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Total k-Rainbow domination numbers in graphs

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Abstract: Let $k \geq 1$ be an integer, and let G be a graph. A *k-rainbow dominating function* (or a *k-RDF*) of G is a function f from the vertex set $V(G)$ to the family of all subsets of $\{1, 2, \ldots, k\}$ such that for every $v \in V(G)$ with $f(v) = \emptyset$, the condition $\bigcup_{u \in N_G(v)} f(u) = \{1, 2, \ldots, k\}$ is fulfilled, where $N_G(v)$ is the open neighborhood of v. The *weight* of a k-RDF f of G is the value $\omega(f) = \sum_{v \in V(G)} |f(v)|$. A k-rainbow dominating function f in a graph with no isolated vertex is called a *total k-rainbow dominating function* if the subgraph of G induced by the set $\{v \in V(G) \mid f(v) \neq \emptyset\}$ has no isolated vertices. The *total k-rainbow domination number* of G, denoted by $\gamma_{trk}(G)$, is the minimum weight of a total k-rainbow dominating function on G. The total 1-rainbow domination is the same as the total domination. In this paper we initiate the study of total k-rainbow domination number and we investigate its basic properties. In particular, we present some sharp bounds on the total k-rainbow domination number and we determine the total k-rainbow domination number of some classes of graphs.

Keywords: total k-rainbow dominating function; total k-rainbow domination number

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1. Introduction and preliminaries

For terminology and notation on graph theory not given here, the reader is referred to [7, 8]. In this paper, G is a simple graph without isolated vertices, with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$ and the size $|E|$ of G is denoted by $m = m(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. The *minimum degree* and the *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A leaf is a vertex of degree one and a stem is a vertex adjacent to a leaf. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v\in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. The *complement* $\overline{G}$ of G is the simple graph with vertex set $V(G)$ defined by $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. We write K_n for the *complete graph* of order n, C_n for a *cycle* of order n and P_n for a *path* of order n. A *matching* M of a graph G is a subset of the edges E, such that no two edges in M have a common vertex. The *matching number* $\alpha'(G)$ of G is the maximum cardinality of a matching in G.

Let $k \geq 1$ be an integer, and set $[k] := \{1, 2, \ldots, k\}$. A function $f : V(G) \to 2^{[k]}$ is a *k-rainbow dominating function* (or a *k-RDF*) of G if for every vertex $v \in V(G)$ with $f(v) = \emptyset$, the condition $\bigcup_{u\in N_G(v)} f(u) = [k]$ is fulfilled. The *weight* of a k-RDF f on G is the value $\omega(f) := \sum_{v\in V(G)} |f(v)|$. The *$k$-rainbow domination number* of G, denoted by $\gamma_{rk}(G)$, is the minimum weight of a k-RDF on G. A k-RDF f on G is called a *γ_{rk}-function* if $\omega(f) = \gamma_{rk}(G)$. This concept was introduced by Brešar, Henning and Rall [1] and has been studied by several authors [2–6, 13, 15–20].

A set S of vertices of a graph G with minimum degree $\delta(G) > 0$ is a *total dominating set* if $N(S) = V(G)$. The minimum cardinality of a total dominating set, denoted by $\gamma_t(G)$, is called the *total domination number* of G. A *$\gamma_t(G)$-set* is a total dominating set of G of cardinality $\gamma_t(G)$. The literature on this subject has been surveyed in [10, 11].

In a graph G if we think of each vertex x as the possible location for a guard capable of protecting each vertex in its closed neighborhood $N[x]$, then the total domination concept in a graph represents situations in which every location requires the presence of one guard in a neighboring location.

Here we assume a more complex situation that, for example, there are different types of guards and it is required that each location which is occupied by no guard has all types of guards in its neighborhood and every location which is occupied by at least one guard requires the presence of one guard in a neighboring location. Here, we introduced the total k-rainbow domination concept to consider such situation. A k-rainbow dominating function f on G, is called a *total k-rainbow dominating function* (or a *Tk-RDF*) if the subgraph of G induced by the set $\{v \in V(G) \mid f(v) \neq \emptyset\}$ has no isolated vertex. The *total k-rainbow domination number* of G, denoted by $\gamma_{trk}(G)$, is the minimum weight of a Tk-RDF of G. A Tk-RDF f of G is a *γ_{trk}-function* if $\omega(f) = \gamma_{trk}(G)$. Note that $\gamma_{tr1}(G)$ is equal to the classical total domination number, denoted by $\gamma_t(G)$. If $G_1, G_2, \ldots, G_s$ are the components of G, then $\gamma_{trk}(G) = \sum_{i=1}^{s} \gamma_{trk}(G_i)$.

Hence, it is sufficient to study $\gamma_{trk}(G)$ for connected graphs.
Since every Tk-RDF f of a graph G is a kRDF of G, we have

$$\gamma_{rk}(G) \leq \gamma_{trk}(G). \tag{1}$$

In this paper, we initiate the study of the total k-rainbow domination number and we investigate its basic properties. In particular, we present some sharp bounds for the total k-rainbow domination number and determine its value for some classes of graphs.
For any graph G with $\delta(G) \geq 1$ and any γ_{trk}-function f on G, let $V_i^f = \{v : |f(v)| = i\}$ for each $i = 0, 1, \ldots, k$.

Observation 1. Let G be a graph of order n and let f be a γ_{trk}-function on G. Then the following holds.

(i) $n = \sum_{i=0}^{k} |V_i^f|$,

(ii) $\gamma_{trk}(G) = \sum_{i=1}^{k} i|V_i^f|$, and

(iii) $|V_0^f| \geq \sum_{i=2}^{k}(i-1)|V_i^f|$.

