Complexity and approximation ratio of semitotal domination in graphs

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Abstract: A set \( S \subseteq V(G) \) is a semitotal dominating set of a graph \( G \) if it is a dominating set of \( G \) and every vertex in \( S \) is within distance 2 of another vertex of \( S \). The semitotal domination number \( \gamma_{t2}(G) \) is the minimum cardinality of a semitotal dominating set of \( G \). We show that the semitotal domination problem is APX-complete for bounded-degree graphs, and the semitotal domination problem in any graph of maximum degree \( \Delta \) can be approximated with an approximation ratio of \( 2 + \ln(\Delta - 1) \).

Keywords: semitotal domination, APX-complete, NP-complete

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1. Terminology and introduction

In this paper, we shall only consider graphs without multiple edges or loops or isolated vertices. For a graph \( G \), \( S \subseteq V(G) \), \( v \in V(G) \), the open neighborhood of \( v \) in \( S \) is denoted by \( N_S(v) \) (or simply \( N(v) \)), i.e. \( N_S(v) = \{ u : uv \in E(G), u \in S \} \).

Domination and its variations in graphs have attracted considerable attention \([3, 5, 6]\). A set \( S \subseteq V(G) \) is a dominating set of a graph \( G \) if every vertex in \( V(G) \setminus S \) is adjacent to a vertex in \( S \). The domination number \( \gamma(G) \) is the minimum cardinality of a dominating set of \( G \). A set \( S \subseteq V(G) \) is a semitotal dominating set of a graph \( G \) if it is a dominating set of \( G \) and every vertex in \( S \) is within distance 2 of another vertex of \( S \). The semitotal domination number \( \gamma_{t2}(G) \) is the minimum cardinality of a semitotal dominating set of \( G \).

The semitotal domination problem consists of finding the semitotal domination number of a graph \( G \). It has been proved to be NP-complete and was claimed that there
is a linear-time algorithm for trees [4]. Henning studied the semitotal domination in cubic claw-free graphs and proposed a conjecture, and soon the conjecture was confirmed in [8]. In this paper, we continue studying the complexity of the semitotal domination problem and extend these studies by investigating the approximation hardness of the semitotal domination problem in graphs.

2. NP-completeness of semitotal domination

Goddard et al. proved that the semitotal domination problem is NP-complete for general graphs, where the semitotal domination problem is stated as follows.

**Semitotal Domination Problem**

*Input:* A graph $G$, and an integer $k$.

*Question:* Is there a semitotal dominating set of $G$ with cardinality at most $k$?

We show that it is NP-complete for bipartite, or chordal, or planar graphs via reduction from the dominating set problem.

**Theorem 1.** The semitotal domination problem is NP-complete for bipartite, or chordal, or planar graphs.

**Proof.** It is clear the semitotal domination problem is in NP, since it is easy to verify a yes instance of the semitotal domination problem in polynomial time. Now let us show how to transform any instance $(G, k)$ of $\text{DOM}$ into an instance $(G', k')$ of the semitotal domination problem so that $G$ has a dominating set of order $k$ if and only if $G'$ has a semitotal dominating set of order $k'$.

Let $G$ be an arbitrary graph, we construct a graph $G'$ as follows. For each vertex $v \in V(G)$, we build a star $K^v_{1,4} = \{w_{v,1}, w_{v,2}, w_{v,3}, w_{v,4}\}$ centered at $w_{v,1}$, and add the star $K^v_{1,4}$ and connect $w_{v,1}$ to $v$. That is to say, $V(G') = V(G) \cup \{w_{v,1}, w_{v,2}, w_{v,3}, w_{v,4} : v \in V(G)\}$, and $E(G') = E(G) \cup \{w_{v,1}w_{v,2}, w_{v,1}w_{v,3}, w_{v,1}w_{v,4}, w_{v,1}v : v \in V(G)\}$.

Suppose that $G$ has a dominating set of $D$, then we have $D' = D \cup \{w_{v,1} : v \in V(G)\}$ is a semitotal dominating set of $G'$. It can be seen that $|D'| = |D| + |V(G)|$.

Conversely, suppose that $G'$ has a semitotal dominating set $D'$. Then we can obtain a semitotal dominating set $D''$ such that $|D''| \leq |D'|$ and $D'' \supseteq \{w_{v,1} : v \in V(G)\}$ and $D'' \cap \{w_{v,2}, w_{v,3}, w_{v,4} : v \in V(G)\} = \emptyset$. Now, we claim that $D = D'' \setminus \{w_{v,1} : v \in V(G)\}$ is a dominating set of $G$. It can be seen that $|D| = |D''| - |V(G)|$.

