Roman domination excellent graphs: trees

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Abstract: A Roman dominating function (RDF) on a graph \( G = (V, E) \) is a labeling \( f: V \to \{0, 1, 2\} \) such that every vertex with label 0 has a neighbor with label 2. The weight of \( f \) is the value \( f(V) = \sum_{v \in V} f(v) \). The Roman domination number, \( \gamma_R(G) \), of \( G \) is the minimum weight of an RDF on \( G \). An RDF of minimum weight is called a \( \gamma_R \)-function. A graph \( G \) is said to be \( \gamma_R \)-excellent if for each vertex \( x \in V \) there is a \( \gamma_R \)-function \( h_x \) on \( G \) with \( h_x(x) \neq 0 \). We present a constructive characterization of \( \gamma_R \)-excellent trees using labelings. A graph \( G \) is said to be in class \( UVR \) if \( \gamma(G - v) = \gamma(G) \) for each \( v \in V \), where \( \gamma(G) \) is the domination number of \( G \). We show that each tree in \( UVR \) is \( \gamma_R \)-excellent.

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1. Introduction and preliminaries

For basic notation and graph theory terminology not explicitly defined here, we in general follow Haynes et al. [9]. Specifically, let \( G \) be a simple graph with vertex set \( V(G) \) and edge set \( E(G) \). A spanning subgraph for \( G \) is a subgraph of \( G \) which contains every vertex of \( G \). In a graph \( G \), for a subset \( S \subseteq V(G) \) the subgraph induced by \( S \) is the graph \( \langle S \rangle \) with vertex set \( S \) and edge set \( \{xy \in E(G) \mid x, y \in S\} \). The complement \( \bar{G} \) of \( G \) is the graph whose vertex set is \( V(G) \) and whose edges are the pairs of nonadjacent vertices of \( G \). We write \( K_n \) for the complete graph of order \( n \) and \( P_n \) for the path on \( n \) vertices. Let \( C_m \) denote the cycle of length \( m \). For any vertex \( x \) of a graph \( G \), \( N_G(x) \) denotes the set of all neighbors of \( x \) in \( G \), \( N_G[x] = N_G(x) \cup \{x\} \) and the degree of \( x \) is \( \deg_G(x) = |N_G(x)| \). The minimum and maximum degrees of a graph \( G \) are denoted by \( \delta(G) \) and \( \Delta(G) \), respectively. For a subset \( S \) of vertices, let
$N_G[S] = \bigcup_{v \in S} N_G[v]$. The **external private neighborhood** \( epn(v, S) \) of \( v \in S \) is defined by \( epn(v, S) = \{ u \in V(G) - S \mid N_G(u) \cap S = \{v\} \} \). A **leaf** is a vertex of degree one and a **support vertex** is a vertex adjacent to a leaf. If \( F \) and \( H \) are disjoint graphs, \( v_F \in V(F) \) and \( v_H \in V(H) \), then the **coalescence** \((F \cdot H)(v_F, v_H : v)\) of \( F \) and \( H \) via \( v_F \) and \( v_H \), is the graph obtained from the union of \( F \) and \( H \) by identifying \( v_F \) and \( v_H \) in a vertex labeled \( v \). If \( F \) and \( H \) are graphs with exactly one vertex in common, say \( x \), then the **coalescence** \((F \cdot H)(x)\) of \( F \) and \( H \) via \( x \) is the union of \( F \) and \( H \).

Let \( Y \) be a finite set of integers which has positive as well as non-positive elements. Denote by \( P(Y) \) the collection of all subsets of \( Y \). Given a graph \( G \), for a \( Y \)-valued function \( f : V(G) \to Y \) and a subset \( S \) of \( V(G) \) we define \( f(S) = \sum_{v \in S} f(v) \). The **weight** of \( f \) is \( f(V(G)) \). A \( Y \)-**valued Roman dominating function** on a graph \( G \) is a function \( f : V(G) \to Y \) satisfying the conditions: (a) \( f(N_G[v]) \geq 1 \) for each \( v \in V(G) \), and (b) if \( v \in V(G) \) and \( f(v) \leq 0 \), then there is \( u_v \in N_G(v) \) with \( f(u_v) = \max\{k \mid k \in Y\} \). For a \( Y \)-valued Roman dominating function \( f \) on a graph \( G \), where \( Y = \{r_1, r_2, \ldots, r_k\} \) and \( r_1 < r_2 < \cdots < r_k \), let \( V^f_i = \{ v \in V(G) \mid f(v) = r_i \} \) for \( i = 1, \ldots, k \). Since these \( k \) sets determine \( f \), we can equivalently write \( f = (V^f_1; V^f_2; \ldots; V^f_k) \). If \( f \) is \( Y \)-valued Roman dominating function on a graph \( G \) and \( H \) is a subgraph of \( G \), then we denote the restriction of \( f \) on \( H \) by \( f|_H \). The **\( Y \)-Roman domination number** of a graph \( G \), denoted \( \gamma^Y_R(G) \), is defined to be the minimum weight of a \( Y \)-valued dominating function on \( G \). As examples, let us mention: (a) the domination number \( \gamma(G) = \gamma^\{0,1\}_R(G) \), (b) the minus domination number \([6], Y = \{-1, 0, 1\} \), (c) the signed domination number \([5], Y = \{-1, 1\} \), (d) the Roman domination number \(\gamma_R(G) = \gamma^\{0,1,2\}_R(G) \)[4], and (e) the signed Roman domination number \([1], Y = \{-1, 1, 2\} \).

A \( Y \)-valued Roman dominating function \( f \) on \( G \) with weight \( \gamma^Y_R(G) \) is called a \( \gamma^Y_R \)-function on \( G \).

Now we introduce a new partition of a vertex set of a graph, which plays a key role in the paper. In determining this partition, all \( \gamma^Y_R \)-functions of a graph are necessary. For each \( X \in P(Y) \) we define the set \( V^X(G) \) as consisting of all \( v \in V(G) \) with \( \{ f(v) \mid f \text{ is a } \gamma^Y_R \text{-function on } G \} = X \). Then all members of the family \( \{ V^X(G) \}_{X \in P(Y)} \) clearly form a partition of \( V(G) \). We call this partition the **\( Y \)-partition of \( G \)**.

Fricke et al. [7] in 2002 began the study of graphs, which are excellent with respect to various graph parameters. Let us concentrate here on the parameter \( \gamma^Y_R \). A vertex \( v \in V(G) \) is said to be (a) **\( \gamma^Y_R \)-good**, if \( h(v) \geq 1 \) for some \( \gamma^Y_R \)-function \( h \) on \( G \), and (b) **\( \gamma^Y_R \)-bad** otherwise. A graph \( G \) is said to be **\( \gamma^Y_R \)-excellent** if all vertices of \( G \) are **\( \gamma^Y_R \)-good**. Any vertex-transitive graph is **\( \gamma^Y_R \)-excellent**. Note that when \( \gamma^Y_R \equiv \gamma \), the set of all \( \gamma \)-good and the set of all \( \gamma \)-bad vertices of a graph \( G \) form the **\( \gamma \)-partition of \( G \)**. For further results on this topic see e.g. \([2, 10–15]\).

In this paper we begin an investigation of **\( \gamma^Y_R \)-excellent graphs** in the case when \( Y = \{0, 1, 2\} \). In what follows we shall write \( \gamma^Y_R \) instead of \( \gamma^\{0,1,2\}_R \), and we shall abbreviate a \( \{0,1,2\} \)-valued Roman dominating function to an **RD-function**. Let us describe all members of the \( \gamma^Y_R \)-partition of any graph \( G \) (we write \( V^i(G) \), \( V^{ij}(G) \) and \( V^{ijk}(G) \) instead of \( V^{\{i\}}(G) \), \( V^{\{i,j\}}(G) \) and \( V^{\{i,j,k\}}(G) \), respectively).

(i) \( V^i(G) = \{ x \in V(G) \mid f(x) = i \} \) for each \( \gamma^Y_R \)-function \( f \) on \( G \), \( i = 1, 2, 3; \).
\( \gamma_{R}(G) = \{ x \in V(G) \mid \text{there are } \gamma_{R}\text{-functions } f_{x}, g_{x}, h_{x} \text{ on } G \text{ with} \\
\quad f_{x}(x) = 0, g_{x}(x) = 1 \text{ and } h_{x}(x) = 2 \} \); \\
\( V^{012}(G) = \{ x \in V(G) - V^{012}(G) \mid \text{there are } \gamma_{R}\text{-functions } f_{x} \text{ and } g_{x} \text{ on } G \\
\quad \text{with } f_{x}(x) = i \text{ and } g_{x}(x) = j, 0 \leq i < j \leq 2 \} \). 

Clearly a graph \( G \) is \( \gamma_{R}\)-excellent if and only if \( V^{0}(G) = \emptyset \).

It is often of interest to known how the value of a graph parameter is affected when a small change is made in a graph. In this connection, Hansberg, Jafari Rad and Volkmann studied in [8] changing and unchanging of the Roman domination number of a graph when a vertex is deleted, or an edge is added.

**Lemma 1.** ([8]) Let \( v \) be a vertex of a graph \( G \). Then \( \gamma_{R}(G - v) < \gamma_{R}(G) \) if and only if there is a \( \gamma_{R}\)-function \( f = (V_{0}, V_{1}, V_{2}) \) on \( G \) such that \( v \in V_{1} \). If \( \gamma_{R}(G - v) < \gamma_{R}(G) \) then \( \gamma_{R}(G - v) = \gamma_{R}(G) - 1 \).

Lemma 1 implies that \( V^{1}(G), V^{01}(G), V^{12}(G), V^{012}(G) \) form a partition of \( V^{-}(G) = \{ x \in V(G) \mid \gamma_{R}(G - x) + 1 = \gamma(G) \} \).

**Lemma 2.** ([8]) Let \( x \) and \( y \) be non-adjacent vertices of a graph \( G \). Then \( \gamma_{R}(G) \geq \gamma_{R}(G + xy) \geq \gamma_{R}(G) - 1 \). Moreover, \( \gamma_{R}(G + xy) = \gamma_{R}(G) - 1 \) if and only if there is a \( \gamma_{R}\)-function \( f \) on \( G \) such that \( \{ f(x), f(y) \} = \{ 1, 2 \} \).

The same authors defined the following two classes of graphs:

(i) \( R_{CVR} \) is the class of graphs \( G \) such that \( \gamma_{R}(G - v) < \gamma_{R}(G) \) for all \( v \in V(G) \).

(ii) \( R_{CEA} \) is the class of graphs \( G \) such that \( \gamma_{R}(G + e) < \gamma_{R}(G) \) for all \( e \in E(G) \).

**Remark 1.** By Lemmas 1 and 2 it easy follows that:

(i) each graph in \( R_{CVR} \cup R_{CEA} \) is \( \gamma_{R}\)-excellent,

(ii) if \( G \) is a \( \gamma_{R}\)-excellent graph, \( e \in E(G) \) and \( \gamma_{R}(G) = \gamma_{R}(G + e) \), then \( G + e \) is \( \gamma_{R}\)-excellent,

(iii) each graph (in particular each \( \gamma_{R}\)-excellent graph) is a spanning subgraph of a graph in \( R_{CEA} \) with the same Roman domination number.

