Communications in Combinatorics and Optimization
Vol. 3 No. 1, 2018 pp.25-35
DOI: 10.22049/CCO.2017.26018.1067

CCO
Commun. Comb. Optim.

Product version of reciprocal degree distance of composite graphs

K. Pattabiraman[1]

[1]Department of Mathematics, Annamalai University, Annamalainagar 608 002, India
pramank@gmail.com

Received: 7 September 2017; Accepted: 21 October 2017
Published Online: 25 October 2017

Communicated by ***Ivan Gutman***

Abstract: In this paper, we present the upper bounds for the product version of reciprocal degree distance of the tensor product, join and strong product of two graphs in terms of other graph invariants including the Harary index and Zagreb indices.

Keywords: Degree distance, reciprocal degree distance, composite graph

AMS Subject classification: 05C12, 05C76

1. Introduction

All the graphs considered in this paper are simple and connected. For vertices $u, v \in V(G)$, the *distance* between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path in G and let $d_G(v)$ be the *degree* of a vertex $v \in V(G)$. For two simple graphs G and H their *tensor product,* denoted by $G \times H$, has vertex set $V(G) \times V(H)$ in which (g_1, h_1) and (g_2, h_2) are adjacent whenever g_1g_2 is an edge in G and h_1h_2 is an edge in H. Note that if G and H are connected graphs, then $G \times H$ is connected only if at least one of the graph is non-bipartite. The *strong product* of graphs G and H, denoted by $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H) = \{(u, v) : u \in V(G), v \in V(H)\}$ and $(u, x)(v, y)$ is an edge whenever (i) $u = v$ and $xy \in E(H)$, or (ii) $uv \in E(G)$ and $x = y$, or (iii) $uv \in E(G)$ and $xy \in E(H)$. The *join* $G + H$ of graphs G and H is obtained from the disjoint union of the graphs G and H, where each vertex of G is adjacent to each vertex of H.
A *topological index* of a graph is a real number related to the graph; it does not depend

on labeling or pictorial representation of a graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, biological and other properties of chemical compounds. There exist several types of such indices, especially those based on vertex and edge distances. One of the most intensively studied topological indices is the Wiener index; for other related topological indices see [2].
Let G be a connected graph. The *Wiener index* of G is defined as $W(G) = \frac{1}{2}\sum\limits_{u,\, v\, \in V(G)} d_G(u,v)$ where the summation goes over all pairs of distinct vertices of G. Similarly, the *Harary index* of G is defined as $H(G) = \frac{1}{2}\sum\limits_{u,\, v\, \in V(G)} \frac{1}{d_G(u,v)}$. Gutman et al. [7, 8] were introduced the *product version of Wiener index* as follows $W^*(G) = \prod\limits_{\{u,v\}\subseteq V(G)} d_G(u,v)$. Dobrynin and Kochetova [4] and Gutman [6] independently proposed a vertex-degree-weighted version of Wiener index called degree distance or Schultz molecular topological index, which is defined for a connected graph G as $DD(G) = \frac{1}{2}\sum\limits_{u,v\in V(G)} (d_G(u) + d_G(v))d_G(u,v)$, where $d_G(u)$ is the degree of the vertex u in G. Note that the degree distance is a degree-weight version of the Wiener index. Hua and Zhang [10] introduced a new graph invariant named reciprocal degree distance, which can be seen as a degree-weight version of Harary index, that is,

$$H_A(G) = \frac{1}{2}\sum_{u,v\in V(G),\, u\neq v} \frac{(d_G(u) + d_G(v))}{d_G(u,v)}.$$

Hua and Zhang [10] have obtained lower and upper bounds for the reciprocal degree distance of graph in terms of other graph invariants including the degree distance, Harary index, the first Zagreb index, the first Zagreb coindex, pendent vertices, independence number, chromatic number and vertex and edge-connectivity. In this sequence, the product version of *reciprocal degree distance* is defined as

$$H_A^*(G) = \prod_{\{u,v\}\subseteq V(G),\, u\neq v} \frac{d_G(u) + d_G(v)}{d_G(u,v)}.$$

The *first Zagreb index* and *second Zagerb index* are defined as

$$M_1(G) = \sum_{u\in V(G)} d_G(u)^2 \quad \text{and} \quad M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v).$$

Similarly, the *first Zagreb coindex* and *second Zagerb coindex* are defined as

$$\overline{M}_1(G) = \sum_{uv\notin E(G)} (d_G(u) + d_G(v)) \quad \text{and} \quad \overline{M}_2(G) = \sum_{uv\notin E(G)} d_G(u)d_G(v).$$

The Zagreb indices are found to have applications in QSPR and QSAR studies as well, see [3]. Various topological indices on tensor product, strong product have been studied by several authors [1, 5, 9, 11–15].
In this paper, we present upper bounds for the product version of reciprocal degree distance of the tensor product, join and strong product of two graphs in terms of other graph invariants including the Harary index and Zagreb indices.