Since $G$ is bipartite (resp. chordal, planar), $G'$ is also bipartite (resp. chordal, planar). Note that the dominating set problem is NP-complete for bipartite, or chordal, or planar graphs, so the semitotal dominating set problem is also NP-complete for such graphs.

\[\square\]
3. APX-completeness of semitotal domination

The notation of \( L \)-reduction can be found in [1, 2, 7]. Given two \( NP \) optimization problems \( G_1 \) and \( G_2 \) and a polynomial time transformation \( h \) from instances of \( G_1 \) to instances of \( G_2 \), \( h \) is said to be an \( L \)-reduction if there are positive constants \( \alpha \) and \( \beta \) such that for every instance \( x \) of \( G_1 \), we have

\[
(1) \quad \text{opt}_{G_1}(h(x)) \leq \alpha \cdot \text{opt}_{G_2}(x);
\]

\[
(2) \quad \text{for every feasible solution } y \text{ of } h(x) \text{ with objective value } m_G(h(x), y) = c_2 \text{ we can in polynomial time find a solution } y \text{ of } x \text{ with } m_{G_1}(x, y) = c_1 \text{ such that } |\text{opt}_{G_1}(x) - c_1| \leq \beta \cdot |\text{opt}_{G_2}(h(x)) - c_2|.
\]

To show that a problem \( P \in APX \) is APX-complete, we need to show that there is an \( L \)-reduction from some APX-complete problem to \( P \). The following problem was proved to be APX-complete (see [2]):

\[
\begin{align*}
\text{MIN DOM SET-B} \\
\text{Input:} & \quad \text{A graph } G = (V, E) \text{ with degree at most } B. \\
\text{Solution:} & \quad \text{A dominating set } S \text{ of } G. \\
\text{Measure:} & \quad \text{Cardinality of the dominating set } S.
\end{align*}
\]

Now we consider the following problem with \( B \in \{3, 4\} \):

\[
\begin{align*}
\text{SEMITOTAL DOM-B} \\
\text{Input:} & \quad \text{A graph } G = (V, E) \text{ with degree at most } B. \\
\text{Solution:} & \quad \text{A semitotal dominating set } S \text{ of } G. \\
\text{Measure:} & \quad \text{Cardinality of the semitotal dominating set } S.
\end{align*}
\]

**Theorem 2.**

i) \( \text{SEMITOTAL DOM-4 is APX-complete} \);

ii) \( \text{SEMITOTAL DOM-3 is APX-complete} \).

**Proof.** i) It is clear that \( \text{SEMITOTAL DOM-4} \in APX \), and so we just have to show \( \text{SEMITOTAL DOM-4} \) is APX-hard. We will construct an \( L \)-reduction \( f \) from MIN DOM-3 for cubic graphs to SEMITOTAL DOM-4 for graphs with maximum degree 4. Given a cubic graph \( G \), we construct a graph \( G' \) with maximum degree 4 in the following way. For each vertex \( v \) with \( N(v) = \{a, b, c\} \), we split the vertex \( v \) and transform to the gadget depicted in Fig. 1 (b).
Let $D$ is a dominating set of $G$, we construct a vertex set $TD$ of $G'$ in the following way:

- If $v \in D$, then $v_2, v_4, v_6$ are put to $TD$.
- If $v \notin D$, then we have that one of $\{a, b, c\}$ is in $D$ and thus $v_2, v_6$ are put to $TD$.

It can be checked that $TD$ is a semitotal dominating set of $G'$ and $|TD| = |D| + 2|V(G)|$. In particular, we have $|TD^*| \leq |D^*| + 2|V(G)|$, where $TD^*$ is a minimum semitotal dominating set of $G'$ and $D^*$ is a minimum dominating set of $G$. It is well known that $\gamma(G) \geq \frac{|V(G)|}{\Delta + 1}$, so we have $|V(G)| \leq 5|D^*|$. Therefore, we have $|TD^*| \leq |D^*| + 2|V(G)| \leq 11|D^*|$. Let $TD'$ be a semitotal dominating set of $G'$. We construct a vertex set $D'$ of $G$ as follows. Let $T(v) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ for any $v \in V(G)$ and $s(v) = |T(v) \cap TD'|$. If $s(v) = 3$, we put $v$ to $D'$. We claim that $D'$ is a dominating set of $G$. $|D'| \leq |TD'| - 2|V(G)|$. In particular, we have $|D^*| \leq |TD^*| - 2|V(G)|$ and thus $|D^*| = |TD^*| - 2|V(G)|$. In addition, $|D| - |D^*| \leq |TD' - |TD^*|$. As a result, $f$ is an L-reduction with $\alpha = 11$ and $\beta = 1$.

ii) It is clear that SEMITOTAL DOM-3 $\in APX$, and so we just have to show SEMITOTAL DOM-3 is APX-hard. Since we have shown that SEMITOTAL DOM-4 is APX-complete, we now consider the following L-reduction $g$ from SEMITOTAL DOM-4 to SEMITOTAL DOM-3.