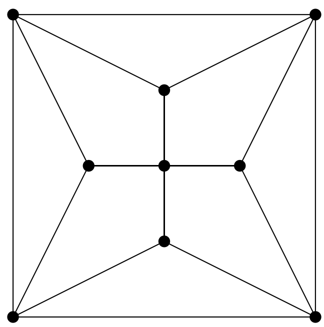

Figure 1. **The graph F**

Theorem A. *([9]) If G is a planar graph with* $\mathrm{daim}(G) = 2$, *then* $\gamma(G) \leq 2$ *or* $G = F$ *where F is the graph illustrated in Figure 1.*

2. Basic properties and bounds

In this section we present basic properties of the total k-rainbow domination number of a graph and give some sharp bounds on the total k-rainbow domination number. First, we study the relation between total domination number and total k-rainbow domination number.

Theorem 2. Let $k \geq 2$ be an integer and let G be a connected graph of order $n \geq k$. Then

$$\gamma_t(G) \leq \gamma_{trk}(G) \leq k\gamma_t(G).$$

Moreover,

(a) The left equality holds if and only if G has a $\gamma_t(G)$-set D such that the induced subgraph $G[D]$ is 1-regular and D can be partitioned into k nonempty subsets $D_1, D_2, \ldots, D_k$ so that $V(G) - D \subseteq N(D_i)$ for $i = 1, 2, \ldots, k$.

(b) The right equality holds if and only if G has a $\gamma_{trk}(G)$-function f such that for each $v \in V(G)$, either $f(v) = \{1, 2, \ldots, k\}$ or $f(v) = \emptyset$.

Proof. To prove the lower bound, let f be a $\gamma_{trk}(G)$-function and let $V_0 = \{v \in V(G) \mid f(v) = \emptyset\}$. It is easy to verify that $V(G) - V_0$ is a total dominating set of G and so $\gamma_t(G) \leq |V(G) - V_0| \leq \omega(f) = \gamma_{trk}(G)$. To prove the upper bound, let S be a $\gamma_t(G)$-set and define $g : V(G) \to 2^{[k]}$ by $g(x) = \{1, 2, \ldots, k\}$ for $x \in S$ and $g(x) = \emptyset$ for $x \in V(G) - S$. Obviously g is a Tk-RDF of G and hence $\gamma_{trk}(G) \leq \omega(g) = k\gamma_t(G)$.

(a) Assume that $\gamma_t(G) = \gamma_{trk}(G)$. If $\gamma_t(G) = \gamma_{trk}(G) = n$, then it is not hard to see that G is a 1-regular graph and the assertion is trivial. We may assume, therefore, that $\gamma_t(G) = \gamma_{trk}(G) < n$. Let f be a $\gamma_{trk}(G)$-function, $D_0 = \{v \in V(G) \mid f(v) = \emptyset\}$ $D_i = \{v \in V(G) \mid f(v) = \{i\}\}$ for each $i \in \{1, \ldots, k\}$ and $D_{k+1} = V(G) \setminus \bigcup_{i=0}^{k} D_i$. Since $\gamma_{trk}(G) < n$, we have $D_0 \neq \emptyset$. It is easy to see that $D = \cup_{i=1}^{k+1} D_i$ is a total dominating set of G and so

$$\gamma_t(G) \leq \sum_{i=1}^{k+1} |D_i| \leq \sum_{i=1}^{k} |D_i| + 2|D_{k+1}| \leq \omega(f) = \gamma_{trk}(G). \tag{2}$$

Since $\gamma_t(G) = \gamma_{trk}(G)$, we have equality throughout the inequality chain (2). Hence $|D_{k+1}| = 0$ and $\gamma_t(G) = \sum_{i=1}^{k} |D_i|$ implying that D is a $\gamma_t(G)$-set. Therefore every vertex in D_0 has at least one neighbor in D_i for each $i \in \{1, 2, \ldots, k\}$. Thus $D_i \neq \emptyset$ and $D_0 = V(G) \setminus D \subseteq N(D_i)$ for each $i \in \{1, 2, \ldots, k\}$. If the subgraph of G induced by D, $G[D]$, has a component of order at least 3, say G_1, then for any leaf $v \in V(G_1)$ in some spanning tree of G_1, $D - \{v\}$ is a total dominating set of G that leads to a contradiction. Thus $G[D]$ is 1-regular graph and the assertion holds.

Conversely, let G has a $\gamma_t(G)$-set D such that the induced subgraph $G[D]$ is 1-regular and D can be partitioned into k nonempty subsets $D_1, D_2, \ldots, D_k$ so that $V(G) - D \subseteq N(D_i)$ for each $i = 1, 2, \ldots, k$. Define $g : V(G) \to 2^{[k]}$ by $g(x) = \{i\}$ for each $x \in D_i$ $(i = 1, 2, \ldots, k)$ and $g(x) = \emptyset$ otherwise. Clearly g is a total k-rainbow dominating function of G of weight $|D|$ and so $\gamma_{trk}(G) \leq \gamma_t(G)$. On the other hand, as proven earlier, $\gamma_t(G) \leq \gamma_{trk}(G)$. It follows that $\gamma_t(G) = \gamma_{trk}(G)$.

(b) Assume that $\gamma_{trk}(G) = k\gamma_t(G)$. Let D be a $\gamma_t(G)$-set and define $f : V(G) \to 2^{[k]}$ by $f(x) = \{1, 2, \ldots, k\}$ for each $x \in D$ and $f(x) = \emptyset$ otherwise. Obviously f is a total k-rainbow dominating function of G of weight $k\gamma_t(G)$. This implies that f is a $\gamma_{trk}(G)$-function such that for each $v \in V(G)$, either $f(v) = \{1, 2, \ldots, k\}$ or $f(v) = \emptyset$.