Denote by \( G_{n,k} \) the family of all mutually non-isomorphic \( n \)-order \( \gamma_{R}\)-excellent connected graphs having the Roman domination number equal to \( k \). With the family \( G_{n,k} \), we associate the poset \( \mathbb{RE}_{n,k} = (G_{n,k}, \prec) \) with the order \( \prec \) given by \( H_{1} \prec H_{2} \) if and only if \( H_{2} \) has a spanning subgraph which is isomorphic to \( H_{1} \) (see [16] for terminology on posets). Remark 1 shows that all maximal elements of \( \mathbb{RE}_{n,k} \) are in \( R_{CEA} \). Here we concentrate on the set of all minimal elements of \( \mathbb{RE}_{n,k} \). Clearly a graph \( H \in G_{n,k} \) is a minimal element of \( \mathbb{RE}_{n,k} \) if and only if for each \( e \in E(H) \) at
least one of the following holds: (a) $H - e$ is not connected, (b) $\gamma_R(H) \neq \gamma_R(H - e)$, and (c) $H - e$ is not $\gamma_R$-excellent. All trees in $G_{n,k}$ are obviously minimal elements of $\mathcal{RE}_{n,k}$.

The remainder of this paper is organized as follows. In Section 2, we formulate our main result, namely, a constructive characterization of $\gamma_R$-excellent trees. We present a proof of this result in Sections 3 and 4. Applications of our main result are given in Sections 5 and 6. We conclude in Section 7 with some open problems.

We end this section with the following useful result.

**Lemma 3.** ([4]) Let $f = (V_0^f; V_1^f; V_2^f)$ be any $\gamma_R$-function on a graph $G$. Then each component of a graph $\langle V_1^f \rangle$ has order at most 2 and no edge of $G$ joins $V_1^f$ and $V_2^f$.

In most cases Lemmas 1, 2 and 3 will be used in the sequel without specific reference.

### 2. The main result

In this section, we present a constructive characterization of $\gamma_R$-excellent trees using labelings. We define a **labeling** of a tree $T$ as a function $S : V(T) \rightarrow \{A, B, C, D\}$. A labeled tree is denoted by a pair $(T, S)$. The label of a vertex $v$ is also called its **status**, denoted $sta_T(v : S)$ or $sta_T(v)$ if the labeling $S$ is clear from context. We denote the sets of vertices of status $A, B, C$ and $D$ by $S_A(T), S_B(T), S_C(T)$ and $S_D(T)$, respectively. In all figures in this paper we use • for a vertex of status $A$, ◦ for a vertex of status $B$, ♦ for a vertex of status $C$, and ○ for a vertex of status $D$. If $H$ is a subgraph of $T$, then we denote the restriction of $S$ on $H$ by $S|_H$.

![Figure 1](image)

**Figure 1.** All trees with $|L_B \cup L_C| \leq 2$.

To state a characterization of $\gamma_R$-excellent trees, we introduce four types of operations. Let $\mathcal{F}$ be the family of labeled trees $(T, S)$ that can be obtained from a
sequence of labeled trees \( \tau : (T^1, S^1), \ldots, (T^j, S^j), (j \geq 1) \), such that \((T^1, S^1)\) is in \((H_1, I^1), \ldots, (H_5, I^5)\) (see Figure 1) and \((T, S) = (T^j, S^j)\), and, if \( j \geq 2 \), \((T^{i+1}, S^{i+1})\) can be obtained recursively from \((T^i, S^i)\) by one of the operations \(O_1, O_2, O_3\) and \(O_4\) listed below; in this case \(\tau\) is said to be a \(\mathcal{T}\)-sequence of \(T\). When the context is clear we shall write \(T \in \mathcal{T}\) instead of \((T, S) \in \mathcal{T}\).

![Figure 2. \((F, J)\)-graphs](image)

**Operation** \(O_1\). The labeled tree \((T^{i+1}, S^{i+1})\) is obtained from \((T^i, S^i)\) and \((F, J) \in \{(F_1, J^1), (F_2, J^2), (F_3, J^3)\}\) (see Figure 2) by adding the edge \(ux\), where \(u \in V(T_i), x \in V(F)\) and \(sta_T(u) = sta_F(x) = C\).

**Operation** \(O_2\). The labeled tree \((T^{i+1}, S^{i+1})\) is obtained from \((T^i, S^i)\) and \((F_4, J^4)\) (see Figure 2) by adding the edge \(ux\), where \(u \in V(T_i), x \in V(F_4)\), \(sta_T(u) = D\), and \(sta_{F_4}(x) = C\).

**Operation** \(O_3\). The labeled tree \((T^{i+1}, S^{i+1})\) is obtained from \((T^i, S^i)\) and \((H_k, I^k)\), \(k \in \{2, 3, \ldots, 7\}\) (see Figure 1), in such a way that \(T^{i+1} = (T^i \cdot H_k)(u, v : u)\), where \(sta_{T^i}(u) = sta_{H_k}(v) = A\), and \(sta_{T^{i+1}}(u) = A\).

**Operation** \(O_4\). The labeled tree \((T^{i+1}, S^{i+1})\) is obtained from \((T^i, S^i)\) and \((H_k, I^k)\), \(k \in \{3, 4, 6\}\) (see Figure 1), in such a way that \(T^{i+1} = (T^i \cdot H_k)(u, v : u)\), where \(sta_{T^i}(u) = D\), \(sta_{H_k}(v) = A\), and \(sta_{T^{i+1}}(u) = D\).

Remark that if \(y \in V(T^i)\) and \(i \leq k \leq j\), then \(sta_{T^i}(y) = sta_{T^j}(y)\). Now we are prepared to state the main result.

**Theorem 1.** Let \(T\) be a tree of order at least 2. Then \(T\) is \(\gamma_R\)-excellent if and only if there is a labeling \(S : V(T) \rightarrow \{A, B, C, D\}\) such that \((T, S)\) is in \(\mathcal{T}\). Moreover, if \((T, S) \in \mathcal{T}\) then
\[
(P_s) \quad S_B(T) = \{x \in V^{02}(T) \mid \deg(x) = 2 \text{ and } |N(x) \cap V^{02}(T)| = 1\}, \quad S_A(T) = V^{01}(T), \quad S_D(T) = V^{012}(T), \quad \text{and } S_C(T) = V^{02}(T) - S_B(T).
\]

3. Preparation for the proof of Theorem 1

3.1. Coalescence

We shall concentrate on the coalescence of two graphs via a vertex in \(V^{01}\) and derive the properties which will be needed for the proof of our main result.
Proposition 1. Let $G = (G_1 \cdot G_2)(x)$ be a connected graph and $x \in V^{01}(G)$. Then the following holds.

(i) If $f$ is a $\gamma_R$-function on $G$ and $f(x) = 1$, then $f|_{G_i}$ is a $\gamma_R$-function on $G_i$, and $f|_{G_i-x}$ is a $\gamma_R$-function on $G_i-x$, $i = 1, 2$.

(ii) $\gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1$.

(iii) If $h$ is a $\gamma_R$-function on $G$ and $h(x) = 0$, then exactly one of the following holds:

(iii.1) $h|_{G_1}$ is a $\gamma_R$-function on $G_1$, $h|_{G_2-x}$ is a $\gamma_R$-function on $G_2-x$, and $h|_{G_2}$ is no RD-function on $G_2$;

(iii.2) $h|_{G_1-x}$ is a $\gamma_R$-function on $G_1-x$, $h|_{G_1}$ is no RD-function on $G_1$, and $h|_{G_2}$ is a $\gamma_R$-function on $G_2$.

(iv) Either $\{x\} = V^{01}(G_1) \cap V^{01}(G_2)$ or $\{x\} = V^{01}(G_i) \cap V^1(G_j)$, where $\{i, j\} = \{1, 2\}$.

Proof. (i) and (ii): Since $f(x) = 1$, $f|_{G_i}$ is an RD-function on $G_i$, and $f|_{G_i-x}$ is an RD-function on $G_i-x$, $i = 1, 2$. Assume $g_1$ is a $\gamma_R$-function on $G_1$ with $g_1(V(G_1)) < f|_{G_1}(V(G_1))$. Define an RD-function $f'$ as follows: $f'(u) = g_1(u)$ for all $u \in V(G_1)$ and $f'(u) = f(u)$ when $u \in V(G_2-x)$. Then $f'(V(G)) = g_1(V(G_1)) + f|_{G_2-x}(V(G_2-x)) < f(V(G))$, a contradiction. Thus, $f|_{G_i}$ is a $\gamma_R$-function on $G_i$, $i = 1, 2$. Now, Lemma 1 implies that $f|_{G_i-x}$ is a $\gamma_R$-function on $G_i-x$, $i = 1, 2$. Hence $\gamma_R(G) = f|_{G_1}(V(G_1)) + f|_{G_2}(V(G_2)) - f(x) = \gamma_R(G_1) + \gamma_R(G_2) - 1$.

(iii) First note that $h(x) = 0$ implies $h|_{G_i}$ is an RD-function on $G_i$ for some $i \in \{1, 2\}$, say $i = 1$. If $h|_{G_2}$ is an RD-function on $G_2$ then $\gamma_R(G) = h(V(G)) \geq \gamma_R(G_1) + \gamma_R(G_2)$, a contradiction with (ii). Thus, $h|_{G_2-x}$ is an RD-function on $G_2-x$. Now we have $\gamma_R(G_1) + \gamma_R(G_2) - 1 = \gamma_R(G) = h(V(G)) = h|_{G_1}(V(G_1)) + h|_{G_2-x}(V(G_2-x)) \geq \gamma_R(G_1) + (\gamma_R(G_2) - 1)$. Hence $h|_{G_1}$ is a $\gamma_R$-function on $G_1$ and $h|_{G_2-x}$ is a $\gamma_R$-function on $G_2-x$.

(iv) Let $f_1$ be a $\gamma_R$-function on $G_1$. Assume first that $f_1(x) = 2$. Define an RD-function $g$ on $G$ as follows: $g(u) = f_1(u)$ when $u \in V(G_1)$ and $g(u) = f(u)$ when $u \in V(G_2-x)$, where $f$ is defined as in (i). The weight of $g$ is $\gamma_R(G_1) + (\gamma_R(G_2) + 1) - 2 = \gamma_R(G)$. But $g(x) = 2$ and $x \in V^{01}(G_1)$, a contradiction. Thus $f_1(x) \neq 2$. Now by (i) we have $x \in V^1(G_i) \cup V^{01}(G_i)$, $i = 1, 2$, and by (iii), $x \in V^{01}(G_j)$ for some $j \in \{1, 2\}$.

Proposition 2. Let $G = (G_1 \cdot G_2)(x)$, where $G_1$ and $G_2$ are connected graphs and $\{x\} = V^{01}(G_1) \cap V^{01}(G_2)$.

(i) If $f_i$ is a $\gamma_R$-function on $G_i$ with $f_i(x) = 1$, $i = 1, 2$, then the function $f : V(G) \rightarrow \{0, 1, 2\}$ with $f|_{G_i} = f_i$, $i = 1, 2$, is a $\gamma_R$-function on $G$.

(ii) $\gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1$.

(iii) Let $V_R = \{V^0, V^1, V^2, V^{01}, V^{02}, V^{12}, V^{012}\}$. Then for any $A \in V_R$, $A(G_1) \cup A(G_2) = A(G)$. 

\[ \square \]
Proof. (i) and (ii): Note that $f$ is an RD-function on $G$ and $\gamma_R(G) \leq f(V(G)) = f_1(V(G_1)) + f_2(V(G_2)) - f(x) = \gamma_R(G_1) + \gamma_R(G_2) - 1$. Now let $h$ be any $\gamma_R$-function on $G$.

Case 1: $h(x) \geq 1$. Then $h|_{G_i}$ is an RD-function on $G_i$, $i = 1, 2$. If $h(x) = 2$ then since $x \in V^{01}(G_1) \cap V^{01}(G_2)$, $h|_{G_i}$ is no $\gamma_R$-function on $G_i$, $i = 1, 2$. Hence $\gamma_R(G) \geq (\gamma_R(G_1) + 1) + (\gamma_R(G_2) + 1) - h(x) = \gamma_R(G_1) + \gamma_R(G_2)$, a contradiction. If $h(x) = 1$ then $\gamma_R(G) = (h(V(G)) = h(V(G_1)) + h(V(G_2)) - h(x) \geq \gamma_R(G_1) + \gamma_R(G_2) - 1$. Thus $h(x) = 1$, $\gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1$ and $f$ is a $\gamma_R$-function on $G$.