Given a graph $G$ with maximum degree 4, we construct a graph $G'$ with maximum degree 3 in the following way. For each vertex $u$ with degree 4 and $N(u) = \{a, b, c, d\}$, we split the vertex $u$ and transform to the gadget depicted in Fig. 1 (a). Let $TD^1$ be a semitotal dominating set of $G$, we construct a vertex set $TD^2$ of $G'$ in the following way:

- If $u \in TD^1$ and $d(u) \leq 3$, then $u$ is put to $TD^2$.
- If $u \in TD^1$ and $d(u) = 4$, then $u_1, u_3, u_5$ are put to $TD^2$.
- If $u \notin TD^1$ and $d(u) = 4$, then $u_2, u_3$ are put to $TD^2$.

It can be checked that $TD^2$ is a semitotal dominating set of $G'$ and $|TD^2| = |TD^1| + 2k$, where $k$ is the number of vertices with degree 4 in $G$. In particular, $|TD^2^*| = 2k + 2$. The gadget is in the limit subcase of the following L-reduction $h$ from the problem of finding a minimum dominating set of degree 4 in $G$. Since $G$ is $\alpha$-dominated, we have $|TD^2| = |TD^1| + \alpha$, where $TD^1$ is the dominating set of $G$. In particular, $|TD^2^*| = 2k + 2\alpha$. Therefore, we have $|TD^2^*| = 2k + 2\alpha$.
\[ |TD^{1*}| + 2k, \text{ where } TD^{1*} \text{ is a minimum semitotal dominating set of } G \text{ and } TD^{2*} \text{ is a minimum semitotal dominating set of } G'. \text{ Since } \Delta(G) \leq 4, \text{ we have } 5|TD^{1}| \geq |V(G)| \geq k, \text{ and so } |TD^{2}| \leq |TD^{1}| + 2k \leq 11|TD^{1}|. \]

Let \( TD^2 \) be a semitotal dominating set of \( G' \). We construct a vertex set \( TD^1 \) of \( G \) as follows. For any \( u \in V(G) \) with \( d(u) = 4 \), let \( T(u) = \{u_1, u_2, u_3, u_4, u_5\} \) and \( s(u) = |T(u) \cap TD^2| \). For any \( u \in V(G) \) with \( d(u) \leq 3 \), let \( T(u) = \{u\} \) and \( s(u) = |T(u) \cap TD^2| \). If \( s(u) = 3 \) or \( s(u) = 1 \), we put \( u \) to \( TD^1 \). We claim that \( TD^1 \) is a semitotal dominating set of \( G \).

\[ |TD^1| \leq |TD^2| - 2k, \text{ where } k \text{ is the number of vertices with degree } 4 \text{ in } G. \text{ In particular, we have } |TD^{1*}| \leq |TD^{2*}| - 2k \text{ and thus } |TD^{1*}| = |TD^{2*}| - 2k \text{ In addition, } |TD^1| - |TD^{1*}| \leq |TD^2| - |TD^{2*}|. \text{ As a result, } g \text{ is an } L\text{-reduction with } \alpha = 11 \text{ and } \beta = 1. \]  

4. Approximation ratio of semitotal domination

Given a graph \( G \), let \( v \in V(G) \) and \( A \) be a family of subset of \( V(G) \), we define \( F(A, v) = \{S : S \cap N[v] \neq \emptyset, S \in A\} \) and \( f(A, v) = |F(A, v)| \).

Algorithm 1: GreedySemiTotalDom\((G)\);
Output: \( D \);
begin
1: \( A \leftarrow \{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\} \); 
2: \( B \leftarrow \emptyset \); 
3: \( D \leftarrow \emptyset \); 
4: \( i \leftarrow 0 \); 
5: \( A_i \leftarrow A \); 
6: while\( (A \neq \emptyset) \) 
7: begin
8: find a vertex \( v \in V(G) \setminus D \) which maximizes \( f(A, v) \); 
9: \( T \leftarrow \{S : S \in A, S \cap N[v] \neq \emptyset\} \); 
10: \( A \leftarrow A \setminus T \); 
11: if \( (\forall b \in B, b \cap N[v] = \emptyset) \) 
12: \( A \leftarrow A \cup \{N[v]\} \); 
13: endif 
14: \( B \leftarrow B \cup \{N[v]\} \); 
15: \( D \leftarrow D \cup \{v\} \); 
16: \( i \leftarrow i + 1 \); 
17: \( A_i \leftarrow A \); 
18: end 
19: \( g \leftarrow i \); 
20:end.

Remark: Although it seems that \( A_i \) is not used in Algorithm 1, it will be used in the analysis of the approximation ratio of Algorithm 1 for finding the semitotal
domination number of a graph in Theorem 3.