Conversely, let G have a $\gamma_{trk}(G)$-function f such that for each $v \in V(G)$, either $f(v) = \{1, 2, \ldots, k\}$ or $f(v) = \emptyset$. Let $D = \{v \in V(G) \mid f(v) = \{1, 2, \ldots, k\}\}$. Then clearly $\gamma_{trk}(G) = k|D|$ and D is a total dominating set of G. Hence $k\gamma_t(G) \leq k|D| = \gamma_{trk}(G)$. On the other hand, as proven earlier, $\gamma_{trk}(G) \leq k\gamma_t(G)$. Hence $k\gamma_t(G) = \gamma_{trk}(G)$ and the proof is complete.

□

Proposition 1. Let $k \geq 1$ be an integer. If G is a connected graph of order $n \geq 2$, then

$$\min\{k, n\} \leq \gamma_{trk}(G) \leq n.$$

In particular, $\gamma_{trn}(G) = n$.

Proof. Let f be a $\gamma_{trk}(G)$-function. If there exists a vertex v such that $f(v) = \emptyset$, then the definition yields to $f(N(v)) = [k]$ and thus $k \leq \gamma_{trk}(G)$. If $|f(v)| \geq 1$ for all vertices $v \in V(G)$, then $n \leq \gamma_{trk}(G)$, and the first inequality is proved.
Now consider the function g, defined by $g(v) = \{1\}$ for each $v \in V(G)$. Clearly, g is a total k-rainbow dominating function of weight n, and so $\gamma_{trk}(G) \leq n$. □

Next result provide a sufficient condition to have $\gamma_{trk}(G) = n$.

Theorem 3. Let G be a graph of order $n \geq 2$ with $k > \Delta(G)^2 - \Delta(G)$. Then $\gamma_{trk}(G) = n$.

Proof. Suppose, to the contrary, that $\gamma_{trk}(G) < n$. Let f be a $\gamma_{trk}(G)$-function such that $|V(G) - V_0^f|$ is as small as possible. Since $\gamma_{trk}(G) < n$, there is a vertex $v \in V(G)$ such that $f(v) = \emptyset$. We conclude from $k > \Delta(G)^2 - \Delta(G)$ and $\bigcup_{w \in N_G(v)} f(w) = [k]$ that v has a neighbor u for which $|f(u)| \geq \Delta(G)$. Suppose without loss of generality that $\{1, 2, \ldots, \Delta(G)\} \subseteq f(u)$. Let $N(u) = \{u_1, u_2 \ldots, u_{\deg(u)}\}$ where $u_1 = v$. Since f is a Tk-RDF, we may assume $f(u_{\deg(u)}) \neq \emptyset$. Define the function $g : V(G) \to 2^{[k]}$ by $g(u_i) = f(u_i) \cup \{i\}$ for $1 \leq i \leq \deg(u) - 1$, $g(u) = f(u) \setminus \{1, 2, \ldots, \deg(u) - 1\}$, and $g(x) = f(x)$ otherwise. Clearly, g is a Tk-RDF of G contradicting the choice of f. Therefore, $\gamma_{trk}(G) = n$. □

Next result is an immediate consequence of Theorem 3.

Corollary 1. If $k \geq 3$, then $\gamma_{trk}(C_n) = \gamma_{trk}(P_n) = n$.

Now we characterize all graphs G with $\gamma_{trk}(G) = k$.

Theorem 4. Let $k \geq 1$ be an integer, and let G be a graph of order $n \geq k$. Then $\gamma_{trk}(G) = k$ if and only if $n = k$ or $n > k$ and there exists a set $A = \{v_1, v_2, \ldots, v_t\} \subset V(G)$ with $2 \leq t \leq k$ such that the induced subgraph $G[A]$ has no isolated vertex and $V(G) - A \subseteq N(v_i)$ for $1 \leq i \leq t$.

Proof. Suppose that $\gamma_{trk}(G) = k$. Let f be a $\gamma_{trk}(G)$-function, and let $V_0 = \{v : |f(v)| = 0\}$. If $V_0 = \emptyset$, then $n = k$. If $V_0 \neq \emptyset$, then let $v \in V_0$. By definition, we have $\bigcup_{u \in N(v)} f(u) = [k]$. Now let $v_1, v_2, \ldots, v_t \in N(v)$ be all vertices in $N(v)$ with the property that $|f(v_i)| \neq 0$ for $1 \leq i \leq t$. Then the condition $\gamma_{trk}(G) = k$ implies that $\sum_{i=1}^{t} |f(v_i)| = k$, $2 \leq t \leq k$, $G[\{v_1, v_2, \ldots, v_t\}]$ has no isolated vertex, and $V(G) - \{v_1, v_2, \ldots, v_t\} \subseteq N(v_i)$ for each $i \in \{1, 2, \ldots, t\}$.
Conversely, let G satisfies in the condition. Applying Proposition 1, we have $\gamma_{trk}(G) \geq k$. If $n = k$, then obviously $\gamma_{trk}(G) = k$. Now let $n > k$. Define the function $f : V(D) \to 2^{[k]}$ by $f(v_i) = \{i\}$ for $1 \leq i \leq t-1$, $f(v_t) = \{t, t+1, \ldots, k\}$ and $f(x) = \emptyset$ otherwise. Clearly, f is a total k-rainbow dominating function on G of weight k and so $\gamma_{trk}(G) \leq k$. Thus $\gamma_{trk}(G) = k$ and the proof is complete. □

Corollary 2. Let G be a connected graph of order $n \geq 2$. Then $\gamma_{tr2}(G) = 2$ if and only if $G = K_2$ or $G = K_2 \vee H$ for some graph H of order $n - 2$.