2. Tensor product

In this section, we compute the product version of the reciprocal degree distance of $G \times K_r$.
The proof of the following lemma follows easily from the properties and structure of $G \times K_r$. The lemma is used in the proof of the main theorem of this section.

Lemma 1. Let G be a connected graph on $n \geq 2$ vertices. For any pair of vertices x_{ij}, $x_{kp} \in V(G \times K_r)$, $r \geq 3$, $i, k \in \{1, 2, \ldots, n\}$ $j, p \in \{1, 2, \ldots, r\}$. Then
(i) If $u_i u_k \in E(G)$, then

$$d_{G\times K_r}(x_{ij}, x_{kp}) = \begin{cases} 1 & \text{if} \quad j \neq p, \\ 2 & \text{if} \quad j = p \text{ and } u_i u_k \text{ is on a triangle of } G, \\ 3 & \text{if} \quad j = p \text{ and } u_i u_k \text{ is not on a triangle of } G. \end{cases}$$

(ii) If $u_i u_k \notin E(G)$, then $d_{G\times K_r}(x_{ij}, x_{kp}) = d_G(u_i, u_k)$.
(iii) $d_{G\times K_r}(x_{ij}, x_{ip}) = 2$.

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(K_r) = \{v_1, v_2, \ldots, v_r\}$. Let x_{ij} denote the vertex (u_i, v_j) of $G \times K_r$. We only prove the case when $u_i u_k \notin E(G)$, $i \neq k$ and $j = p$. The proofs for other cases are similar.
We may assume $j = 1$. Let $P = u_i u_{s_1} u_{s_2} \ldots u_{s_p} u_k$ be the shortest path of length $p + 1$ between u_i and u_k in G. From P we have a (x_{i1}, x_{k1})-path $P_1 = x_{i1} x_{s_1 2} \ldots x_{s_{p-1} 2} x_{s_p 3} x_{k1}$ if the length of P is odd, and $P_1 = x_{i1} x_{s_1 2} \ldots x_{s_{p-1} 2} x_{s_p 2} x_{k1}$ if the length of P is even.
Obviously, the length of P_1 is $p+1$, and thus $d_{G\times K_r}(x_{i1}, x_{k1}) \leq p+1 \leq d_G(u_i, u_k)$. If there were a (x_{i1}, x_{k1})-path in $G \times K_r$ that is shorter than $p+1$ then it is easy to find a (u_i, u_k)-path in G that is also shorter than $p+1$ in contrast to $d_G(u_i, u_k) = p+1$. □

Remark 1. (Arithmetic Geometric Inequality) Let $a_1, a_2, \ldots, a_n$ be non-negative numbers. Then $\sqrt[n]{a_1 a_2 \ldots a_n} \leq \frac{a_1 + a_2 + \ldots + a_n}{n}$.

Theorem 1. Let G be a connected graph with $n \geq 2$ vertices and m edges. Then

$$H_A^*(G \times K_r) \leq \frac{(r-1)^{5nr} m^{nr}}{n^{3nr}} \Big[H_A(G)(H_A(G) - \frac{M_1(G)}{2} - t) \Big]^{nr},$$

where $t = \sum\limits_{u_i u_k \in E_2} \frac{d_G(u_i)+d_G(u_k)}{6}$ and $r \geq 3$.

Proof. Set $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(K_r) = \{v_1, v_2, \ldots, v_r\}$. Let x_{ij} denote the vertex (u_i, v_j) of $G \times K_r$. The degree of the vertex x_{ij} in $G \times K_r$ is $d_G(u_i)d_{K_r}(v_j)$, that is $d_{G\times K_r}(x_{ij}) = (r-1)d_G(u_i)$. By the definition of H_A^*, we have

$$\begin{aligned} H_A^*(G\times K_r) &= \prod_{x_{ij},\, x_{kp} \,\in\, V(G\times K_r)} \frac{d_{G\times K_r}(x_{ij}) + d_{G\times K_r}(x_{kp})}{d_{G\times K_r}(x_{ij}, x_{kp})} \\ &= \prod_{i=0}^{n-1} \prod_{\substack{j,\,p=0 \\ j\neq p}}^{r-1} \frac{d_{G\times K_r}(x_{ij}) + d_{G\times K_r}(x_{ip})}{d_{G\times K_r}(x_{ij}, x_{ip})} \times \prod_{\substack{i,\,k=0 \\ i\neq k}}^{n-1} \prod_{j=0}^{r-1} \frac{d_{G\times K_r}(x_{ij}) + d_{G\times K_r}(x_{kj})}{d_{G\times K_r}(x_{ij}, x_{kj})} \times \\ &\quad \prod_{\substack{i,\,k=0 \\ i\neq k}}^{n-1} \prod_{\substack{j,\,p=0 \\ j\neq p}}^{r-1} \frac{d_{G\times K_r}(x_{ij}) + d_{G\times K_r}(x_{kp})}{d_{G\times K_r}(x_{ij}, x_{kp})}. \end{aligned} \tag{1}$$