Case 2: $h(x) = 0$. Then at least one of $h|_{G_1}$ and $h|_{G_2}$ is an RD-function, say the first. If $h|_{G_2}$ is an RD-function on $G_2$ then $h(V(G)) \geq \gamma_R(G_1) + \gamma_R(G_2)$, a contradiction. Hence $h|_{G_2-x}$ is a $\gamma_R$-function on $G_2 - x$. But then $\gamma_R(G) = h(V(G)) \geq \gamma_R(G_1) + \gamma_R(G_2)$, $G_2 - x \geq \gamma_R(G_1) + \gamma_R(G_2) - 1 \geq \gamma_R(G)$. Thus, (i) and (ii) hold.

(iii): Let $g_1$ be a $\gamma_R$-function on $G_1$ with $g_1(x) = 0$, and $g_2$ a $\gamma_R$-function on $G_2 - x$. Then the RD-function $g$ on $G$ for which $g|_{G_1} = g_1$ and $g|_{G_2-x} = g_2$ has weight $g_1(V(G_1)) + g_2(V(G_2-x)) = \gamma_R(G_1) + \gamma_R(G_2) - x = \gamma_R(G_1) + \gamma_R(G_2) - 1 = \gamma_R(G)$. Hence by (i), $x \in V^{01}(G) \cup V^{012}(G)$. However, by Case 1 it follows that $h(x) \neq 2$ for any $\gamma_R$-function $h$ on $G$. Thus $x \in V^{01}(G)$.

Let $y \in V(G_1-x)$, $l_1$ a $\gamma_R$-function on $G_1$, and $h$ a $\gamma_R$-function on $G$. We shall prove that the following holds.

Claim 4.1 There are a $\gamma_R$-function $l$ on $G$, and a $\gamma_R$-function $h_1$ on $G_1$ such that $l(y) = l_1(y)$ and $h_1(y) = h(y)$.

Define an RD-function $l$ on $G$ as $l|_{G_1} = l_1$ and $l|_{G_2-x} = l_2$, where $l_2$ is a $\gamma_R$-function on $G_2 - x$. Since $l(V(G)) = \gamma_R(G_1) + \gamma_R(G_2) - x = \gamma_R(G)$, $l$ is a $\gamma_R$-function on $G$ and $l(y) = l_1(y)$.

Assume now that there is no $\gamma_R$-function $h_1$ on $G_1$ with $h_1(y) = h(y)$. Proposition 1 implies that, $h|_{G_1-x}$ is a $\gamma_R$-function on $G_1 - x$. But then the function $h': V(G_1) \to \{0, 1, 2\}$ defined as $h'(u) = 1$ when $u = x$ and $h'(u) = h|_{G_1}(u)$ otherwise, is a $\gamma_R$-function on $G_1$ with $h'(y) = h|_{G_1}(y)$, a contradiction.

By Claim 4.1 and since $x \in V^{01}(G)$, $A(G_1) = A(G) \cap V(G_1)$ for any $A \in V_R$. By symmetry, $A(G_2) = A(G) \cap V(G_2)$. Therefore $A(G_1) \cup A(G_2) = A(G)$ for any $A \in V_R$. \hfill \Box

Lemma 4. Let $G = (G_1 \cdot G_2)(x)$, where $G_1$ and $G_2$ are connected graphs and $\{x\} = V^{012}(G_1) \cap V^{012}(G_2)$. Then $\gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1$ and $x \in V^{012}(G)$.

Proof. Let $f_i$ be a $\gamma_R$-function on $G_i$ with $f_i(x) = 1$, $i = 1, 2$. Then the function $f$ defined as $f|_{G_i} = f_i$ is an RD-function on $G_i$, $i = 1, 2$. Hence $\gamma_R(G) \leq f(V(G)) = \gamma_R(G_1) + \gamma_R(G_2) - 1$. Let now $h$ be any $\gamma_R$-function on $G$.

Case 1: $h(x) = 2$. \hfill \Box
Since \( x \in V^{012}(G_1) \cap V^{01}(G_2) \), \( h|_{G_1} \) is a \( \gamma_R \)-function on \( G_1 \) and \( h|_{G_2} \) is an RD-function on \( G_2 \) of weight more than \( \gamma_R(G_2) \). Hence \( \gamma_R(G) = h(V(G)) \geq \gamma_R(G_1) + (\gamma_R(G_2) + 1) - h(x) \). Thus \( \gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1 \).

**Case 2:** \( h(x) = 1 \).
Then obviously \( h|_{G_1} \) and \( h|_{G_2} \) are \( \gamma_R \)-functions. Hence \( \gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1 \).

**Case 3:** \( h(x) = 0 \).
Hence at least one of \( h|_{G_1} \) and \( h|_{G_2} \) is a \( \gamma_R \)-function. If both \( h|_{G_1} \) and \( h|_{G_2} \) are \( \gamma_R \)-functions, then \( \gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) \), a contradiction. Hence either \( h|_{G_1} \) and \( h|_{G_2-x} \) are \( \gamma_R \)-functions, or \( h|_{G_1-x} \) and \( h|_{G_2} \) are \( \gamma_R \)-functions. Since \( \{x\} = V^{012}(G_1) \cap V^{01}(G_2) \), in both cases we have \( \gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1 \).
Thus, \( \gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1 \) and \( x \in V^{012}(G) \).

**3.2. Three lemmas for trees**

**Lemma 5.** Let \( T \) be a \( \gamma_R \)-excellent tree of order at least 2. Then \( V(T) = V^{01}(T) \cup V^{012}(T) \cup V^{02}(T) \).

**Proof.** Let \( x \in V(T), y \in N(x) \) and \( f \) a \( \gamma_R \)-function on \( T \). Suppose \( x \in V^1(T) \). If \( f(y) = 1 \), then the RD-function \( g \) on \( T \) defined as \( g(x) = 2, g(y) = 0 \) and \( g(u) = f(u) \) for all \( u \in V(T) - \{x, y\} \) is a \( \gamma_R \)-function on \( T \), a contradiction. But then \( N(x) \subseteq V^0(T) \), which is impossible.
Suppose now \( x \in V^2(T) \cup V^{12}(T) \). Hence \( x \) is not a leaf. Choose a \( \gamma_R \)-function \( h \) on \( T \) such that (a) \( h(x) = 2 \), and (b) \( k = |epn[x, V^k_2]| \) to be as small as possible. Let \( epn[x, V^k_2] = \{y_1, y_2, \ldots, y_k\} \) and denote by \( T_i \) the connected component of \( T - x \), which contains \( y_i \). Hence \( h(y_i) = 0 \) for all \( i \leq k \). Since \( T \) is \( \gamma_R \)-excellent, there is a \( \gamma_R \)-function \( f_k \) on \( T \) with \( f_k(y_i) \neq 0 \). Since \( x \in V^2(T) \cup V^{12}(T) \), \( f_k(x) \neq 0 \). If \( f_k(y_k) = 1 \) then \( f_k(x) = 1 \), which easily implies \( x \in V^{012}(T) \), a contradiction. Hence \( f_k(y_k) = f_k(x) = 2 \). Define a \( \gamma_R \)-function \( l \) on \( T \) as \( l|_{T_k} = f_k|_{T_k} \) and \( l(u) = h(u) \) for all \( u \in V(T) - V(T_k) \). But \( |epn[x, V^k_2]| < k \), a contradiction with the choice of \( h \). Thus \( V^1(T) \cup V^2(T) \cup V^{12}(T) \) is empty, and the required follows.

**Lemma 6.** Let \( T \) be a tree and \( V^-(T) \) is not empty. Then each component of \( \langle V^-(T) \rangle \) is either \( K_1 \) or \( K_2 \).

**Proof.** Assume that \( P : x_1, x_2, x_3 \) is a path in \( T \) and \( x_1, x_2, x_3 \in V^-(T) \). Then there is a \( \gamma_R \)-function \( f_i \) on \( T \) with \( f_i(x_i) = 1, i = 1, 2, 3 \) (by Lemma 1). Denote by \( T_j \) the connected component of \( T - x_2x_j \) that contains \( x_j, j = 1, 3 \). Then \( f_2|_{T_j} \) and \( f_3|_{T_j} \) are \( \gamma_R \)-functions on \( T_j, j = 1, 3 \). Now define a \( \gamma_R \)-function \( h \) on \( T \) such that \( h|_{T_j} = f_j|_{T_j}, j = 1, 3, \) and \( h(u) = f_2(u) \) when \( u \in V(T) - (V(T_1) \cup V(T_3)) \). But \( h(x_1) = h(x_2) = h(x_3) = 1 \), a contradiction.

**Lemma 7.** Let \( T \) be a \( \gamma_R \)-excellent tree of order at least 2.
(i) If $x \in V^{012}(T)$, then $x$ is adjacent to exactly one vertex in $V^-(T)$, say $y_1$, and $y_1 \in V^{012}(T)$.

(ii) Let $x \in V^{02}(T)$. If $\deg(x) \geq 3$ then $x$ has exactly 2 neighbors in $V^-(T)$. If $\deg(x) = 2$ then either $N_T(x) \subseteq V^{012}(T)$ or there is a path $u, x, y, z$ in $T$ such that $u, z \in V^{01}(T)$, $y \in V^{02}(T)$ and $\deg(y) = 2$.

(iii) $V^{01}(T)$ is either empty or independent.

**Proof.** Let $x \in V^{012}(T) \cup V^{02}(T)$ and $N(x) = \{y_1, y_2, \ldots, y_r\}$. If $x$ is a leaf, then clearly $x, y_1 \in V^{012}(T)$. So, let $r \geq 2$. Denote by $T_i$ the connected component of $T - x$ which contains $y_i$, $i \geq 1$. Choose a $\gamma_R$-function $h$ on $T$ such that (a) $h(x) = 2$, and (b) $k = |epm[x, V^h_2]|$ to be as small as possible. Let without loss of generality $epm[x, V^h_2] = \{y_1, y_2, \ldots, y_k\}$. By the definition of $h$ it immediately follows that (c) $h|_{T_j}$ is a $\gamma_R$-function on $T_j$ for all $j \geq k + 1$, (d) for each $i \in \{1, \ldots, k\}$, $h|_{T_i}$ is no RD-function on $T_i$, and (e) $h|_{T_i - y_i}$ is a $\gamma_R$-function on $T_i - y_i$.

Hence $\gamma_R(T_i) \leq \gamma_R(T_i - y_i) + 1$ for all $i \in \{1, \ldots, k\}$. Assume that the equality does not hold for some $i \leq k$. Define an RD-function $h_i$ on $T$ as follows: $h_i(u) = h(u)$ when $u \in V(T) - V(T_i)$ and $h_i|_{T_i} = h_i'$, where $h'_i$ is some $\gamma_R$-function on $T_i$. But then either $h_i$ has weight less than $\gamma_R(T)$ or $h_i$ is a $\gamma_R$-function on $T$ with $epm[x, V^h_{2i}] = epm[x, V^h_{2}] - \{y_i\}$. In both cases we have a contradiction. Thus $\gamma_R(T_i) = \gamma_R(T_i - y_i) + 1$ for all $i \in \{1, \ldots, k\}$. Therefore $\gamma_R(T) = h(V(T)) = 2 + \Sigma_{i=1}^k(\gamma_R(T_i) - 1) + \Sigma_{j=k+1}^r\gamma_R(T_j) = 2 - k + \Sigma_{i=1}^r\gamma_R(T_i) = 2 - k + \gamma_R(T - x)$. Thus $\gamma_R(T) = 2 - k + \gamma_R(T - x)$.