Theorem 5. Let G be a graph of order at least two and $k' > k$. Then

$$\gamma_{trk'}(G) \leq \gamma_{trk}(G) + (k' - k) \left\lfloor \frac{\gamma_{trk}(G)}{k} \right\rfloor.$$

Proof. Let f a $\gamma_{trk}(G)$-function. Assume n_i is the number of vertices $v \in V(G)$ for which $i \in f(v)$ for each $1 \leq i \leq k$. Assume without loss of generality that $n_1 \geq n_2 \geq \cdots \geq n_k$. Clearly the function $g : V(G) \to 2^{[k]}$ defined by $g(v) = f(v) \cup \{k+1, \ldots, k'\}$ when $k \in f(v)$ and $g(v) = f(v)$ otherwise, is a Tk'-RDF of G and so

$$\gamma_{trk'}(G) \leq \omega(g) = \gamma_{trk}(G) + (k' - k)n_k \leq \gamma_{trk}(G) + (k' - k) \left\lfloor \frac{\gamma_{trk}(G)}{k} \right\rfloor.$$

□

Corollary 3. Let $k' > k$ be two positive integers and let G be a graph of order at least two. Then

$$\gamma_{trk'}(G) \leq k' \left\lfloor \frac{\gamma_{trk}(G)}{k} \right\rfloor.$$

In particular, $\gamma_{trk'}(G) \leq k'\gamma_t(G)$.

Now we present lower and upper bound on the total k-rainbow domination number of a graph in terms of its order, minimum degree and k.

Proposition 2. Let $k \geq 1$ be an integer. If G is a connected graph of order $n \geq 2$, then

$$\gamma_{trk}(G) \leq n - \delta(G) + k.$$

Proof. If $\delta(G) = k$, then the result is true by Proposition 1. Assume that $\delta(G) > k$. Let v be a vertex of minimum $\delta(G)$ and let $u \in N(v)$. Define $f : V(G) \to 2^{[k]}$ by $f(u) = \{1\}$, $f(v) = [k]$, $f(x) = \emptyset$ if $x \in N(v) - \{u\}$ and $f(x) = \{1\}$ otherwise. Clearly, f is a total k-rainbow dominating function of G of weight $n-\delta(G)+k$ yielding $\gamma_{trk}(G) \leq n - \delta(G) + k$. □

Corollary 4. Let $k \geq 1$ be an integer. If G is a graph of order n with $\gamma_{trk}(G) = n$, then $k \geq \delta(G)$.

Theorem 6. Let $k \geq 1$ be an integer. If G is a graph of order $n \geq 2$, then

$$\gamma_{trk}(G) \geq \left\lceil \frac{kn}{\Delta(G) + k - 1} \right\rceil.$$

Proof. Let f be a $\gamma_{trk}(G)$-function, and let $V_i = \{v : |f(v)| = i\}$ for $i = 0, 1, \ldots, k$. Then $\gamma_{trk}(G) = |V_1| + 2|V_2| + \cdots + k|V_k|$ and $n = |V_0| + |V_1| + \cdots + |V_k|$. Let $F = (V(G) - V_0, V_0)$ be the set of edges with one end point in $V(G) - V_0$ and the other end point in V_0. Since f is a $\gamma_{trk}(G)$-function, we have

$$\begin{array}{rcl} k|V_0| & \leq & \sum_{xy \in F,\ x \in V(G)-V_0} |f(x)| \\ & \leq & (\Delta(G) - 1)(|V_1| + 2|V_2| + \ldots + k|V_k|) \\ & = & \gamma_{trk}(G)(\Delta(G) - 1). \end{array} \tag{3}$$

Now it follows from (3) that

$$\begin{array}{rcl} (\Delta(G) + k - 1)\gamma_{trk}(G) & = & (\Delta(G) - 1)\gamma_{trk}(G) + k\gamma_{trk}(G) \\ & \geq & k|V_0| + k(|V_1| + 2|V_2| + \ldots + k|V_k|) \\ & = & k(|V_0| + |V_1| + \ldots + |V_k|) + \\ & & k(|V_2| + 2|V_3| + \ldots + (k-1)|V_k|) \\ & = & kn + k(|V_2| + 2|V_3| + \ldots + (k-1)|V_k|) \\ & \geq & kn, \end{array}$$

and this leads to the desired bound. □

The special case $k = 1$ of Theorem 6 can be found in [11].

Theorem 7. For every positive integer k and every connected graph G of order at least two,

$$\gamma_{trk}(G) \leq (k+1)\gamma(G).$$

Furthermore, if the equality holds then every minimum dominating set of G is an efficient dominating set and each vertex belonging to a minimum dominating set has degree at least $k-1$.