We shall calculate the sums of (1) are separately.
First we compute $\prod\limits_{i=0}^{n-1} \prod\limits_{\substack{j,\,p=0 \\ j\neq p}}^{r-1} \frac{d_{G\times K_r}(x_{ij}) + d_{G\times K_r}(x_{ip})}{d_{G\times K_r}(x_{ij}, x_{ip})}$.

$$\begin{aligned} \prod_{i=0}^{n-1} \prod_{\substack{j,\,p=0 \\ j\neq p}}^{r-1} \frac{d_{G\times K_r}(x_{ij}) + d_{G\times K_r}(x_{ip})}{d_{G\times K_r}(x_{ij}, x_{ip})} &= \prod_{i=0}^{n-1} \prod_{\substack{j,\,p=0 \\ j\neq p}}^{r-1} \frac{2(r-1)d_G(u_i)}{2} \quad \text{(by Lemma 1)} \\ &\leq \left[\frac{\frac{1}{2}\sum\limits_{i=0}^{n-1}\sum\limits_{\substack{j,\,p=0 \\ j\neq p}}^{r-1}(r-1)d_G(u_i)}{nr}\right]^{nr} \quad \text{(by Rem. 1)} \\ &= \left[\frac{2r(r-1)^2 m}{2nr}\right]^{nr} \\ &= \left[\frac{(r-1)^2 m}{n}\right]^{nr}. \end{aligned} \tag{2}$$

Next we compute $\prod\limits_{j=0}^{r-1} \prod\limits_{\substack{i,\,k=0 \\ i\neq k}}^{n-1} \frac{d_{G\times K_r}(x_{ij}) + d_{G\times K_r}(x_{kj})}{d_{G\times K_r}(x_{ij}, x_{kj})}$. By Remark 1, we have

$$\begin{aligned} \prod_{j=0}^{r-1} \prod_{\substack{i,\,k=0 \\ i\neq k}}^{n-1} \frac{d_{G\times K_r}(x_{ij}) + d_{G\times K_r}(x_{kj})}{d_{G\times K_r}(x_{ij}, x_{kj})} &\leq \left[\frac{\frac{1}{2}\sum\limits_{j=0}^{r-1}\sum\limits_{\substack{i,\,k=0 \\ i\neq k}}^{n-1}\frac{d_{G\times K_r}(x_{ij})+d_{G\times K_r}(x_{kj})}{d_{G\times K_r}(x_{ij},x_{kj})}}{nr}\right]^{nr} \\ &= \left[\frac{\frac{1}{2}\sum\limits_{j=0}^{r-1} S}{nr}\right]^{nr}. \end{aligned} \tag{3}$$

Now we compute S. For that, let $E_1 = \{uv \in E(G) \,|\, uv \text{ is on a } C_3 \text{ in } G\}$ and $E_2 = E(G) - E_1$.