(i) Since $\gamma_R(T - x) + 1 = \gamma_R(T)$, $k = 1$. We already know that $h|_{T_j}$ is a $\gamma_R$-function on $T_j$, $j \geq 2$. Assume that $y_j \in V^{012}(T) \cup V^{01}(T)$ for some $j \geq 2$. Then there is a $\gamma_R$-function $l$ on $T$ with $l(y_j) = 1$. Clearly $l|_{T_j}$ is a $\gamma_R$-function on $T_j$. Now define a $\gamma_R$-function $h''$ on $T$ as follows: $h''(u) = h(u)$ when $u \in V(T) - V(T_j)$ and $h''(y_j) = 1$ and $xy_j \in E(G)$, which is impossible. Thus, $y_2, y_3, \ldots, y_r \in V^{02}(T)$. Define now $\gamma_R$-functions $h_1$ and $h_2$ on $T$ as follows: $h_1(u) = h_2(u) = h(u)$ for all $u \in V(T) - \{x, y_1\}$, $h_1(x) = h_1(y_1) = 1$, $h_2(x) = 0$ and $h_2(y_1) = 2$. Thus $y_1 \in V^{01}(T)$.

(ii) Since $\gamma_R(T - x) = \gamma_R(T)$, $k = 2$. Recall that $h|_{T_j}$ is a $\gamma_R$-function on $T_j$, $j \geq 3$, and $\gamma_R(T_i - y_i) = \gamma_R(T_i) - 1$ for $i = 1, 2$. Hence there is a $\gamma_R$-function $f_i$ on $T_i$ with $f_i(y_i) = 1$, $i = 1, 2$.

Suppose first that $r \geq 3$. As in the proof of (i), we obtain $y_3, \ldots, y_r \in V^{02}(T)$. Hence there is a $\gamma_R$-function $g$ on $T$ such that $g(y_3) = 2$. By the choice of $h$, $g(x) = 0$. Then $g|_{T_i}$ is a $\gamma_R$-function on $T_i$, $i = 1, 2$. Define now a $\gamma_R$-function $g'$ on $T$ as $g'|_{T_i} = f_i$, $i = 1, 2$, and $g'(u) = g(u)$ when $u \in V(T) - (V(T_1) \cup V(T_2))$. Since $g'(y_1) = g'(y_2) = 1, y_1, y_2 \in V^-(T)$.

So, let $r = 2$ and let $f$ be a $\gamma_R$-function on $T$ with $f(x) = 0$. Then there is $y_s$ such that $f(y_s) = 2$, say $s = 2$. Hence $y_2 \in V^{02}(T) \cup V^{012}(T)$ and $f|_{T_1}$ is a $\gamma_R$-function on $T_1$. Define the $\gamma_R$-function $l$ on $T$ as $l|_{T_1} = f_1$ and $l(u) = f(u)$ when $u \in V(T) - V(T_1)$. Since $l(y_1) = 1, y_1 \in V^{01}(T) \cup V^{012}(T)$. 
Assume first that $y_1 \in V_{012}(T)$. Then there is a $\gamma_R$-function $f'$ on $T$ with $f'(y_1) = 2$. Since $x \in V_{02}(T)$ and $\deg(x) = 2$, $f'(x) = 0$. Hence $f'|_{T_2}$ is a $\gamma_R$-function on $T_2$. But then we can choose $f'$ so that $f'|_{T_2} = f_2$. Thus $y_2 \in V_{012}(T)$.

So let $y_1 \in V_{01}(T)$ and suppose $y_2 \in V_{012}(T)$. Then there is a $\gamma_R$-function $f''$ on $T$ with $f''(y_2) = 1$. Since $x \in V_{02}(T)$, $f''(x) = 0$ and $f''(y_1) = 2$, a contradiction. Thus, if $y_1 \in V_{01}(T)$ then $y_2 \in V_{02}(T)$.

Finally, let us consider a path $y_1, x, y_2, z \in T$, where $y_1 \in V_{01}(T)$, $x, y_2 \in V_{02}(T)$ and $\deg(x) = 2$. Assume to the contrary that $N(y_2) = \{z_1, z_2, \ldots, z_s = x\}$ with $s \geq 3$. Denote by $T_{y_i}$ the connected component of $T - y_2$ that contains $z_p, p = 1, 2, \ldots, s$. By applying results proved above for $x \in V_{02}(T)$ with $\deg(x) \geq 3$ to $y_2$, we obtain that (a) $y_2$ has exactly 2 neighbors in $V^-(T)$, say, without loss of generality, $z_1, z_2 \in V^-(T)$, and (b) $\gamma_R(T_{z_i} - z_i) = \gamma_R(T_{z_i}) - 1$, where $i = 1, 2$. Recall now that: $h(x) = 2$, $h|_{T_i}$ is no RD-function on $T_i$ and $h|_{T_i - y_i}$ is a $\gamma_R$-function on $T_i - y_i$, $i = 1, 2$. Hence $h(y_i) = h(y_2) = 0$ and $h|_{T_j}$ is a $\gamma_R$-function on $T_j$, $j \leq s - 1$. Since $\gamma_R(T_j, z_i) = \gamma_R(T_{z_i}) - 1$, $i = 1, 2$, additionally we can choose $h$ so that $h(z_1) = h(z_2) = 1$. But then the function $h_1$ defined as $h_1(u) = h(u)$ when $u \in V(T) - \{y_1, x, y_2, z_1, z_2\}$ and $h_1(y_1) = h_1(x) = 1$, $h_1(y_2) = 2$, $h_1(z_1) = h(z_2) = 0$ is a $\gamma_R$-function on $T$. Now $h_1(x) = 1$, $h_1(y_2) = 2$ and $xy_2 \in E(G)$ lead to a contradiction. Thus, $N(y_2) = \{x, z\}$.

Suppose $z \notin V_{01}(T)$. Then there is a $\gamma_R$-function $h_4$ on $T$ with $h_4(z) = 2$. If $h_4(y_2) = 2$, then $h_4(x) = 0$ and the function $h_5$ on $T$ defined as $h_5(x) = h_5(y_2) = 1$ and $h_5(u) = h_4(u)$ otherwise, is a $\gamma_R$-function on $T$, a contradiction. Hence $h_4(y_2) = 0$ and since $y_1 \in V_{01}(T)$, $h_4(x) = 2$ and $h_4(y_1) = 0$. But then the function $h_6$ on $T$ defined as $h_6(x) = h_6(y_1) = 1$ and $h_6(u) = h_4(u)$ otherwise, is a $\gamma_R$-function on $T$, a contradiction. Therefore $z \in V_{01}(T)$, and we are done.

(iii) Assume that $u_1, u_2 \in V_{01}(T)$ are adjacent. Let $T_{u_i}$ be the component of $T - u_1u_2$ that contains $u_i$, $i = 1, 2$. Let $g_i$ be a $\gamma_R$-function on $T$ with $g_i(u_i) = 1$, $i = 1, 2$. Hence $g_1(T_{u_1})$ is a $\gamma_R$-function on $T_{u_1}$, $i, j = 1, 2$. Thus $\gamma_R(T) = \gamma_R(T_{u_1}) + \gamma_R(T_{u_2})$.

Define now a $\gamma_R$-function $g_3$ on $T$ as $g_3|_{T_i} = g_i|_{T_i}$, $i = 1, 2$. But then a function $g_4$ defined as $g_4(u) = g_3(u)$ when $u \in V(T) - \{u_1, u_2\}$, $g_4(u_1) = 2$ and $g_4(u_2) = 0$ is a $\gamma_R$-function on $T$, contradicting $u_1 \in V_{01}(T)$. Thus $V_{01}(T)$ is independent.

4. Proof of the main result

Proof of Theorem 1. Let $T$ be a $\gamma_R$-excellent tree. First, we shall prove the following statement.

($P_2$) There is a labeling $L : V(T) \rightarrow \{A, B, C, D\}$ such that (a) $L_A(T)$ is either empty or independent, (b) each component of $\langle L_B(T) \rangle$ and $\langle L_D(T) \rangle$ is isomorphic to $K_2$, (c) each element of $L_B(T)$ has degree 2 and it is adjacent to exactly one vertex in $L_A(T)$, (d) each vertex $v$ in $L_C(T)$ has exactly 2 neighbors in $L_A(T) \cup L_D(T)$, and if $\deg(v) = 2$ then both neighbors of $v$ are in $L_D(T)$.

By Lemma 5 we know that $V(T) = V_{01}(T) \cup V_{012}(T) \cup V_{02}(T)$. Define a labeling $L : V(T) \rightarrow \{A, B, C, D\}$ by $L_A(T) = V_{01}(T)$, $L_D(T) = V_{012}(T)$, $L_B(T) = \{x \in$
\(V^{02}(T)\mid \text{deg}(x) = 2\) and \(|N(x) \cap V^{02}(T)| = 1\), and \(L_C(T) = V^{02}(T) - L_B(T)\). The validity of \((P_2)\) immediately follows by Lemma 7.

Denote by \(\mathcal{T}\) the family of all labeled, as in \((P_2)\), trees \(T\). We shall show that if \((T, L) \in \mathcal{T}_1\) then \((T, L) \in \mathcal{T}\).

**I** Proof of \((T, L) \in \mathcal{T}_1 \Rightarrow (T, L) \in \mathcal{T}\).

Let \((T, L) \in \mathcal{T}_1\). The following claim is immediate.

**Claim 1.1**

(i) Each leaf of \(T\) is in \(L_A(T) \cup L_D(T)\).

(ii) If \(v\) is a support vertex of \(T\), then \(v\) is adjacent to at most 2 leaves.

(iii) If \(u_1\) and \(u_2\) are leaves adjacent to the same support vertex, then \(u_1, u_2 \in L_A(T)\).

We now proceed by induction on \(k = |L_B \cup L_C|\). The base case, \(k \leq 2\), is an immediate consequence of the following easy claim, the proof of which is omitted.

**Claim 1.2** (see Fig.1)

(i) If \(k = 0\) then \((T, L) = (H_1, I^1)\).

(ii) If \(k = 1\) then \((T, L)\) is obtained from \((H_1, I_1)\) by operation \(O_2\), i.e. \((T, L) = (H_{11}, I^{11})\).

(iii) If \(k = 2\) then either \((T, L)\) is \((H_r, I^r)\) with \(r \in \{2, 3, 4, 5\}\), or \((T, L)\) is obtained from \((H_{11}, I^{11})\) by operation \(O_1\) or by operation \(O_2\) (see the graphs \((H_s, I^s)\) where \(s \in \{6, 7, 8, 9, 10\}\).

Let \(k \geq 3\) and suppose that each tree \((H, L') \in \mathcal{T}_1\) with \(|L'_B(H) \cup L'_C(H)| < k\) is in \(\mathcal{T}\).

Let now \((T, L) \in \mathcal{T}_1\) and \(k = |L_B(T) \cup L_C(T)|\). To prove the required result, it suffices to show that \(T\) has a subtree, say \(U\), such that \((U, L|_U) \in \mathcal{T}_1\) and \((T, L)\) is obtained from \((U, L|_U)\) by one of operations \(O_1, O_2, O_3\) and \(O_4\). Consider any diametral path \(P : x_1, x_2, \ldots, x_n\) in \(T\). Clearly \(x_1\) is a leaf. Denote by \(x_1^1, x_1^2, \ldots\) all neighbors of \(x_i\), which do not belong to \(P\), \(2 \leq i \leq n - 1\).

**Case 1:** \(\text{sta}(x_1) = A\) and \(\text{sta}(x_2) = B\).

Then \(\text{deg}(x_1) = 1\), \(\text{deg}(x_2) = \text{deg}(x_3) = 2\), \(\text{sta}(x_3) = B\) and \(\text{sta}(x_4) = A\). Thus \(T\) is obtained from \(T - \{x_1, x_2, x_3\} \in \mathcal{T}_1\) and a copy of \(H_2\) by operation \(O_3\) (via \(x_4\)).