Proof. By Proposition 1, we may assume $(k+1)\gamma(G) < n$. Let $D = \{u_1, \ldots, u_{\gamma(G)}\}$ be a dominating set of G and let v_i be a neighbor of u_i for each i. Define the function $f : V(G) \to 2^{[k]}$ by $f(u_i) = [k], f(v_i) = \{1\}$ for $i = 1, \ldots, \gamma(G)$ and $f(x) = \emptyset$ otherwise. It is easy to see that f is a Tk-RDF of G and so $\gamma_{trk}(G) \leq \omega(f) \leq (k+1)\gamma(G)$.
Now let $\gamma_{trk}(G) = (k+1)\gamma(G)$. Suppose $D = \{u_1, \ldots, u_{\gamma(G)}\}$ is a $\gamma(G)$-set. Let v_i be a neighbor of u_i for each i. If there exists an edge $u_iu_j \in E(G)$, then the function $g : V(G) \to 2^{[k]}$ defined by $g(v_i) = g(v_j) = \emptyset, g(v) = [k]$ for $v \in D$, $g(v_l) = \{1\}$ for $l \in \{1, \ldots, \gamma(G)\} - \{i, j\}$ and $g(x) = \emptyset$ otherwise, is a Tk-RDF of G of weight less than $\gamma_{trk}(G)$, a contradiction. So D is independent. If there is a vertex $z \in V(G) \setminus D$ with two neighbors in D, say u_i, u_j, then the function $h : V(G) \to 2^{[k]}$ defined by $h(z) = \{1\}, h(v) = [k]$ for $v \in D$, $h(v_l) = \{1\}$ for $l \in \{1, \ldots, \gamma(G)\} - \{i, j\}$ and $h(x) = \emptyset$ otherwise, is a Tk-RDF of G of weight less than $\gamma_{trk}(G)$, a contradiction. Thus D is an efficient dominating set of G.
On the other hand, if a vertex $u_i \in D$ has degree at most $k-2$, then the function $f : V(G) \to 2^{[k]}$ defined by $f(x) = \{1\}$ for $x \in N[u_i]$, $f(u_j) = [k], f(v_j) = \{1\}$ for $j \in \{1, \ldots, \gamma(G)\} - \{i\}$, and $f(x) = \emptyset$ otherwise, is a Tk-RDF of G of weight less than $\gamma_{trk}(G)$, a contradiction. This completes the proof. □

Let G be the graph obtained from a cycle $C_r = (u_1u_2 \ldots u_r)$ and r copies of $K_{1,k+1}$ by joining u_i to the central vertex of ith copy of $K_{1,k+1}$. It is easy to verify that $\gamma(G) = r$ and $\gamma_{trk}(G) = r(k+1)$. This example shows that the upper bound of Theorem 7 is sharp.
For planar graphs with diameter two or graphs having two disjoint minimum dominating sets, we will improve the upper bound given in Theorem 7.

Theorem 8. If G is a connected planar graph with $\mathrm{diam}(G) = 2$ different from F, then

$$\gamma_{trk}(G) \leq \min\{n, 2k+1\}.$$

Proof. Let G be a planar graph with $\mathrm{diam}(G) = 2$ different from F. Then $\gamma(G) \leq 2$ by Proposition A. By Proposition 1, it is enough to prove $\gamma_{trk}(G) \leq 2\gamma(G) + 1$. Let $D = \{x, y\}$ be a minimum dominating set. Since $\mathrm{diam}(G) = 2$, we have $d(x, y) \leq 2$. Define the function $f : V(G) \to 2^{[k]}$ by $f(x) = f(y) = [k]$ and $f(x) = \emptyset$ otherwise if $d(x, y) = 1$, and by $f(x) = f(y) = [k], f(z) = \{1\}$ for a vertex $z \in N(x) \cap N(y)$ and $f(x) = \emptyset$ otherwise when $d(x, y) = 2$. It is easy to see that f is a Tk-RDF of G and so $\gamma_{trk}(G) \leq \omega(f) = 2k + 1$. □

Let G be a graph without isolated vertices and let D be a γ-set of G. By the well known Ore's Theorem [14], $V(G) - D$ contains a dominating set D' of G. Any dominating set $D' \subseteq V(G) - D$ is called an *inverse dominating set* with respect to D. The minimum cardinality of all inverse dominating sets of G is called the *inverse domination* number and is denoted by $\gamma'(G)$. An inverse dominating set D' is called a γ'-set of G if $|D'| = \gamma'(G)$. Clearly, $\gamma(G) \leq \gamma'(G)$. The inverse domination number was introduced by Kulli and Sigarkanti [12].

Theorem 9. Let G be a graph without isolated vertices. Then

$$\gamma_{trk}(G) \leq (k-1)\gamma(G) + \gamma'(G).$$

Proof. Let D_1 be a γ-set of G and D_2 a γ'-set of G with respect to D_1. Define the function $f : V(G) \to 2^{[k]}$ by $f(v) = [k-1]$ for $v \in D_1$, $f(v) = \{k\}$ for $v \in D_2$ and $f(x) = \emptyset$ otherwise. Clearly f is a Tk-RDF of G. Therefore $\gamma_{trk}(G) \leq \omega(f) = (k-1)|D_1| + |D_2| = (k-1)\gamma(G) + \gamma'(G)$. □

This bound is sharp for K_n $(n \geq 2)$ and $K_{m,n}$ $(m \geq n \geq 2)$.

Corollary 5. For every positive integer k and every graph G having two disjoint minimum dominating sets,

$$\gamma_{trk}(G) \leq k\gamma(G).$$

Next, we present an upper bound on the total k-rainbow domination number in terms of k-rainbow domination number.