$$\begin{aligned}
S &= \Big(\sum_{\substack{i,\,k=0\\ i\neq k\\ u_iu_k\notin E(G)}}^{n-1} + \sum_{\substack{i,\,k=0\\ i\neq k\\ u_iu_k\in E_1}}^{n-1} + \sum_{\substack{i,\,k=0\\ i\neq k\\ u_iu_k\in E_2}}^{n-1} \Big)\Big(\frac{d_{G\times K_r}(x_{ij}) + d_{G\times K_r}(x_{kj})}{d_{G\times K_r}(x_{ij}, x_{kj})}\Big)\\
&= \Bigg(\sum_{\substack{i,\,k=0\\ i\neq k\\ u_iu_k\notin E(G)}}^{n-1} \frac{(r-1)(d_G(u_i)+d_G(u_k))}{d_G(u_i,u_k)} + \sum_{\substack{i,\,k=0\\ i\neq k\\ u_iu_k\in E_1}}^{n-1} \frac{(r-1)(d_G(u_i)+d_G(u_k))}{2} +\\
&\quad \sum_{\substack{i,\,k=0\\ i\neq k\\ u_iu_k\in E_2}}^{n-1} \frac{(r-1)(d_G(u_i)+d_G(u_k))}{3}\Bigg) \quad \text{(by Lemma 1)}\\
&= (r-1)\Bigg\{\Bigg(\sum_{\substack{i,\,k=0\\ i\neq k\\ u_iu_k\notin E(G)}}^{n-1} \frac{d_G(u_i)+d_G(u_k)}{d_G(u_i,u_k)} + \sum_{\substack{i,\,k=0\\ i\neq k\\ u_iu_k\in E_1}}^{n-1} \frac{d_G(u_i)+d_G(u_k)}{d_G(u_i,u_k)} +\\
&\quad \sum_{\substack{i,\,k=0\\ i\neq k\\ u_iu_k\in E_2}}^{n-1} \frac{d_G(u_i)+d_G(u_k)}{d_G(u_i,u_k)}\Bigg) - \sum_{\substack{i,\,k=0\\ i\neq k\\ u_iu_k\in E_1}}^{n-1} \frac{d_G(u_i)+d_G(u_k)}{2} - 2\sum_{\substack{i,\,k=0\\ i\neq k\\ u_iu_k\in E_2}}^{n-1} \frac{d_G(u_i)+d_G(u_k)}{3}\Bigg\}\\
&= (r-1)\Bigg\{2H_A(G) - \sum_{\substack{i,\,k=0\\ i\neq k\\ u_iu_k\in E(G)}}^{n-1} \frac{d_G(u_i)+d_G(u_k)}{2} - \sum_{\substack{i,\,k=0\\ i\neq k\\ u_iu_k\in E_2}}^{n-1} \frac{d_G(u_i)+d_G(u_k)}{6}\Bigg\}\\
&= (r-1)^2\Bigg\{2H_A(G) - M_1(G) - \sum_{u_iu_k\in E_2} \frac{d_G(u_i)+d_G(u_k)}{3}\Bigg\}. \qquad (4)
\end{aligned}$$

Now summing (4) over $j = 0, 1, \ldots, r-1$, we get,

$$\sum_{j=0}^{r-1} S = r(r-1)\Big(2H_A(G) - M_1(G) - \sum_{u_iu_k\in E_2} \frac{d_G(u_i)+d_G(u_k)}{3}\Big). \qquad (5)$$

Hence

$$\begin{aligned}
\prod_{j=0}^{r-1}\prod_{\substack{i,\,k=0\\ i\neq k}}^{n-1} \frac{d_{G\times K_r}(x_{ij}) + d_{G\times K_r}(x_{kj})}{d_{G\times K_r}(x_{ij}, x_{kj})} &\leq \left[\frac{\frac{r(r-1)}{2}\Big(2H_A(G) - M_1(G) - \sum\limits_{u_iu_k\in E_2} \frac{d_G(u_i)+d_G(u_k)}{3}\Big)}{nr}\right]^{nr}\\
&= \left[\frac{(r-1)\Big(H_A(G) - \frac{M_1(G)}{2} - \sum\limits_{u_iu_k\in E_2} \frac{d_G(u_i)+d_G(u_k)}{6}\Big)}{n}\right]^{nr}. \qquad (6)
\end{aligned}$$

Next we compute $\prod\limits_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\prod\limits_{\substack{j,\,p=0\\ j\neq p}}^{r-1} \frac{d_{G\times K_r}(x_{ij})+d_{G\times K_r}(x_{kp})}{d_{G\times K_r}(x_{ij},x_{kp})}$. By Lemma 1 and Remark 1, we have

$$\prod_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\prod_{\substack{j,\,p=0,\\ j\neq p}}^{r-1}\frac{d_{G\times K_r}(x_{ij})+d_{G\times K_r}(x_{kp}))}{d_{G\times K_r}(x_{ij},x_{kp}}\leq\left[\frac{\frac{1}{2}\sum\limits_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\sum\limits_{\substack{j,\,p=0,\\ j\neq p}}^{r-1}\frac{(r-1)(d_G(u_i)+d_G(u_k))}{d_G(u_i,u_k)}}{nr}\right]^{nr},$$
$$=\left[\frac{r(r-1)^2H_A(G)}{nr}\right]^{nr}$$
$$=\left[\frac{(r-1)^2H_A(G)}{n}\right]^{nr}. \quad (7)$$

Using (1) and the sums $\mathbf{A_1}$,$\mathbf{A_2}$ and $\mathbf{A_3}$ in (2),(5) and (7), respectively, we have,

$$H_A^*(G\times K_r)\leq\frac{(r-1)^{5nr}m^{nr}}{n^{3nr}}\Big[H_A(G)(H_A(G)-\frac{M_1(G)}{2}-t)\Big]^{nr}$$

where $t=\sum\limits_{u_iu_k\in E_2}\frac{d_G(u_i)+d_G(u_k)}{6}$. □

Using Theorem 1, we have the following corollaries.