**Case 2:** \(\text{sta}(x_1) = A\) and \(\text{sta}(x_2) = C\).

Hence \(\text{deg}(x_2) \geq 3\). By the choice of \(P\), \(\text{deg}(x_2) = 3\), \(x_2^1\) is a leaf, \(\text{sta}(x_2^1) = A\), and \(\text{sta}(x_3) = C\). If \(\text{deg}(x_3) \geq 4\) then \(T\) is obtained from \(T - \{x_2^1, x_1, x_2\} \in \mathcal{T}_1\) and a copy of \(F_1\) by operation \(O_1\). So, let \(\text{deg}(x_3) = 3\). Assume first that \(\text{sta}(x_4) = A\). Then either \(x_3^1\) is a leaf of status \(A\) or \(x_3^1\) is a support vertex, \(\text{deg}(x_3^1) = 2\), and both \(x_3^1\) and its leaf-neighbor have status \(D\). Thus, \(T\) is obtained from \(T - (N[x_2] \cup N[x_3]) \in \mathcal{T}_1\) and a copy of \(H_3\) or \(H_4\), respectively, by operation \(O_3\) (via \(x_4\)). Finally let \(\text{sta}(x_4) = D\).

By the choice of \(P\), either \(x_3^1\) is a leaf of status \(A\) and then \(T\) is obtained from
In what follows, let \( sta(x_1) = D \). Hence \( \text{deg}(x_2) = 2 \), \( \text{sta}(x_2) = D \) and \( \text{sta}(x_3) = C \). If \( \text{deg}(x_3) = 2 \) then \( T \) is obtained from \( T - N[x_2] \in \mathcal{T}_1 \) and a copy of \( F_4 \) by operation \( O_2 \).

**Case 3:** \( \text{deg}(x_3) = 3 \) and \( \text{sta}(x_4) \in \{A, D\} \).

In this case \( \text{sta}(x_3^1) = C \), \( x_3^1 \) is a support vertex, \( \text{deg}(x_3^1) = 3 \), and the leaf neighbors of \( x_3^1 \) have status \( A \). Now (a) if \( \text{sta}(x_4) = A \) then \( T \) is obtained from \( T - (N[x_2] \cup N[x_3^1]) \in \mathcal{T}_1 \) and a copy of \( H_4 \) by operation \( O_3 \) (via \( x_4 \)), and (b) if \( \text{sta}(x_4) = D \) then \( T \) is obtained from \( T - (N[x_2] \cup N[x_3^1]) \in \mathcal{T}_1 \) and a copy of \( H_4 \) by operation \( O_4 \) (via \( x_4 \)).

**Case 4:** \( \text{deg}(x_3) = 3 \), \( \text{sta}(x_4) = C \) and \( \text{sta}(x_3^1) = A \).

Hence \( x_3^1 \) is a leaf. If \( \text{deg}(x_4) = 3 \) and \( \text{sta}(x_5) = \text{sta}(x_4^1) = D \), or \( \text{deg}(x_4) \geq 4 \), then \( T \) is obtained from \( T - \{x_1, x_2, x_3, x_3^1\} \in \mathcal{T}_1 \) and a copy of \( F_2 \) by operation \( O_1 \). So, let \( \text{deg}(x_4) = 3 \) and the status of at least one of \( x_5 \) and \( x_4^1 \) is \( A \). Assume first that \( \text{sta}(x_4^1) = A \). Hence \( x_4^1 \) is a leaf (by the choice of \( P \)). If \( \text{sta}(x_5) = A \) then \( T \) is obtained from a copy of \( H_4 \) and a tree in \( \mathcal{T}_1 \) by operation \( O_3 \) (via \( x_5 \)). If \( \text{sta}(x_5) = D \) then \( T \) is obtained from a copy of \( H_4 \) and a tree in \( \mathcal{T}_1 \) by operation \( O_4 \) (via \( x_5 \)). Second, let \( \text{sta}(x_4^1) = D \). Hence \( \text{sta}(x_5) = A \), \( \text{deg}(x_4^1) = 2 \) and the status of the leaf-neighbor of \( x_4^1 \) is \( D \). But then \( T \) is obtained from a copy of \( H_5 \) and a tree in \( \mathcal{T}_1 \) by operation \( O_3 \) (via \( x_5 \)).

**Case 5:** \( \text{deg}(x_3) = 3 \), \( \text{sta}(x_4) = C \) and \( \text{sta}(x_3^1) = D \).

Hence \( \text{deg}(x_3^1) = 2 \), \( x_3^1 \) is a support vertex, and the leaf-neighbor of \( x_3^1 \) has status \( D \). If \( \text{deg}(x_4) \geq 4 \) or \( \text{sta}(x_5) = \text{sta}(x_4^1) = D \), then \( T \) is obtained from \( T - N[x_2, x_3^1] \in \mathcal{T}_1 \) and a copy of \( F_3 \) by operation \( O_1 \). So, let \( \text{deg}(x_4) = 3 \) and at least one of \( x_5 \) and \( x_4^1 \) has status \( A \). Assume \( \text{sta}(x_4^1) = A \). Hence \( x_4^1 \) is a leaf. If \( \text{sta}(x_5) = A \) then \( T \) is obtained from \( T - N[x_2, x_3^1, x_4^1] \in \mathcal{T}_1 \) and a copy of \( H_6 \) by operation \( O_3 \) (via \( x_5 \)). If \( \text{sta}(x_5) = D \) then \( T \) is obtained from \( T - N[x_2, x_3^1, x_4^1] \in \mathcal{T}_1 \) and a copy of \( H_6 \) by operation \( O_4 \) (via \( x_5 \)). Now let \( \text{sta}(x_4^1) = D \). Hence \( \text{sta}(x_5) = A \) and then \( T \) is obtained from a copy of \( H_7 \) and a tree in \( \mathcal{T}_1 \) by operation \( O_3 \) (via \( x_5 \)).

**Case 6:** \( \text{deg}(x_3) \geq 4 \).

Hence \( x_3 \) has a neighbor, say \( y \), such that \( y \neq x_4 \) and \( \text{sta}(y) = C \). By the choice of \( P \), \( y \) is a support vertex which is adjacent to exactly 2 leaves, say \( z_1 \) and \( z_2 \), and \( \text{sta}(z_1) = \text{sta}(z_2) = A \). But then \( T \) is obtained from \( T - \{y, z_1, z_2\} \in \mathcal{T}_1 \) and a copy of \( F_1 \) by operation \( O_1 \).

By Claim 2.1, there are no other possibilities.

**(II)** \( (T, S) \in \mathcal{T} \Rightarrow (T, S) \in \mathcal{T}_1 \). Obvious.

It remains the following.

**(III)** Proof of \( (T, S) \in \mathcal{T} \Rightarrow T \) is \( \gamma_R \)-excellent and \( (P_1) \) holds.
Let \((T, S) \in \mathcal{F}\). We know that \((T, S) \in \mathcal{F}_1\). We now proceed by induction on \(k = |S_B \cup S_C|\). First let \(k \leq 2\). By Claim 1.2, \(T \in \mathcal{F} = \{H_1, \ldots, H_{11}\}\). It is easy to see that all elements of \(\mathcal{F}\) are \(\gamma_R\)-excellent graphs and \((P_1)\) holds for each \(T \in \mathcal{F}\).

Let \(k \geq 3\) and suppose that if \((H, S') \in \mathcal{F}\) and \(|S'_B(H) \cup S'_C(H)| < k\), then \(H\) is \(\gamma_R\)-excellent and \((P_1)\) holds with \((T, S)\) replaced by \((H, S')\). So, let \((T, S) \in \mathcal{F}\) and \(k = |S_B(T) \cup S_C(T)|\). Then there is a \(\mathcal{F}\)-sequence \(\tau : (T^1, S^1), \ldots, (T^{j-1}, S^{j-1}), (T, S)\). By induction hypothesis, \(T^{j-1}\) is \(\gamma_R\)-excellent and \((P_1)\) holds with \((T, S)\) replaced by \((T^{j-1}, S^{j-1})\). We consider four possibilities depending on whether \(T\) is obtained from \(T^{j-1}\) by operation \(O_1, O_2, O_3\) or \(O_4\).

**Case 7**: \(T\) is obtained from \(T^{j-1} \in \mathcal{F}\) and \(F_a\) by operation \(O_1\), \(a \in \{1, 2, 3\}\). Hence \(T\) is obtained after adding the edge \(ux\) to the union of \(T^{j-1}\) and \(F_a\), where \(\text{sta}_{T^{j-1}}(u) = \text{sta}_{F_a}(x) = C\) (see Fig. 2). First note that \(\gamma_R(F_a) = a + 1\), and \(F_2\) and \(F_3\) are \(\gamma_R\)-excellent graphs. Since \(\gamma_R(F_a - x) = \gamma_R(F_a)\) and \(u \in V^{02}(T^{j-1})\), Lemma 2 implies \(\gamma_R(T) = \gamma_R(T^{j-1}) + \gamma_R(F_a)\). Hence for any \(\gamma_R\)-function \(g\) on \(T\), the weight of \(g|_{F_a}\) is not more than \(\gamma_R(F_a)\). But then \(g(x) \neq 1\) and either \(g|_{F_a}\) is a \(\gamma_R\)-function on \(F_a\) or \(g|_{F_a - x}\) is a \(\gamma_R\)-function on \(F_a - x\). By inspection of all \(\gamma_R\)-functions on \(F_a\) and \(F_a - x\), we obtain

\[(\alpha_1)\quad S_A(T) \cap V(F_a) = V^{01}(T) \cap V(F_a), S_B(T) \cap V(F_a) = \emptyset, \{x\} = S_C(T) \cap V(F_a) = V^{02}(T) \cap V(F_a), \text{ and } S_D(T) \cap V(F_a) = V^{012}(T) \cap V(F_a).\]

By the definition of operation \(O_1\) it immediately follows

\[(\alpha_2)\quad S_X(T) \cap V(T^{j-1}) = S_X^{j-1}(T^{j-1}), \text{ for all } X \in \{A, B, C, D\}.\]

Let \(f_1\) be a \(\gamma_R\)-function on \(T^{j-1}\) and \(f_2\) a \(\gamma_R\)-function on \(F_a\). Then the RD-function \(f\) on \(T\) defined as \(f|_{T^{j-1}} = f_1\) and \(f|_{F_a} = f_2\) is a \(\gamma_R\)-function on \(T\). Since \(f_1\) was chosen arbitrarily, we have

\[(\alpha_3)\quad V^{01}(T^{j-1}) \subseteq V^{01}(T) \cup V^{012}(T), \quad V^{02}(T^{j-1}) \subseteq V^{02}(T) \cup V^{012}(T), \quad \text{and } V^{012}(T^{j-1}) \subseteq V^{012}(T).\]

By \((\alpha_1)\) and \((\alpha_3)\) we conclude that \(T\) is \(\gamma_R\)-excellent.