Proposition 3. Let $k \geq 2$ be an integer and G be a connected graph of order at least two. Then

$$\gamma_{trk}(G) \leq 2\gamma_{rk}(G) - 1.$$

Furthermore, this bound is sharp for each k

Proof. By Proposition 1, we may assume that $2\gamma_{rk}(G) - 1 < n$. Let f be a γ_{rk}-function and let $V(G) \setminus V_0^f = \{v_1, \ldots, v_t\}$. Note that $V_0^f = \{v \mid f(v) = \emptyset\} \neq \emptyset$. Let $B_1, \ldots, B_r$ be the components of $G[\{v_1, \ldots, v_t\}]$. Since G is connected, we can chose a vertex w_i for each i such that $f(w_i) = \emptyset$ and w_i has a neighbor in B_i. Define $g : V(G) \to 2^{[k]}$ by $g(w_i) = \{1\}$ for $1 \leq i \leq r$ and $g(x) = f(v)$ otherwise. It is easy to see that g is a Tk-RDF of G and so

$$\gamma_{trk}(G) \leq \omega(g) = r + \gamma_{rk}(G) \leq t + \gamma_{rk}(G).$$

If $r < t$ or $t < \gamma_{rk}(G)$, then the recent inequality chain leads to $\gamma_{trk}(G) \leq 2\gamma_{rk}(G) - 1$, as desired. Assume that $r = t = \gamma_{rk}(G)$. This implies that $|V(B_i)| = |f(v_i)| = 1$ for each i and so each vertex of V_0^f is adjacent to at least k vertices in $\{v_1, \ldots, v_t\}$. Let

$u \in V_0^f$ and let u be adjacent to $v_1, \ldots, v_k$. Then the function $g : V(G) \to 2^{[k]}$ by $g(u) = g(w_i) = \{1\}$ for $k+1 \le i \le r$ and $g(x) = f(v)$ otherwise, is a Tk-RDF of G and so

$$\gamma_{trk}(G) \le \omega(g) = t + (t-k) + 1 = 2t - (k-1) = 2\gamma_{rk}(G) - (k-1) \le 2\gamma_{rk}(G) - 1.$$

To prove the sharpness, let $m \ge 2k$ and let G be the graph obtained from two complete bipartite graphs $K_{m,k}, K_{m,k-1}$ with partite sets $(\{x_1, \ldots, x_k\}, \{y_1, \ldots, y_m\})$ and $(\{x'_1, \ldots, x'_{k-1}\}, \{z_1, \ldots, z_m\})$, respectively, by identifying x_i and x'_i for $i = 1, \ldots, k-1$, and joining a pendant edge $z_i z'_i$ for each $1 \le i \le m$. It is not hard to see that $\gamma_{rk}(G) = k + m$ and $\gamma_{trk}(G) = k + 2m + 1$ and the proof is complete. □

Finally, we establish an upper bound on the total 2-rainbow domination number of a graph in terms of its order, maximum degree and matching number. The *private neighborhood* $pn(v, S)$ of $v \in S$ is defined by $pn(v, S) = N(v) - N(S - \{v\})$. Each vertex in $pn(v, S)$ is called a *private neighbor* of v.

Theorem 10. Let G be a connected graph of order $n \ge 4$ different from star. Then

$$\gamma_{tr2}(G) \le n - \Delta(G) + \alpha'(G).$$

Proof. Since G is not a star, we have $\alpha'(G) \ge 2$. Suppose v is a vertex of maximum degree $\Delta(G)$ and let $X = V(G) \setminus N_G[v]$. Assume that S is the set consisting of all isolated vertices of the induced subgraph $G[X]$. If $X = \emptyset$, then clearly $\gamma_{tr2}(G) = 3$ and $\Delta(G) = n - 1$. This implies that $\gamma_{tr2}(G) \le n - \Delta(G) + 2$ as desired. Assume that $X \ne \emptyset$. If $S = \emptyset$, then let $u \in N(v)$ and define $f : V(G) \to 2^{[2]}$ by $f(v) = \{1, 2\}, f(u) = \{1\}, f(x) = \{1\}$ for $x \in X$ and $f(x) = \emptyset$ otherwise. Obviously, f is an T2-RDF of G and so $\gamma_{tr2}(G) \le n - \Delta(G) + 2$ that leads to the desired bound..
Suppose $S \ne \emptyset$. Since $\delta(G) \ge 1$, every vertex $s \in S$ is adjacent to at least one vertex of $N(v)$. Assume S' is the smallest subset of $N(v)$ that dominates S. By the choice of S', each vertex $u \in S'$ has a private neighbor $u' \in S$ with respect to S', so $|S'| \le |S|$. Hence $M = \{uu' | u \in S'\}$ is a matching in G.
If $S' = N(v)$ and $X = S$, then the function $f : V(G) \to 2^{[2]}$ defined by $f(v) = \{1\}, f(x) = \{1, 2\}$ for $x \in S'$ and $f(x) = \emptyset$ otherwise, is an T2-RDF of G and we have

$$\gamma_{tr2}(G) \le 2|S'| + 1 \le |S| + |S'| + 1 \le n - \Delta(G) - 1 + 1 + |S'| \le n - \Delta(G) + \alpha'(G).$$

If $S' = N(v)$ and $S \subsetneqq X$, then let $uu' \in E(G[X - S])$. Then $M \cup \{uu'\}$ is a matching of G and so $|S'| \le \alpha'(G) - 1$. Define the function $f : V(G) \to 2^{[2]}$ by $f(v) = \{1\}, f(x) = \{1, 2\}$ for $x \in S'$, $f(x) = \{1\}$ for $x \in X - S$ and $f(x) = \emptyset$ otherwise. Clearly, f is an T2-RDF of G and this implies that

$$\gamma_{tr2}(G) \le 2|S'| + 1 + n - 1 - \Delta(G) - |S| \le |S'| + n - \Delta(G) \le n - \Delta(G) + \alpha'(G) - 1.$$

If $S' \subsetneqq N(v)$ and $X = S$, then $M \cup \{vz\}$ is a matching of G for each $z \in N(v) - S'$, and so $|S'| \leq \alpha'(G) - 1$. Clearly, the function $f : V(G) \to 2^{[2]}$ by $f(v) = \{1, 2\}, f(x) = \{1, 2\}$ for $x \in S'$ and $f(x) = \emptyset$ otherwise, is an T2-RDF of G and so

$$\gamma_{tr2}(G) \leq 2|S'| + 2 \leq 2 + |S| + \alpha'(G) - 1 \leq n - \Delta(G) + \alpha'(G).$$