Corollary 1. Let G be a connected graph on $n\geq 2$ vertices with m edges. If each edge of G is on a C_3, then

$$H_A^*(G\times K_r)=\leq\frac{(r-1)^{5nr}m^{nr}}{n^{3nr}}\Big[H_A(G)(H_A(G)-\frac{M_1(G)}{2})\Big]^{nr}$$

where $r\geq 3$.

For a triangle-free graph, $\sum\limits_{u_iu_k\in E_2}d_G(u_i)d_G(u_k)=M_2(G)$.

Corollary 2. If G is a connected triangle free graph on $n\geq 2$ vertices and m edges, then

$$H_A^*(G\times K_r)\leq\frac{(r-1)^{5nr}m^{nr}}{n^{3nr}}\Big[H_A(G)(H_A(G)-\frac{2M_1(G)}{3})\Big]^{nr}$$

where $r\geq 3$.

By direct calculations we obtain expressions for the values of the Harary indices of K_n and C_n. $H(K_n)=\frac{n(n-1)}{2}$ and $H(C_n)=n\Big(\sum\limits_{i=1}^{\frac{n}{2}}\frac{1}{i}\Big)-1$ when n is even, and $n\Big(\sum\limits_{i=1}^{\frac{n-1}{2}}\frac{1}{i}\Big)$ otherwise. Similarly, $H_A(K_n)=n(n-1)^2$ and $H_A(C_n)=4H(C_n)$.
Using Corollaries 1 and 2, we obtain the H_A^* of the graphs $K_n\times K_r$ and $C_n\times K_r$.

Example 1. *(i)* $H_A^*(K_n\times K_r)\leq\frac{(r-1)^{5nr}(n-1)^{3nr}}{2^{nr}}$.
(ii)

$$H_A^*(C_n\times K_r)\leq\begin{cases}\frac{(r-1)^{15r}(48)^{3r}}{3^{6r}}, & \text{if } n=3,\\ \frac{(16)^{nr}(r-1)^{5nr}}{n^{2nr}}\Big[(H(C_n))^2-\frac{2nH(C_n)}{3}\Big]^{nr}, & \text{if } n>3.\end{cases}$$

3. Join of graphs

In this section, we compute the product version of reciprocal degree distance of join of two graphs.

Theorem 2. *Let G_1 be a graph of order n and size p and let G_2 be a graph of order m and size q. Then* $H_A^*(G_1+G_2) \leq \Big[(M_1(G_1)+2mp)(M_1(G_2)+2nq)(\overline{M}_1(G_1)+m(n(n-1)-2p))(\overline{M}_1(G_2)+n(m(m-1)-2q))(2mp+2nq+mn(m+n))\Big]^{nm}$.

Proof. Let $V(G_1)=\{u_1,u_2,\ldots,u_n\}$ and $V(G_2)=\{v_1,v_2,\ldots,v_m\}$. By definition of the join of two graphs, one can see that, $d_{G_1+G_2}(x) = \begin{cases} d_{G_1}(x)+|V(G_2)|, & \text{if } x\in V(G_1) \\ d_{G_2}(x)+|V(G_1)|, & \text{if } x\in V(G_2) \end{cases}$ and