Now we shall prove that

\[(\alpha_4)\quad V^{01}(T) \cap V(T^{j-1}) = V^{01}(T^{j-1}), \quad V^{02}(T) \cap V(T^{j-1}) = V^{02}(T^{j-1}), \quad \text{and } V^{012}(T) \cap V(T^{j-1}) = V^{012}(T^{j-1}).\]

Assume there is a vertex \(z \in V^{02}(T^{j-1}) \cap V^{012}(T)\). By Lemma 7, \(z\) is adjacent to at most 2 elements of \(V^{-1}(T^{j-1})\). Now by \((\alpha_3)\) and since \(\Delta((V^{-1}(T))) \leq 1\) (by Lemma 6), \(z\) is adjacent to exactly one element of \(V^{-1}(T^{j-1})\). But then Lemma 7 implies that there is a path \(z_1, z, z_2, z_3\) in \(T^{j-1}\) such that \(\text{deg}_{T^{j-1}}(z) = \text{deg}_{T^{j-1}}(z_2) = 2, z, z_2 \in V^{02}(T^{j-1})\) and \(z_1, z_3 \in V^{01}(T^{j-1})\). Since \((P_1)\) is true for \(T^{j-1}\), \(\text{sta}_{T^{j-1}}(z_1) = \text{sta}_{T^{j-1}}(z_3) = A\), and \(\text{sta}_{T^{j-1}}(z) = \text{sta}_{T^{j-1}}(z_2) = B\). Clearly, at least one of \(z_1\) and \(z_3\) is a cut-vertex. Denote by \(Q\) the graph \(\langle\{z_1, z, z_2, z_3\}\rangle\) and let the vertices of \(Q\)
are labeled as in $T^{j-1}$. Let $U_s$ be the connected component of $T - \{z, z_2\}$, which contains $z_s$, $s = 1, 3$.

Assume first that $T^1$ is a subtree of $U \in \{U_1, U_3\}$. Then there is $i$ such that $T^i$ is obtained from $T^{i-1}$ and $Q$ by operation $O_3$. Hence $T^{i-1}$ is a subtree of $U$. Recall that if $y \in V(T^r)$ and $r \leq s \leq j - 1$, then $sta_{T^s}(y) = sta_{T^r}(y)$. Using this fact, we can choose $\tau$ so that $T^{i-1} = U$. Therefore $U$ is in $\mathcal{J}$. Suppose that neither $z_1$ nor $z_3$ is a leaf of $T^{j-1}$. Define $R^s = T^{i+s} - (V(T^{i-1}) \cup \{z, z_2\})$, $s = 1, 2, \ldots, j-1 - i$. Since clearly $R^1$ is in $\{H_2, H_3, \ldots, H_7\}$, the sequence $R^1, R^2, \ldots, R^{j-1-i}$ is a $\mathcal{J}$-sequence of $U'$, where $\{U, U'\} = \{U_1, U_2\}$. Thus, both $U_1$ and $U_3$ are in $\mathcal{J}$, and $sta_{U_1}(z_1) = A$. By the induction hypothesis, $z_1 \in V^{01}(U_1)$.

Suppose now that $u \in V(U_3)$. Consider the sequence of trees $U_3, U_4, U_5$, where $U_4$ is obtained from $U_3$ and $Q$ by operation $O_4$ (via $z_3$), and $U_5$ is obtained from $U_4$ and $F_a$ by operation $O_1$. Clearly $U_5$ is in $\mathcal{J}$, $sta_{U_5}(z_1) = A$ and by the induction hypothesis, $z_1 \in V^{01}(U_5)$. Since $T = (U_5 \cdot U_1)(z_1)$ and $\{z_1\} = V^{01}(U_1) \cap V^{01}(U_5)$, by Proposition 2 we have $z_1 \in V^{01}(T)$. But then Lemma 7 implies $z_2 \in V^{02}(T)$, a contradiction.

Now let $u \in V(U_1)$. Denote by $U_2$ the graph obtained from $U_1$ and $F_a$ by operation $O_3$. Then $U_2$ is in $\mathcal{J}$, $sta_{U_2}(z_1) = A$, and by induction hypothesis, $z_1 \in V^{01}(U_2)$. Define also the graph $U_6$ as obtained from $U_3$ and $Q$ by operation $O_3$, i.e. $U_6 = (U_3 \cdot Q)(z_3$). Then $U_6$ is in $\mathcal{J}$, $sta_{U_6}(z_1) = A$ and by induction hypothesis, $z_1 \in V^{01}(U_6)$.

Now by Proposition 2, $z_1 \in V^{01}(T)$, which leads to $z_2 \in V^{02}(T)$ (by Lemma 7), a contradiction.

Thus, in all cases we have a contradiction. Therefore $V^{02}(T^{j-1}) \subseteq V^{02}(T)$ when both $z_1$ and $z_3$ are cut-vertices. If $z_1$ or $z_3$ is a leaf, then, by similar arguments, we can obtain the same result.

Let now $T^1 \equiv Q$. Then $T^2$ is obtained from $T^1$ and $H_k$ by operation $O_3$. Consider the sequence of trees $\tau_1 : T^1_1 = H_k, T^2, T^3, \ldots, T^{j-1}$. Clearly $\tau_1$ is a $\mathcal{J}$-sequence of $T^{j-1}$ and $T^1_1 \neq Q$. Therefore we are in the previous case. Thus, $V^{02}(T^{j-1}) = V(T^{j-1}) \cap V^{02}(T)$.

Assume now that there is a vertex $w \in V^{01}(T^{j-1}) \cap V^{012}(T)$. By Lemma 7(i) $w$ has a neighbor in $T$, say $w'$, such that $w' \in V^{012}(T)$. Since $w \neq u$, $w' \in V(T^{j-1})$. But all neighbors of $w$ in $T^{j-1}$ are in $V^{02}(T^{j-1})$ (by Lemma 7 applied to $T^{j-1}$ and $w$). Since $V^{02}(T^{j-1}) = V(T^{j-1}) \cap V^{02}(T)$, we obtain a contradiction.

Thus $(\alpha_4)$ is true.

Now we are prepared to prove that $(P_1)$ is valid. Using, in the chain of equalities below, consecutively $(\alpha_2)$, the induction hypothesis, $(\alpha_1)$ and $(\alpha_4)$, we obtain

$$S_A(T) = S_{A^{-1}}(T^{j-1}) \cup (S_A(T) \cap V(F_a)) = V^{01}(T^{j-1}) \cup (V^{01}(T) \cap V(F_a)) = V^{01}(T),$$
and similarly, \( S_D(T) = V^{12}(T) \). Since \( u \notin S_B(T) \) and \( S_B(T) \cap V(F_a) = \emptyset \), we have

\[
S_B(T) = S_B(T) \cap V(T^{j-1}) \overset{(a_2)}{=} S_B^{-1}(T^{j-1}) = \{ t \in V^{02}(T^{j-1}) \mid deg_{T^{j-1}}(t) = 2 \text{ and } |N_{T^{j-1}}(t) \cap V^{02}(T^{j-1})| = 1 \}
\]

\[
\overset{(a_4)}{=} \{ t \in V^{02}(T) \cap V(T^{j-1}) \mid deg_T(t) = 2 \text{ and } |N_T(t) \cap V^{02}(T)| = 1 \}
\[= \{ t \in V^{02}(T) \mid deg_T(t) = 2 \text{ and } |N_T(t) \cap V^{02}(T)| = 1 \}.
\]

The last equality follows from \( deg_T(x) > 2 \) and \( \{ x \} = V^{02}(T) \cap V(F_a) \) (see \( (a_1) \)). Now the equality \( S_C(T) = V^{02}(T) - S_B(T) \) is obvious. Thus, \( (P_1) \) holds and we are done.

**Case 8:** \( T \) is obtained from \( T^{j-1} \in \mathcal{F} \) by operation \( O_2 \).

Clearly, \( \gamma_R(F_4) = \gamma_R(F_4 - x) = 2 \). By Lemma 2, \( \gamma_R(T) = \gamma_R(T^{j-1}) + \gamma_R(H_4) \). Let \( f_1 \) be a \( \gamma_R \)-function on \( T^{j-1} \) and \( f_2 \) a \( \gamma_R \)-function on \( F_4 \). Then the function \( f \) defined as \( f|_{T^{j-1}} = f_1 \) and \( f|_{F_4} = f_2 \) is a \( \gamma_R \)-function on \( T \). Therefore \( V^{02}(T^{j-1}) \subseteq V^{02}(F_4) \), \( V^{01}(T^{j-1}) \subseteq V^{01}(T) \cup V^{02}(T) \), and \( V^{02}(T^{j-1}) \subseteq V^{02}(T) \cup V^{02}(T) \).

Assume that there is \( y \in V^{08}(T^{j-1}) \cap V^{02}(T), s \in \{1, 2\} \), and let \( f' \) be a \( \gamma_R \)-function on \( T \) with \( f'(y) = r \notin \{0, s\} \). If \( f'|_{T^{j-1}} \) is an RD-function on \( T^{j-1} \), then \( f'|_{T^{j-1}}(V(T^{j-1})) > \gamma_R(T^{j-1}) \) and \( f'|_{E_4}(V(F_4)) \geq 2 \). This leads to \( f'(V(T)) > \gamma_R(T) \), a contradiction. Hence \( f'|_{T^{j-1}} \) is no RD-function on \( T^{j-1} \) and \( f'|_{T^{j-1} - u} \) is a \( \gamma_R \)-function on \( T^{j-1} - u \). Define now an RD-function \( f'' \) on \( T^{j-1} \) as \( f''|_{T^{j-1} - u} = f'|_{T^{j-1} - u} \) and \( f''(u) = 1 \). Since \( u \in V^{-}(T^{j-1}) \), \( f'' \) is a \( \gamma_R \)-function on \( T^{j-1} \) with \( f''(y) = r \notin \{0, s\} \), a contradiction with \( y \in V^{08}(T^{j-1}) \). Thus

\[
\overset{(a_5)}{=} V^{02}(T^{j-1}) = V^{02}(T) \cap V(T^{j-1}), \quad V^{01}(T^{j-1}) = V^{01}(T) \cap V(T^{j-1}), \quad V^{02}(T^{j-1}) = V^{02}(T) \cap V(T^{j-1}).
\]

Let \( x, x_1, x_2 \) be a path in \( F_4 \), \( h_1 \) a \( \gamma_R \)-function on \( T^{j-1} \) with \( h_1(u) = 2 \), and \( h_2 \) a \( \gamma_R \)-function on \( T^{j-1} - u \). Define \( \gamma_R \)-functions \( g_1, \ldots, g_4 \) on \( T \) as follows:

- \( g_1|_{T^{j-1}} = h_1, \quad g_1(x) = g_1(x) = 0 \) and \( g_1(x_1) = 2 \);
- \( g_2|_{T^{j-1}} = h_1, \quad g_2(x) = 0 \) and \( g_2(x_1) = g_2(x_2) = 1 \);
- \( g_3|_{T^{j-1}} = h_1, \quad g_3(x) = g_3(x_1) = 0 \) and \( g_3(x_2) = 2 \);
- \( g_4|_{T^{j-1} - u} = h_2, \quad g_4(u) = g_4(x_1) = 0, \quad g(x) = 2 \) and \( g_4(x_2) = 1 \).

This, \( (a_5) \) and Lemma 6 allows us to conclude that \( T \) is \( \gamma_R \)-excellent, \( x_1, x_2 \in V^{12}(T) \) and \( x \in V^{02}(T) \).

By induction hypothesis, \( (P_1) \) holds with \( (T, S) \) replaced by \( (T^{j-1}, S^{j-1}) \). Then Since \( u \notin S_B(T) \) and \( S_B(T) \cap V(F_4) = \emptyset \), we have

\[
S_B(T) = S_B^{-1}(T^{j-1}) = \{ t \in V^{02}(T^{j-1}) \mid deg_{T^{j-1}}(t) = 2 \text{ and } |N_{T^{j-1}}(t) \cap V^{02}(T^{j-1})| = 1 \}
\[= \{ t \in V^{02}(T) \mid deg_T(t) = 2 \text{ and } |N_T(t) \cap V^{02}(T)| = 1 \}.
\]
Claim 1.3. The next claim is obvious.

Let $V$ be a function on $H_k$ and $g$ be another function on $H_k$ such that $S(H_k) = S(T) = 1$. Hence $S_X(T) = S_X(H_k) = 1$. Let $S = \{v \in V(T) : \deg_T(v) = 1 \}$, then $S$ is an equality as required, because

$V(T) = V(H_k) = 1$. Similarly we obtain $S(T) = 1$. We also have

$$S_B(T) = S_B(H_k)$$

as required, because $V(T) = V(H_k) = 1$. Now the equality $S(T) = 1$ is obvious.