Assume now that $S' \subsetneqq N(v)$ and $S \subsetneqq X$. Suppose that $z \in N(v) - S'$ and $uu' \in E(G[X - S])$. Then clearly $M \cup \{vz, uu'\}$ is a matching of G and so $|S'| \leq \alpha'(G) - 2$. Define the function $f : V(G) \to 2^{[2]}$ by $f(v) = \{1, 2\}, f(x) = \{1, 2\}$ for $x \in S'$, $f(x) = \{1\}$ for $x \in X - S$ and $f(x) = \emptyset$ otherwise. It is easy to see that f is an T2-RDF of G and hence

$$\gamma_{tr2}(G) \leq 2|S'| + 2 + n - 1 - \Delta(G) - |S| \leq |S'| + n - \Delta(G) + 1 \leq n - \Delta(G) + \alpha'(G) - 1,$$

and the proof is complete. □

3. Special values of total 2-rainbow domination number

In this section we determine the total 2-rainbow domination number of some classes of graphs including Cycles, paths and ladders. Next result shows that the bound of Theorem 6 is sharp for the special case $k = 2$.

Proposition 4. For $n \geq 3$, $\gamma_{tr2}(C_n) = \lceil \frac{2n}{3} \rceil$.

Proof. By Theorem 6, it is enough to prove $\gamma_{tr2}(C_n) \leq \lceil \frac{2n}{3} \rceil$. Let $C_n = (v_1 v_2 \ldots v_n)$ and define the function $f : V(C_n) \to 2^{[2]}$ by $f(v_{3i+2}) = \{2\}, f(v_{3i+3}) = \emptyset$ for $0 \leq i \leq \lfloor \frac{n}{3} \rfloor - 1$ and $f(x) = \{1\}$ otherwise. Clearly, f is an T2-RDF of C_n of weight $\lceil \frac{2n}{3} \rceil$ and so $\gamma_{tr2}(C_n) = \lceil \frac{2n}{3} \rceil$. □

Proposition 5. For $n \geq 2$, $\gamma_{tr2}(P_n) = \lceil \frac{2n+2}{3} \rceil$.

Proof. Let $P_n := v_1 v_2, \ldots, v_n$. First we show that $\gamma_{tr2}(P_n) \geq \lceil \frac{2n+2}{3} \rceil$. The result is immediate for $n = 2, 3, 4$. Assume $n \geq 5$ and let f be a γ_{tr2}-function on P_n such that the cardinality of V_0^f is as small as possible. Hence $|f(v_1)| = |f(v_2)| = |f(v_{n-1}) = |f(v_n)| = 1$ and $|f(v_{i-1})| + |f(v_i)| + |f(v_{i+1})| \geq 2$ for each $2 \leq i \leq n-1$. But then V_0^f is a packing in the subpath $v_3, \ldots, v_{n-3}$. Therefore $|V_0^f| \leq \lceil (n-4)/3 \rceil$ which leads to $\gamma_{tr2}(P_n) \geq n - \lceil (n-4)/3 \rceil = \lceil (2n+2)/3 \rceil$. To prove $\gamma_{tr2}(P_n) \leq \lceil \frac{2n+2}{3} \rceil$, define the function $f : V(P_n) \to 2^{[2]}$ by $f(v_{3i}) = \emptyset$ for $1 \leq i \leq \lfloor \frac{n}{3} \rfloor$, $f(v_{3i+1}) = \{1\}, f(v_{3i+2}) = \{2\}$ for $0 \leq i \leq \lfloor \frac{n}{3} \rfloor$ if $n \equiv 2 \pmod 3$ and by $f(v_{3i+3}) = \emptyset, f(v_{3i+1}) = \{1\}, f(v_{3i+2}) = \{2\}$ for $0 \leq i \leq \lfloor \frac{n}{3} \rfloor - 2$ and $f(x) = \{1\}$ otherwise, when $n \equiv 0$ or $1 \pmod 3$. Clearly, f is a T2-RDF of P_n of weight $\lceil \frac{2n+2}{3} \rceil$ and so $\gamma_{tr2}(P_n) \leq \lceil \frac{2n+2}{3} \rceil$. Thus $\gamma_{tr2}(P_n) = \lceil (2n+2)/3 \rceil$. □

Next, we focus on the *ladder* $P_2\Box P_n$, where $G\Box H$ is the *Cartesian product* of two graphs G and H, and determine the value $\gamma_{tr2}(P_2\Box P_n)$. Our motivation derives from Vizing's conjecture, that for any graphs G and H, $\gamma(G\Box H)\geq\gamma(G)\gamma(H)$. The conjecture is still open, and many researchers have studied some domination-like invariants for $G\Box H$.
Throughout our argument, we write $V(P_2\Box P_n)=\{v_i^j\mid i=1,2 \text{ and } 1\leq j\leq n\}$ and assume $E(P_2\Box P_n)=\{v_1^jv_2^j\mid 1\leq j\leq n\}\cup\{v_i^jv_i^{j+1}\mid i=1,2 \text{ and } 1\leq j\leq n-1\}$.

Theorem 11. For $n\geq 2$, $\gamma_{tr2}(P_2\Box P_n)=n+1$.