$$d_{G_1+G_2}(u,v) = \begin{cases} 0, & \text{if } u=v \\ 1, & \text{if } uv\in E(G_1) \text{ or } uv\in E(G_2) \text{ or } (u\in V(G_1) \text{ and } v\in V(G_2)) \\ 2, & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned}
H_A^*(G_1+G_2) &= \prod_{\{u,v\}\subseteq V(G_1+G_2)} \frac{d_{G_1+G_2}(u)+d_{G_1+G_2}(v)}{d_{G_1+G_2}(u,v)} \\
&= \prod_{uv\in E(G_1)} \Big((d_{G_1}(u)+m)+(d_{G_1}(v)+m)\Big) \times \\
&\quad \prod_{uv\notin E(G_1)} \frac{(d_{G_1}(u)+m)+(d_{G_1}(v)+m)}{2} \times \\
&\quad \prod_{uv\in E(G_2)} \Big((d_{G_2}(u)+n)+(d_{G_2}(v)+n)\Big) \times \\
&\quad \prod_{uv\notin E(G_2)} \frac{(d_{G_2}(u)+n)+(d_{G_2}(v)+n)}{2} \times \\
&\quad \prod_{u\in V(G_1),\ v\in V(G_2)} \Big((d_{G_1}(u)+m)+(d_{G_2}(v)+n)\Big) \\
&\leq \left[\frac{\sum\limits_{uv\in E(G_1)} \Big(d_{G_1}(u)+d_{G_1}(v)+2m\Big)}{nm}\right]^{nm} \left[\frac{\sum\limits_{uv\notin E(G_1)} \frac{d_{G_1}(u)+d_{G_1}(v)+2m}{2}}{nm}\right]^{nm} \\
&\quad \left[\frac{\sum\limits_{uv\in E(G_2)} \Big(d_{G_2}(u)+d_{G_2}(v)+2n\Big)}{nm}\right]^{nm} \left[\frac{\sum\limits_{uv\notin E(G_2)} \frac{d_{G_2}(u)+d_{G_2}(v)+2n}{2}}{nm}\right]^{nm} \\
&\quad \left[\frac{\sum\limits_{u\in V(G_1),\ v\in V(G_2)} \Big(d_{G_1}(u)+d_{G_2}(v)+(m+n)\Big)}{nr}\right]^{nm} \quad \text{(by Remark 1)} \\
&= \Big[M_1(G_1)+2mp\Big]^{nm} \Big[M_1(G_2)+2nq\Big]^{nm} \Big[\overline{M}_1(G_1)+m(n(n-1)-2p)\Big]^{nm} \\
&\quad \Big[\overline{M}_1(G_2)+n(m(m-1)-2q)\Big]^{nm} \Big[2mp+2nq+mn(m+n)\Big]^{nm}.
\end{aligned}$$

This completes the proof. ∎

One can observe that $M_1(C_n) = 4n,\ n \geq 3$, $M_1(P_1) = 0$, $M_1(P_n) = 4n-6,\ n > 1$ and $M_1(K_n) = n(n-1)^2$. Similarly, $\overline{M_1}(K_n) = \overline{M_2}(K_n) = 0$. Moreover $M_2(P_n) = 4(n-2)$ and $M_2(C_n) = 4n$. Using Theorem 2, we have the following corollaries.

Corollary 3. Let G be graph on n vertices and p edges. Then $H_A^*(G+K_m) \leq \Big[(M_1(G_1)+2mp)(\overline{M}_1(G_1) + m(n(n-1)-2p))(m(m-1)(m+n-1))(2mp+nm(2m+n-1))\Big]^{nm}$.

Let $K_{n,m}$ be the bipartite graph with two partitions having n and m vertices. Note that $K_{n,m} = \overline{K}_n + \overline{K}_m$.

Corollary 4. $H_A^*(K_{n,m}) = H_A^*(\overline{K}_n + \overline{K}_m) \leq (nm)^{3nm}\Big[(n-1)(m-1)(m+n)\Big]^{nm}$.

4. Strong product

In this section, we obtain the product version of reciprocal degree distance of $G \boxtimes K_r$.

Theorem 3. Let G be a connected graph with n vertices and m edges. Then

$$H_A^*(G \boxtimes K_r) = H_A^*(G \boxtimes K_r) \leq \frac{(r-1)^{2nr}}{n^{3nr}}\Big[(2rm + n(r-1)\Big]^{nr}\Big[rH_A(G) + 2(r-1)H(G))\Big]^{2nr}.$$

Proof. Let $V(G) = \{u_1, u_2, \dots, u_n\}$ and $V(K_r) = \{v_1, v_2, \dots, v_r\}$. Let x_{ij} denote the vertex (u_i, v_j) of $G \boxtimes K_r$. The degree of the vertex x_{ij} in $G \boxtimes K_r$ is $d_G(u_i) + d_{K_r}(v_j) + d_G(u_i)d_{K_r}(v_j)$, that is $d_{G\boxtimes K_r}(x_{ij}) = rd_G(u_i) + (r-1)$. One can observe that for any pair of vertices $x_{ij},\ x_{kp} \in V(G \boxtimes K_r)$, $d_{G\boxtimes K_r}(x_{ij}, x_{ip}) = 1$ and $d_{G\boxtimes K_r}(x_{ij}, x_{kp}) = d_G(u_i, u_k)$.