Case 9: $T$ is obtained from $T^{t_j} \in \mathcal{T}$ by operation $O_3$.

Let $T = T^{t_j} \cdot H_k(u,v : u)$, where $\text{sta}_{T^{t_j}}(u) = \text{sta}_{H_k}(v) = \text{sta}_T(u) = A$ and $k \in \{2, \ldots, 7\}$. Hence $S_X(T) = S_X(T^{t_j}) \cup I_X^k(H_k)$, for any $X \in \{A, B, C, D\}$. We know that $(P_1)$ holds with $(T, S)$ replaced by any of $(T^{t_j}, S^{t_j})$ and $(H_k, I^k)$. Hence $S_A(T) = S_A(T^{t_j}) \cup I_A^k(H_k) = V(T^{t_j}) \cup V(H_k)$. Now, by Proposition 2, applied to $T^{t_j}$ and $H_k$, $S_A(T) = V(T^{t_j})$. Similarly we obtain $S_D(T) = V(T^{t_j})$. We also have

$$S_B(T) = S_B(T^{t_j}) \cup I_B^k(H_k)$$

as required, because $V(T^{t_j}) \cup V(H_k) = V(T) = 1$. Now the equality $S(T) = V(T) - S_B(T)$ is obvious.

Case 10: $T$ is obtained from $T^{t_j} \in \mathcal{T}$ and $H_k \in \mathcal{T}$, $k \in \{3, 4, 6\}$, by operation $O_4$. By induction hypothesis and Lemma 4, we have $\gamma_R(T) = \gamma_R(T^{t_j}) + \gamma_R(H_k) - 1$ and $u \in V(T^{t_j})$. Let $f$ be a $\gamma_R$-function on $T^{t_j}$ and $f_2$ a $\gamma_R$-function on $H_k - v$. Then the function $f$ defined as $f|_{T^{t_j}} = f_1$ and $f|_{H_k - v} = f_2$ is a $\gamma_R$-function on $T$. Therefore $V_0(T^{t_j}) \subseteq V_0(T), V_0(T^{t_j}) \subseteq V_0(T) \cup V_0(T), \text{ and } V_0(T^{t_j}) \subseteq V_0(T) \cup V_0(T)$. Assume that there is $y \in V_0(T^{t_j}) \cap V_0(T), s \in \{1, 2\}$, and let $f'$ be a $\gamma_R$-function on $T$ with $f'(y) = r \notin \{0, s\}$. But then $f'|_{T^{t_j}}$ is no RD-function on $T^{t_j}, f'(u) = 0, f'|_{T^{t_j}-u}$ is a $\gamma_R$-function on $T^{t_j} - u$ and $f'|_{H_k}$ is a $\gamma_R$-function on $H_k$. Define now an RD-function $g_1$ on $T^{t_j}$ as $g_1|_{T^{t_j}-u} = f'|_{T^{t_j}-u}$ and $g_1(u) = 1$. Since $g_1(V(T^{t_j}) - u) = \gamma_R(T^{t_j} - u) + 1 = \gamma_R(T^{t_j}), g_1$ is a $\gamma_R$-function on $T^{t_j}$. But $g_1(y) = r \notin \{0, s\}$, a contradiction. Thus

$$(\alpha_6) V_0(T^{t_j}) = V_0(T) \cap V(T^{t_j}), V_0(T^{t_j}) = V_0(T) \cap V(T^{t_j}), \text{ and } V_0(T^{t_j}) = V_0(T) \cap V(T^{t_j}).$$

The next claim is obvious.

Claim 1.3. Let $x$ be the neighbor of $v$ in $H_k, k \in \{3, 4, 6\}$. Then $\gamma_R(H_3) = 4, \gamma_R(H_4) = 5, \gamma_R(H_6) = 6, \gamma_R(H_k - v) = \gamma_R(H_k - \{v, x\}) = \gamma_R(H_k)$, and $l(x) = 0$ for any $\gamma_R$-function $l$ on $H_k - v$.

Let $h$ be a $\gamma_R$-function on $T$. We know that $u \in V(T^{t_j}), u \in V(T^{t_j}), v \in V_0(H_k)$, and $\gamma_R(T) = \gamma_R(T^{t_j}) + \gamma_R(H_k) - 1$. Then by Claim 1.3 we clearly have:

(a1) If $h(u) = 2$ then at least one of the following holds:

(a1.1) $h|_{H_k - v}$ is a $\gamma_R$-function on $H_k - v$, and

(a1.2) $h|_{H_k - \{v, x\}}$ is a $\gamma_R$-function on $H_k - \{v, x\}$.
Let $l_1, l_2, l_3, l_4$ and $l_5$ be $\gamma_R$-functions on $H_k, H_k - v, H_k - \{v, x\}, T^{j-1} - u$ and $T^{j-1}$, respectively, and let $l_5(u) = 2$. Define the functions $h_1, h_2,$ and $h_3$ on $T$ as follows: (i) $h_1_{|T^{j-1}} = l_5, h_1(x) = 0$ and $h_1_{|H_k - \{v, x\}} = l_3$, (ii) $h_2_{|T^{j-1}} = l_5$ and $h_1_{|H_k - v} = l_2$, and (iii) $h_3_{|T^{j-1} - u} = l_4$ and $h_3_{|H_k} = l_1$. Clearly $h_1, h_2,$ and $h_3$ are $\gamma_R$-functions on $T$. After inspection of all $\gamma_R$-functions of $H_k, H_k - v$ and $H_k - \{v, x\}$, we conclude that $V^{01}(H_k) - \{v\} \subseteq V^{01}(T), V^{02}(H_k) \subseteq V^{02}(T),$ and $V^{012}(H_k) \subseteq V^{012}(T)$. This and $(a_6)$ imply

\[(a_7) \quad V^{012}(T) = V^{012}(T^{j-1}) \cup V^{012}(H_k), \quad V^{02}(T) = V^{02}(T^{j-1}) \cup V^{02}(H_k), \]  
and
\[V^{01}(T) = V^{01}(T^{j-1}) \cup (V^{01}(H_k) - \{v\}).\]

Since $(P_1)$ holds with $T$ replaced by $H_k$ or by $T^{j-1}$ (by induction hypothesis), using $(a_7)$ we obtain that $(P_1)$ is satisfied.

5. Corollaries

The next three results immediately follow by Theorem 1.

**Corollary 1.** If $(T, S_1), (T, S_2) \in \mathcal{T}$ then $S_1 \equiv S_2$.

If $(T, S) \in \mathcal{T}$ then we call $S$ the $\mathcal{T}$-labeling of $T$.

**Corollary 2.** Let $T$ be a $\gamma_R$-excellent tree of order $n \geq 5$, and $S$ the $\mathcal{T}$-labeling of $T$. Then \[\frac{2}{3} \leq |V^{02}(T)| \leq \frac{2}{3}(n-1) \quad \text{and} \quad \frac{2}{3}n \geq |V^{-}(T)| \geq \frac{1}{3}(n+2). \] Moreover,\[(i) \quad \frac{2}{3} \leq |V^{02}(T)| \iff \text{if and only if} \quad (T, S) \text{ has a } \mathcal{T}-\text{sequence} \tau : (T^1, S^1), \ldots, (T^j, S^j), \text{ such that} \quad (T^1, S^1) = (F_3, J^3) \text{ and if } j \geq 2, (T^{j+1}, S^{j+1}) \text{ can be obtained recursively from} \quad (T^j, S^j) \text{ and } (F_3, J^3) \text{ by operation } O_1.\]

\[(ii) \quad |V^{02}(T)| \leq \frac{2}{3}(n-1) \iff \text{if and only if} \quad (T, S) \text{ has a } \mathcal{T}-\text{sequence} \tau : (T^1, S^1), \ldots, (T^j, S^j), \text{ such that} \quad (T^1, S^1) = (H_2, I^2) \text{ and if } j \geq 2, (T^{j+1}, S^{j+1}) \text{ can be obtained recursively from} \quad (T^j, S^j) \text{ and } (H_2, I^2) \text{ by operation } O_3.\]

**Corollary 3.** Let $G$ be an $n$-order $\gamma_R$-excellent connected graph of minimum size. Then either $G = K_3$ or $n \neq 3$ and $G$ is a tree.

6. Special cases

Let $G$ be a graph and $\{a_1, \ldots, a_k\} \subseteq \{0, 1, 2, 01, 02, 12, 012\}$. We say that $G$ is a $R_{a_1, \ldots, a_k}$-graph if $V(G) = \bigcup_{i=1}^k V^{a_i}(G)$ and all $V^{a_1}(G), \ldots, V^{a_k}(G)$ are nonempty. Now let $T$ be a $\gamma_R$-excellent tree of order at least 2. By Theorem 1, we immediately conclude that $T \in R_{012} \cup R_{01,02} \cup R_{02,012} \cup R_{01,02,012}$. Moreover,
labeled tree $(T, S)$.

Remark that once a vertex is assigned a status, this status remains unchanged as the labeling proceeds. Therefore, the following result.

As an immediate consequence of Corollary 1 we obtain:

Corollary 5. If $(T, S_1), (T, S_2) \in \mathcal{F}_{01,02}$ then $S_1 \equiv S_2$.

A graph $G$ is called a 2-corona if each vertex of $G$ is either a support vertex or a leaf, and each support vertex of $G$ is adjacent to exactly 2 leaves. In a labeled 2-corona all leaves have status $A$ and all support vertices have status $C$.

(i) $T \in \mathcal{R}_{012}$ if and only if $T = K_2$, and

(ii) $T \in \mathcal{R}_{01,02,012}$ if and only if none of $S_A(T), S_C(T)$ and $S_D(T)$ is empty, where $S$ is the $\mathcal{F}$-labeling of $T$.

In this section, we turn our attention to the classes $\mathcal{R}_{01,02}$ and $\mathcal{R}_{02,012}$.

6.1. $\mathcal{R}_{01,02}$-graphs.

Here we give necessary and sufficient conditions for a tree to be in $\mathcal{R}_{01,02}$. We define a subfamily $\mathcal{J}_{01,02}$ of $\mathcal{F}$ as follows. A labeled tree $(T, S) \in \mathcal{J}_{01,02}$ if and only if $(T, S)$ can be obtained from a sequence of labeled trees $\tau : (T^1, S^1), \ldots, (T^j, S^j)$, $(j \geq 1)$, such that $(T^1, S^1)$ is in $\{(H_2, I^2), (H_3, I^3)\}$ (see Figure 1) and $(T, S) = (T^1 \cdot S^1)$, and, if $j \geq 2$, $(T^{i+1}, S^{i+1})$ can be obtained recursively from $(T^i, S^i)$ by one of the operations $O_5$ and $O_6$ listed below; in this case $\tau$ is said to be a $\mathcal{J}_{01,02}$-sequence of $T$.

**Operation $O_5$.** The labeled tree $(T^{i+1}, S^{i+1})$ is obtained from $(T^i, S^i)$ and $(F_1, I^1)$ (see Figure 2) by adding the edge $ux$, where $u \in V(T_i), x \in V(F_1)$ and $sta_{T_i}(u) = sta_{F_1}(x) = C$.

**Operation $O_6$.** The labeled tree $(T^{i+1}, S^{i+1})$ is obtained from $(T^i, S^i)$ and $(H_k, I^k)$, $k \in \{2, 3\}$ (see Figure 1), in such a way that $T^{i+1} = (T^i \cdot H_k)(u, v : u)$, where $sta_{T_i}(u) = sta_{H_k}(v) = A$, and $sta_{T_i+1}(u) = A$.