Proof. Define the function $f: V(P_2\Box P_n)\to 2^{[2]}$ by $f(v_1^{4j+1})=f(v_2^{4j+1})=\{1\}$ for $0\leq j\leq\lfloor\frac{n}{4}\rfloor$, $f(v_1^{4j+3})=f(v_2^{4j+3})=\{2\}$ for $0\leq j\leq\lfloor\frac{n-2}{4}\rfloor$ and $f(x)=\emptyset$ otherwise, when n is odd, and by $f(v_1^{4j+1})=f(v_2^{4j+1})=\{1\}$ for $0\leq j\leq\lfloor\frac{n-1}{4}\rfloor$, $f(v_1^{4j+3})=f(v_2^{4j+3})=\{2\}$ for $0\leq j\leq\lfloor\frac{n-3}{4}\rfloor$, $f(v_1^n)=\{1\}$ and $f(x)=\emptyset$ otherwise, if $n\equiv 0 \pmod 4$, and by $f(v_1^{4j+1})=f(v_2^{4j+1})=\{1\}$ for $0\leq j\leq\lfloor\frac{n-1}{4}\rfloor$, $f(v_1^{4j+3})=f(v_2^{4j+3})=\{2\}$ for $0\leq j\leq\lfloor\frac{n-3}{4}\rfloor$, $f(v_1^n)=\{2\}$ and $f(x)=\emptyset$ otherwise when $n\equiv 2 \pmod 4$. It is easy to verify that f is a T2-RDF of $P_2\Box P_n$ with $\omega(f)=n+1$, and hence $\gamma_{tr2}(P_2\Box P_n)\leq n+1$.
We now show that $\gamma_{tr2}(P_2\Box P_n)\geq n+1$. The proof is by induction on n. The result is immediate for $n=2$ by Proposition 4. Thus we may assume that $n\geq 3$. Let f' be a γ_{tr2}-function of $P_2\Box P_n$, and set $I=\{i\mid 1\leq i\leq n \text{ and } f'(v_1^i)=f'(v_2^i)=\emptyset\}$. We choose f' so that $|I|$ is as small as possible. First let $I\neq\emptyset$. If $1\in I$ (the case $n\in I$ is similar), then $f'(v_1^2)=f'(v_2^2)=\{1,2\}$ and by the choice of f' we must have $f'(v_1^3)=f'(v_2^3)=\emptyset$. But then the function $f_1': V(P_2\Box P_n)\to 2^{[2]}$ defined by $f_1'(v_1^2)=f_1'(v_2^2)=\emptyset, f_1'(v_1^1)=f_1'(v_2^1)=\{1\}, f_1'(v_1^3)=f_1'(v_2^3)=\{2\}$ and $f_1'(x)=f'(x)$ otherwise, is a γ_{tr2}-function of $P_2\Box P_n$ which leads to a contradiction. If $m\in I$ for some $2\leq i\leq n-1$, then the function f', restricted to the components of $P_2\Box P_n-\{v_1^m,v_2^m\}$ is a T2-RDF and it follows from the induction hypothesis that $\omega(f')\geq(m-1+1)+(n-m+1)=n+1$ as desired. Henceforth, we may assume that $I=\emptyset$. If $f'(v_1^i)|+|f'(v_2^i)|=1$ for each $1\leq i\leq n$, then we may suppose without loss of generality that $f'(v_1^1)=\{1\}, f(v_2^1)=\emptyset$ and this implies that $f'(v_1^2)|+|f'(v_2^2)|\geq 2$, a contradiction. Thus $f'(v_1^i)|+|f'(v_2^i)|\geq 2$ for some $1\leq i\leq n$. We now conclude from $I=\emptyset$ that $\gamma_{tr2}(P_2\Box P_n)\geq n+1$. Thus $\gamma_{tr2}(P_2\Box P_n)=n+1$ and the proof is complete. □

We end this section by characterizing all graphs G of order n with $\gamma_{t2r}(G)=n$. The *corona* $\mathrm{cor}(H)$ of a graph H, is the graph obtained from H by adding a pendant edge to each vertex of H.

Theorem 12. Let G be connected graph of order $n\geq 2$ without isolated vertices. Then $\gamma_{t2r}(G)=n$ if and only if $G\in\{P_2,P_3\}$ or G is the corona, $\mathrm{cor}(H)$, of some graph H.

Proof. If $G\in\{P_2,P_3\}$ or G is the corona, $\mathrm{cor}(H)$, of some graph H, then clearly $\gamma_{t2r}(G)=n$.

Let $\gamma_{t2r}(G) = n$. If $n \leq 3$, then clearly $G \in \{P_2, P_3\}$ and we are done. Let $n \geq 4$. If G has a vertex x which is neither a leaf or a stem, then for any $y \in N(x)$, the function $f = (\{x\}, \{y\}, V(G) - \{x, y\}, \emptyset)$ is a T2RDF on G of weight $n - 1$, a contradiction. Hence each vertex of G is either a leaf or a stem. Since $\gamma_{t2r}(G) = n$, G is not a star. If G has a stem u which is adjacent to $t \geq 2$ leaves $v_1, \ldots, v_t$ and v is a non leaf neighbor of u, then the function $f = (\{v_1, \ldots, v_t\}, \emptyset, V(G) - \{u\}, \{u\})$ is a T2RDF on G of weight $n - t$, a contradiction again. Therefore, each stem of G is adjacent to at exactly one leaf. Assume H is the graph obtained from G by deleting all leaves of G. Then $G = \mathrm{cor}(H)$ and the proof is complete. □

4. Conclusion

In this paper, we introduced a new variant of the domination problem, called the total k-rainbow domination problem, on graphs. We established some sharp bounds on the total k-rainbow domination number for general graphs and determined this parameter for some classes of graphs. As a further study, it is interesting to establish sharp lower bounds for this parameter and to determine the value of this parameter for some well-known classes of graphs, including complete multipartite graphs, generalized Petersen graphs and cartesian products of various types of graphs.

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