$$\begin{aligned} H_A^*(G \boxtimes K_r) &= \prod_{x_{ij},\, x_{kp} \in V(G\boxtimes K_r)} \frac{d_{G\boxtimes K_r}(x_{ij}) + d_{G\boxtimes K_r}(x_{kp})}{d_{G\boxtimes K_r}(x_{ij}, x_{kp})} \\ &= \prod_{i=0}^{n-1} \prod_{\substack{j,\,p=0 \\ j\neq p}}^{r-1} \frac{d_{G\boxtimes K_r}(x_{ij}) + d_{G\boxtimes K_r}(x_{ip})}{d_{G\boxtimes K_r}(x_{ij}, x_{ip})} \times \\ &\quad \prod_{\substack{i,\,k=0 \\ i\neq k}}^{n-1} \prod_{j=0}^{r-1} \frac{d_{G\boxtimes K_r}(x_{ij}) + d_{G\boxtimes K_r}(x_{kj})}{d_{G\boxtimes K_r}(x_{ij}, x_{kj})} \times \\ &\quad \prod_{\substack{i,\,k=0 \\ i\neq k}}^{n-1} \prod_{\substack{j,\,p=0 \\ j\neq p}}^{r-1} \frac{d_{G\boxtimes K_r}(x_{ij}) + d_{G\boxtimes K_r}(x_{kp})}{d_{G\boxtimes K_r}(x_{ij}, x_{kp})}. \end{aligned} \tag{8}$$

We shall obtain the products of (8) separately. First we compute $\prod\limits_{i=0}^{n-1}\prod\limits_{\substack{j,\,p=0\\ j\neq p}}^{r-1}\frac{d_{G\boxtimes K_r}(x_{ij})+d_{G\boxtimes K_r}(x_{ip})}{d_{G\boxtimes K_r}(x_{ij},x_{ip})}$. By Remark 1, we have

$$\begin{aligned}
\prod_{i=0}^{n-1}\prod_{\substack{j,\,p=0\\ j\neq p}}^{r-1}\frac{d_{G\boxtimes K_r}(x_{ij})+d_{G\boxtimes K_r}(x_{ip})}{d_{G\boxtimes K_r}(x_{ij},x_{ip})} &= \prod_{i=0}^{n-1}\prod_{\substack{j,\,p=0\\ j\neq p}}^{r-1}\Big(2d_G(u_i)+2(r-1)+2(r-1)d_G(u_i)\Big)\\
&\leq \left[\frac{\frac{1}{2}\sum\limits_{i=0}^{n-1}\sum\limits_{\substack{j,\,p=0\\ j\neq p}}^{r-1}\Big(2d_G(u_i)+2(r-1)+2(r-1)d_G(u_i)\Big)}{nr}\right]^{nr}\\
&= \Big[\frac{2r^2(r-1)m+nr(r-1)^2}{nr}\Big]^{nr}\\
&= \Big[\frac{2r(r-1)m+n(r-1)^2}{n}\Big]^{nr}. \qquad (9)
\end{aligned}$$

Next we compute $\prod\limits_{j=0}^{r-1}\prod\limits_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\frac{d_{G\boxtimes K_r}(x_{ij})+d_{G\boxtimes K_r}(x_{kj})}{d_{G\boxtimes K_r}(x_{ij},x_{kj})}$. We have

$$\begin{aligned}
&\prod_{j=0}^{r-1}\prod_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\frac{d_{G\boxtimes K_r}(x_{ij})+d_{G\boxtimes K_r}(x_{kj})}{d_{G\boxtimes K_r}(x_{ij},x_{kj})}\\
&= \prod_{j=0}^{r-1}\prod_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\frac{\Big(d_G(u_i)+(r-1)d_G(u_i)+d_G(u_k)+(r-1)d_G(u_k)+2(r-1)\Big)}{d_G(u_i,u_k)}\\
&\leq \left[\frac{\frac{r}{2}\sum\limits_{j=0}^{r-1}\sum_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\frac{d_G(u_i)+d_G(u_k)}{d_G(u_i,u_k)}+\frac{1}{2}\sum\limits_{j=0}^{r-1}\sum\limits_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\frac{2(r-1)}{d_G(u_i,u_k)}}{nr}\right]^{nr} \quad \text{(by Remark 1)}\\
&= \Big[\frac{r^2H_A(G)+2r(r-1)H(G)}{nr}\Big]^{nr}\\
&= \Big[\frac{rH_A(G)+2(r-1)H(G)}{n}\Big]^{nr}. \qquad (10)
\end{aligned}$$

Finally we determine $\prod\limits_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\prod\limits_{\substack{j,\,p=0,\\ j\neq p}}^{r-1}\frac{d_{G\boxtimes K_r}(x_{ij})+d_{G\boxtimes K_r}(x_{kp})}{d_{G\boxtimes K_r}(x_{ij},x_{kp})}$.