**Theorem 3.** The SEMITOTAL DOM in any graph $G = (V, E)$ of maximum degree $\Delta(\geq 2)$ can be approximated with an approximation ratio of $2 + \ln(\Delta - 1)$.

**Proof.** We proceed with some claims.

**Claim 1.** $|A_{i+1}| \leq |A_i| - 1$ and Algorithm 1 can terminate within finite steps.

**Proof.** If $\exists b \in B$ such that $b \cap N[v] \neq \emptyset$, then we have $T \neq \emptyset$ and thus $|A_{i+1}| \leq |A| - |T| \leq |A_i| - 1$.

If $\forall b \in B$, $b \cap N[v] = \emptyset$, then $\forall v' \in N[v]$ we have $v' \in A_i$. Since $|N[v]| \geq 2$, we have $|T| \geq 2$ and thus $|A_{i+1}| \leq |A_i| - |T| + 1 \leq |A_i| - 1$. Therefore, we have Algorithm 1 can terminate within finite steps.

**Claim 2.** $D$ is a semitotal dominating set of $G$.

**Proof.** Firstly, we have $\forall v \in V(G)$, there exist $v' \in D$ such that $v \in N[v']$. Otherwise, $\{v\} \notin T$ for any iteration of Algorithm 1 (see lines 9 and 10), since $\{v\} \in A_0$ we have $\{v\} \in A_g$, a contradiction.

Secondly, we have $\forall v' \in D$, $\exists v \neq v'$ such that $v \in D$ and $v' \in N[v]$. Otherwise, we assume the vertex $v'$ is selected at the $i$-th iteration of Algorithm 1, then $N[v'] \in A_{i+1}$. Since $A_g = \emptyset$, we have there exists a vertex $v$ such that $N[v] \cap N[v'] \neq \emptyset$, a contradiction.

**Claim 3.** Algorithm 1 has approximation ratio of $2 + \ln(\Delta - 1)$.

**Proof.** Let $m$ be the semitotal domination number of $G$ and $U = \{u_1, u_2, \ldots, u_m\}$ be a semitotal dominating set of $G$ with $|U| = m$. If $|A_0| \leq 2m$, we have Claim 3 holds. Now we only need to consider the case $|A_0| > 2m$.

Note that $g$ equals $|D|$ which is the output of Algorithm 1, we will show that $g \leq m(2 + \ln(\Delta - 1))$.

Firstly, we show that $|A_i| - |A_{i+1}| \geq \frac{|A_i|}{m} - 1$ if $|A_i| \geq 2m$. Since $U = \{u_1, u_2, \ldots, u_m\}$ is a semitotal dominating set of $G$, we have there exist a $j$ such that

$$f(A_i, u_j) \geq \frac{|A_i|}{m} > 1.$$ 

Since $f(A_i, u_j)$ is an integer, we have $\frac{|A_i|}{m} \geq 2$ and $u_j \notin D_i$. Now we have

$$m|A_i| - m|A_{i+1}| \geq |A_i| - m,$$

$$(m - 1)|A_i| \geq m|A_{i+1}| - m.$$
Consequently,
\[ |A_{i+1}| \leq (1 - \frac{1}{m})|A_i| + 1, \]
\[ \leq (1 - \frac{1}{m})(1 - \frac{1}{m}|A_{i-1}| + 1) + 1, \]
\[ \leq 1 + \sum_{j=1}^{i} (1 - \frac{1}{m})^j + (1 - \frac{1}{m})^{i+1}|A_0|. \]

Since \(|A_0| = n\), we have
\[ |A_{i+1}| \leq m \left(1 - (1 - \frac{1}{m})^{i+1}\right) + (1 - \frac{1}{m})^{i+1}n. \] (1)

Since \(|A_0| > 2m\), we have there exists an integer \(i\) such that \(|A_i| \geq 2m\) and \(|A_{i+1}| < 2m\). By inequality (1), we have
\[ 2m \leq |A_i| \leq m \left(1 - (1 - \frac{1}{m})^{i}\right) + (1 - \frac{1}{m})^{i}n, \] (2)

and thus
\[ 2 \leq \left(1 - (1 - \frac{1}{m})^{i}\right) + (1 - \frac{1}{m})^{i} \frac{n}{m}. \] (3)

Since \(\frac{n}{m} \leq \Delta\) and \((1 - \frac{1}{m})^{i} \leq e^{-\frac{i}{m}}\), we have
\[ i \leq m(\ln(\Delta - 1)). \] (4)

Since \(|A_{i+1}| < 2m\), we have \(g - (i + 1) < 2m\). Since \(g - i\) is an integer, we have \(g \leq i + 2m \leq m(2 + \ln(\Delta - 1))\).

\[ \square \]

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