a result, g is an L -reduction with $\alpha = 11$ and $\beta = 1$. \square

4. Approximation ratio of semitotal domination

Given a graph G , let $v \in V(G)$ and \mathcal{A} be a family of subset of $V(G)$, we define $\mathcal{F}(\mathcal{A}, v) = \{S : S \cap N[v] \neq \emptyset, S \in \mathcal{A}\}$ and $f(\mathcal{A}, v) = |\mathcal{F}(\mathcal{A}, v)|$.

Algorithm 1: GreedySemiTotalDom(G);

Output: D ;

begin

1: $\mathcal{A} \leftarrow \{\{v_1\}, \{v_2\}, \dots, \{v_n\}\}$;

2: $\mathcal{B} \leftarrow \emptyset$;

3: $D \leftarrow \emptyset$;

4: $i \leftarrow 0$;

5: $\mathcal{A}_i \leftarrow \mathcal{A}$;

6: **while**($\mathcal{A} \neq \emptyset$)

7: **begin**

8: find a vertex $v \in V(G) \setminus D$ which maximizes $f(\mathcal{A}, v)$;

9: $\mathcal{T} \leftarrow \{S | S \in \mathcal{A}, S \cap N[v] \neq \emptyset\}$;

10: $\mathcal{A} \leftarrow \mathcal{A} \setminus \mathcal{T}$;

11: **if** ($\forall b \in \mathcal{B}, b \cap N[v] = \emptyset$)

12: $\mathcal{A} \leftarrow \mathcal{A} \cup \{N[v]\}$;

13: **endif**

14: $\mathcal{B} \leftarrow \mathcal{B} \cup \{N[v]\}$;

15: $D \leftarrow D \cup \{v\}$;

16: $i \leftarrow i + 1$;

17: $\mathcal{A}_i \leftarrow \mathcal{A}$;

18: **end**

19: $g \leftarrow i$;

20:**end.**

Remark: Although it seems that \mathcal{A}_i is not used in Algorithm 1, it will be used in the analysis of the approximation ratio of Algorithm 1 for finding the semitotal

domination number of a graph in Theorem 3.

Theorem 3. *The SEMITOTAL DOM in any graph $G = (V, E)$ of maximum degree $\Delta(\geq 2)$ can be approximated with an approximation ratio of $2 + \ln(\Delta - 1)$.*

Proof. We proceed with some claims.

Claim 1. $|\mathcal{A}_{i+1}| \leq |\mathcal{A}_i| - 1$ and Algorithm 1 can terminate within finite steps.

Proof. If $\exists b \in \mathcal{B}$ such that $b \cap N[v] \neq \emptyset$, then we have $T \neq \emptyset$ and thus $|\mathcal{A}_{i+1}| \leq |\mathcal{A}| - |\mathcal{T}| \leq |\mathcal{A}_i| - 1$.

If $\forall b \in \mathcal{B}$, $b \cap N[v] = \emptyset$, then $\forall v' \in N[v]$ we have $v' \in \mathcal{A}_i$. Since $|N[v]| \geq 2$, we have $|\mathcal{T}| \geq 2$ and thus $|\mathcal{A}_{i+1}| \leq |\mathcal{A}_i| - |\mathcal{T}| + 1 \leq |\mathcal{A}_i| - 1$. Therefore, we have Algorithm 1 can terminate within finite steps. \square

Claim 2. D is a semitotal dominating set of G .

Proof. Firstly, we have $\forall v \in V(G)$, there exist $v' \in D$ such that $v \in N[v']$. Otherwise, $\{v\} \notin \mathcal{T}$ for any iteration of Algorithm 1 (see lines 9 and 10), since $\{v\} \in A_0$ we have $\{v\} \in A_g$, a contradiction.

Secondly, we have $\forall v' \in D$, $\exists v \neq v'$ such that $v \in D$ and $v' \in N[v]$. Otherwise, we assume the vertex v' is selected at the i -th iteration of Algorithm 1, then $N[v'] \in \mathcal{A}_{i+1}$. Since $A_g = \emptyset$, we have there exists a vertex v such that $N[v] \cap N[v'] \neq \emptyset$, a contradiction. \square

Claim 3. Algorithm 1 has approximation ratio of $2 + \ln(\Delta - 1)$.

Proof. Let m be the semitotal domination number of G and $U = \{u_1, u_2, \dots, u_m\}$ be a semitotal dominating set of G with $|U| = m$. If $|\mathcal{A}_0| \leq 2m$, we have Claim 3 holds. Now we only need to consider the case $|\mathcal{A}_0| > 2m$.

Note that g equals $|D|$ which is the output of Algorithm 1, we will show that $g \leq m(2 + \ln(\Delta - 1))$.

Firstly, we show that $|\mathcal{A}_i| - |\mathcal{A}_{i+1}| \geq \frac{|\mathcal{A}_i|}{m} - 1$ if $|\mathcal{A}_i| \geq 2m$. Since $U = \{u_1, u_2, \dots, u_m\}$ is a semitotal dominating set of G , we have there exist a j such that

$$f(\mathcal{A}_i, u_j) \geq \frac{|\mathcal{A}_i|}{m} > 1.$$

Since $f(\mathcal{A}_i, u_j)$ is an integer, we have $\frac{|\mathcal{A}_i|}{m} \geq 2$ and $u_j \notin D_i$. Now we have

$$\begin{aligned} m|\mathcal{A}_i| - m|\mathcal{A}_{i+1}| &\geq |\mathcal{A}_i| - m, \\ (m-1)|\mathcal{A}_i| &\geq m|\mathcal{A}_{i+1}| - m. \end{aligned}$$

Consequently,

$$\begin{aligned} |\mathcal{A}_{i+1}| &\leq \left(1 - \frac{1}{m}\right)|\mathcal{A}_i| + 1, \\ &\leq \left(1 - \frac{1}{m}\right)\left(1 - \frac{1}{m}|\mathcal{A}_{i-1}| + 1\right) + 1, \\ &\leq 1 + \sum_{j=1}^i \left(1 - \frac{1}{m}\right)^j + \left(1 - \frac{1}{m}\right)^{i+1}|\mathcal{A}_0|. \end{aligned}$$

Since $|\mathcal{A}_0| = n$, we have

$$|\mathcal{A}_{i+1}| \leq m \left(1 - \left(1 - \frac{1}{m}\right)^{i+1}\right) + \left(1 - \frac{1}{m}\right)^{i+1}n. \quad (1)$$

Since $|\mathcal{A}_0| > 2m$, we have there exists an integer i such that $|\mathcal{A}_i| \geq 2m$ and $|\mathcal{A}_{i+1}| < 2m$. By inequality (1), we have

$$2m \leq |\mathcal{A}_i| \leq m \left(1 - \left(1 - \frac{1}{m}\right)^i\right) + \left(1 - \frac{1}{m}\right)^i n, \quad (2)$$

and thus

$$2 \leq \left(1 - \left(1 - \frac{1}{m}\right)^i\right) + \left(1 - \frac{1}{m}\right)^i \frac{n}{m}. \quad (3)$$

Since $\frac{n}{m} \leq \Delta$ and $\left(1 - \frac{1}{m}\right)^i \leq e^{-\frac{i}{m}}$, we have

$$i \leq m(\ln(\Delta - 1)). \quad (4)$$

Since $|\mathcal{A}_{i+1}| < 2m$, we have $g - (i + 1) < 2m$. Since $g - i$ is an integer, we have $g \leq i + 2m \leq m(2 + \ln(\Delta - 1))$. \square

\square

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