$$\begin{aligned}
&\prod_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\prod_{\substack{j,\,p=0,\\ j\neq p}}^{r-1}\frac{d_{G\boxtimes K_r}(x_{ij})+d_{G\boxtimes K_r}(x_{kp})}{d_{G\boxtimes K_r}(x_{ij},x_{kp})}\\
&\leq \left[\frac{\frac{r^2(r-1)}{2}\sum\limits_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\frac{d_G(u_i)+d_G(u_k)}{d_G(u_i,u_k)}+r(r-1)^2\sum\limits_{\substack{i,\,k=0\\ i\neq k}}^{n-1}\frac{1}{d_G(u_i,u_k)}}{nr}\right]^{nr} \quad \text{(by Remark 1)}\\
&= \Big[\frac{r^2(r-1)H_A(G)+2r(r-1)^2H(G)}{nr}\Big]^{nr}\\
&= \Big[\frac{r(r-1)H_A(G)+2(r-1)^2H(G)}{n}\Big]^{nr}. \qquad (11)
\end{aligned}$$

Using (9), (10) and (11) in (8), we have

$$H_A^*(G \boxtimes K_r) = \frac{(r-1)^{2nr}}{n^{3nr}} \Big[(2rm + n(r-1)\Big]^{nr} \Big[rH_A(G) + 2(r-1)H(G))\Big]^{2nr}.$$

□

Using Theorem 3, we obtain the following corollary.

Corollary 5. $H_A^*(C_n \boxtimes K_r) \leq (3r-1)^{nr}\left[\frac{2(r-1)(3r-1)H(C_n)}{n}\right]^{2nr}$.

As an application we present formula for product version of reciprocal degree distance of closed fence graph, $C_n \boxtimes K_2$.

Example 2. By Corollary 5, we have

$$H_M^*(C_n \boxtimes K_2) \leq \begin{cases} 5^{2n}\left[\frac{10}{n}(n\sum\limits_{i=1}^{\frac{n}{2}} \frac{1}{i} - 1)\right]^{4n}, & \text{if } n \text{ is even} \\ 5^{2n}\left[10\sum\limits_{i=1}^{\frac{n-1}{2}} \frac{1}{i}\right]^{4n}, & \text{if } n \text{ is odd.} \end{cases}$$

References

[1] Y. Alizadeh, A. Iranmanesh, and T. Došlić, *Additively weighted Harary index of some composite graphs*, Discrete Math. **313** (2013), no. 1, 26–34.

[2] K.C. Das, B. Zhou, and N. Trinajstić, *Bounds on Harary index*, J. Math. Chem. **46** (2009), no. 4, 1377–1393.

[3] J. Devillers and A.T. Balaban, *Topological indices and related descriptors in QSAR and QSPAR*, Gordon and Breach, Amsterdam, The Netherlands, 2000.

[4] A.A. Dobrynin and A.A. Kochetova, *Degree distance of a graph: A degree analog of the Wiener index*, J. Chem. Inf. Comput. Sci. **34** (1994), no. 5, 1082–1086.

[5] T. Došlić, *Vertex-weighted Wiener polynomials for composite graphs*, Ars Math. Contemp. **1** (2008), no. 1.

[6] I. Gutman, *Selected properties of the Schultz molecular topological index*, J. Chem. Inf. Comput. Sci. **34** (1994), no. 5, 1087–1089.

[7] I. Gutman, W. Linert, I. Lukovits, and Ž. Tomović, *The multiplicative version of the Wiener index*, J. Chem. Inf. Comput. Sci. **40** (2000), no. 1, 113–116.

[8] ———, *On the multiplicative Wiener index and its possible chemical applications*, Monatshefte für Chemie/Chemical Monthly **131** (2000), no. 5, 421–427.

[9] M. Hoji, Z. Luo, and E. Vumar, *Wiener and vertex PI indices of Kronecker products of graphs*, Discrete Appl. Math. **158** (2010), no. 16, 1848–1855.

[10] H. Hua and S. Zhang, *On the reciprocal degree distance of graphs*, Discrete Appl. Math. **160** (2012), no. 7, 1152–1163.

[11] M.H. Khalifeh, H. Yousefi-Azari, and A.R. Ashrafi, *Vertex and edge PI indices of Cartesian product graphs*, Discrete Appl. Math. **156** (2008), no. 10, 1780–1789.

[12] K. Pattabiraman and P. Paulraja, *On some topological indices of the tensor products of graphs*, Discrete Appl. Math. **160** (2012), no. 3, 267–279.

[13] ———, *Wiener and vertex PI indices of the strong product of graphs*, Discuss. Math. Graph Theory **32** (2012), no. 4, 749–769.

[14] K. Xu, K.C. Das, H. Hua, and M.V. Diudea, *Maximal Harary index of unicyclic graphs with given matching number*, Studia Univ. Babes–Bolyai Chem. **58** (2013), 71–86.

[15] K. Xu, J. Wang, and H. Liu, *The Harary index of ordinary and generalized quasi-tree graphs*, J. Appl. Math. Comput. **45** (2014), no. 1-2, 365–374.