Remark that once a vertex is assigned a status, this status remains unchanged as the labeled tree $(T, S)$ is recursively constructed. By the above definitions we see that $S_D(T)$ is empty when $(T, S) \in \mathcal{J}_{01,02}$. So, in this case, it is naturally to consider a labeling $S$ as $S : V(T) \to \{A, B, C\}$. From Theorem 1 we immediately obtain the following result.

Corollary 4. Let $T$ be a tree of order at least 2. Then $T \in \mathcal{R}_{01,02}$ if and only if there is a labeling $S : V(T) \to \{A, B, C\}$ such that $(T, S)$ is in $\mathcal{J}_{01,02}$. Moreover, if $(T, S) \in \mathcal{J}_{01,02}$ then

\[
(P_3) \quad S_B(T) = \{x \in V^{o2}(T) \mid \deg(x) = 2 \text{ and } |N(x) \cap V^{o2}(T)| = 1\}, S_A(T) = V^{o1}(T), \text{ and } S_C(T) = V^{o2}(T) - S_B(T).
\]

As an immediate consequence of Corollary 1 we obtain:

Corollary 5. If $(T, S_1), (T, S_2) \in \mathcal{J}_{01,02}$ then $S_1 \equiv S_2$.
Proposition 3. Every connected $n$-order graph $H$, $n \geq 2$, is an induced subgraph of a $\mathcal{R}_{01,02}$-graph with the domination number equals to $2|V(H)|$.

Proof. Let a graph $G$ be a 2-corona such that the induced subgraph by the set of all support vertices of $G$ is isomorphic to $H$. Let $x$ be a support vertex of $G$ and $y, z$ the leaf neighbors of $x$ in $G$. Then clearly for any $\gamma_R$-function $f$ on $G$, $f(x) + f(y) + f(z) \geq 2$, $f(y) \neq 2 \neq f(z)$ and $f(x) \neq 1$. Define RD-functions $h$ and $g$ on $G$ as follows: (a) $h(u) = 2$ when $u$ is a support vertex of $G$ and $h(u) = 0$, otherwise, and (b) $g(v) = h(v)$ when $v \notin \{x, y, z\}$, and $g(x) = 0$, $g(y) = g(z) = 1$. Therefore $\gamma_R(G) = 2|V(H)|$ and $G$ is in $\mathcal{R}_{01,02}$. \hfill $\Box$

Corollary 6. There does not exist a forbidden subgraph characterization of the class of $\mathcal{R}_{01,02}$-graphs. There does not exist a forbidden subgraph characterization of the class of $\gamma_R$-excellent graphs.

Let $\mathcal{T}_{01,02}$ be the family of all labeled trees $(T, L)$ that can be obtained from a sequence of labeled trees $\lambda: (T^1, L^1), \ldots, (T^j, L^j), \ (j \geq 1)$, such that $(T, L) = (T^j, L^j)$, $(T^1, L^1)$ is either $(H_2, I_2)$ (see Figure 1) or a labeled 2-corona tree, and, if $j \geq 2$, $(T^{i+1}, L^{i+1})$ can be obtained recursively from $(T^i, L^i)$ by one of the operations $O_7$ and $O_8$ listed below; in this case $\lambda$ is said to be a $\mathcal{T}_{01,02}$-sequence of $T$.

Operation $O_7$. The labeled tree $(T^{i+1}, L^{i+1})$ is obtained from $(T^i, L^i)$ and $(H_2, I_2)$, in such a way that $T^{i+1} = (T^i \cdot H_2)(u, v : u)$, where $sta_{T^i}(u) = sta_{H_2}(v) = A$, and $sta_{T^{i+1}}(u) = A$.

Operation $O_8$. The labeled tree $(T^{i+1}, L^{i+1})$ is obtained from $(T^i, L^i)$ and a labeled 2-corona tree, say $U_i$, in such a way that $T^{i+1} = (T^i \cdot U_i)(u, v : u)$, where $sta_{T^i}(u) = sta_{U_i}(v) = A$, and $sta_{T^{i+1}}(u) = A$.

Again, once a vertex is assigned a status, this status remains unchanged as the 2-labeled tree $T$ is recursively constructed.

Theorem 2. For any tree $T$ the following are equivalent.

$(A_1)$ $T$ is in $\mathcal{R}_{01,02}$.

$(A_2)$ There is a labeling $S: V(T) \rightarrow \{A, B, C\}$ such that $(T, S)$ is in $\mathcal{R}_{01,02}$.

$(A_3)$ There is a labeling $L: V(T) \rightarrow \{A, B, C\}$ such that $(T, L)$ is in $\mathcal{T}_{01,02}$.

Proof. $(A_1) \Leftrightarrow (A_2)$: By Corollary 4.

$(A_3) \Rightarrow (A_2)$: Let a tree $(T, L) \in \mathcal{T}_{01,02}$. It is clear that all $\mathcal{T}_{01,02}$-sequences of $(T, L)$ have the same number of elements. Denote this number by $r(T)$. We shall prove that $(T, L) \in \mathcal{T}_{01,02} \Rightarrow (T, L) \in \mathcal{R}_{01,02}$. We proceed by induction on $r(T)$. If $r(T) = 1$ then either
Let a labeled tree \((T, S)\) exist. If \((T, S)\) is a labeled 2-corona tree, or \((T, L) = (H_2, I^2)\). In both cases \((T, L) \in \mathcal{T}_{01,02}\).

We need the following obvious claim.

**Claim 2.1** If \((T', L')\) is a labeled 2-corona tree, \(w \in V(T')\) and \(sta(w) = A\), then either \((T', L')\) is \((H_3, I^3)\) or there is a \(\mathcal{T}\)-sequence \(\tau: (T^1, S^1), \ldots, (T^i, S^i), (l \geq 2)\), such that \((T^1, S^1) = (H_3, I^3)\), \(w \in V(T^1)\), \((T^i, S^i) = (T', L')\), and \((T^{i+1}, S^{i+1})\) can be obtained recursively from \((T^i, S^i)\) and \((F_1, J^1)\) by operation \(O_9\).

Suppose now that each tree \((H, L_H) \in \mathcal{T}_{01,02}\) with \(r(H) < k\) is in \(\mathcal{T}_{01,02}\), where \(k \geq 2\). Let \(\lambda : (T^1, L^1), \ldots, (T^k, L^k)\), be a \(\mathcal{T}_{01,02}\)-sequence of a labeled tree \((T, L) \in \mathcal{T}'_{01,02}\). By the induction hypothesis, \((T^k, L^{k-1})\) is in \(\mathcal{T}_{01,02}\). Let \(\tau_1 : (U^1, S^1), \ldots, (U^m, S^m)\) be a \(\mathcal{T}\)-sequence of \((T^k, L^{k-1})\). Hence \(U^m = T^{k-1}\) and \(S^m = L^{k-1}\). If \((T^k, L_k)\) is obtained from \((T^{k-1}, L^{k-1})\) and \((H_2, I^2)\) by operation \(O_7\), then \((U^1, S^1), \ldots, (U^m, S^m), (T^k, L^k) = (T, L)\) is a \(\mathcal{T}\)-sequence of \((T, L)\). So, let \((T^k, L_k)\) is obtained from \((T^{k-1}, L^{k-1})\) and a labeled 2-corona tree, say \((Q, L_q)\) by operation \(O_8\). Hence \(T^{k-1}\) and \(Q\) have exactly one vertex in common, say \(w\), and \(sta_{T^{k-1}}(w) = sta_Q(w) = sta_T(w) = A\). By Claim 2.1, \((Q, L_q) \in \mathcal{T}_{01,02}\) and it has a \(\mathcal{T}_{01,02}\)-sequence, say \((Q^1, L_{q1}), \ldots, (Q^s, L_{qs})\) such that \(Q^s = Q\), \(L_q = L_q^s\), and \(w \in V(Q^1)\). Denote \(W^{m+i} = (V(U^m) \cup V(Q^1))\) and let a labeling \(S_1, \ldots, S^m+i\) be such that \(S_1^{m+i}|_{U^m} = S^m\) and \(S_1^{m+i}|_{Q^1} = L^q_1\). Then the sequence of labeled trees \((U^1, S^1), \ldots, (U^m, S^m), (W^{m+1}, S^{m+1}), \ldots, (W^{m+s}, S^{m+s}) = (T, L)\) is a \(\mathcal{T}_{01,02}\)-sequence of \((T, L)\).

\((A_2) \Rightarrow (A_3)\):

Let a labeled tree \((T, S) \in \mathcal{T}_{01,02}\). Then \((T, S)\) has a \(\mathcal{T}\)-sequence \(\tau : (T^1, S^1), \ldots, (T^j, S^j) = (T, S)\), where \((T^1, S^1) \in \{(H_2, I^2), (H_3, I^3)\} \subset \mathcal{T}'_{01,02}\). We proceed by induction on \(p(T) = \sum_{z \in \mathcal{C}(T)} deg_T(z)\), where \(\mathcal{C}(T)\) is the set of all cut-vertices of \(T\) that belong to \(S_A(T)\). Assume first \(p(T) = 0\). If \(j = 1\) then we are done. If \(j \geq 2\) then \((T^1, S^1) = (H_3, I^3)\) and \((T^{j+1}, S^{j+1})\) is obtained from \((F_1, J^1)\) and \((T^i, S^i)\) by operation \(O_5\). Thus, \((T, S)\) is a labeled 2-corona tree, which allow us to conclude that \((T, S)\) is in \(\mathcal{T}'_{01,02}\).

Suppose now that \(p(T) = k \geq 1\) and for each labeled tree \((H, S_H) \in \mathcal{T}_{01,02}\) with \(p(H) < k\) is fulfilled \((H, S_H) \in \mathcal{T}'_{01,02}\). Then there is a cut-vertex, say \(z\), such that \(z \in S_A(T)\), (b) \((T, S)\) is a coalescence of 2 graphs, say \((T', S|_{T'})\) and \((T'', S|_{T''})\), via \(z\), and (c) no vertex in \(S_A(T) \cap V(T'')\) is a cut-vertex of \(T''\). Hence \((T', S|_{T'}) \in \mathcal{T}'_{01,02}\) (by induction hypothesis) and \((T'', S|_{T''})\) is either a labeled 2-corona tree or \(H_2\). Thus \((T, S)\) is in \(\mathcal{T}'_{01,02}\).

\(\square\)

6.2. \(\mathcal{R}_{02,012}\)-trees.

Our aim in this section is to present a characterization of \(\mathcal{R}_{02,012}\)-trees. For this purpose, we need the following definitions. Let \(\mathcal{R}_{02,012} \subset \mathcal{T}\) be such that \((T, S) \in \mathcal{R}_{02,012}\) if and only if \((T, S)\) can be obtained from a sequence of labeled trees \(\tau : (T^1, S^1), \ldots, (T^i, S^i), (j \geq 1)\), such that \((T^1, S^1) = (F_3, J^3)\) (see Figure 2) and \((T, S) = (T^i, S^i)\), and, if \(j \geq 2\), \((T^{i+1}, S^{i+1})\) can be obtained recursively from \((T^{i}, S^{i})\) by one of the operations \(O_9\) and \(O_{10}\) listed below.
Operation $O_9$. The labeled tree $(T^{i+1}, S^{i+1})$ is obtained from $(T^i, S^i)$ and $(F_3, J^3)$ by adding the edge $ux$, where $u \in V(T^i)$, $x \in V(F_3)$ and $\text{sta}_{S^i}(u) = \text{sta}_{F_3}(x) = C$.

Operation $O_{10}$. The labeled tree $(T^{i+1}, S^{i+1})$ is obtained from $(T^i, S^i)$ and $(F_4, J^4)$ (see Figure 2) by adding the edge $ux$, where $u \in V(T^i)$, $x \in V(F_4)$, $\text{sta}_{T^i}(u) = D$, and $\text{sta}_{F_4}(x) = C$.

Note that once a vertex is assigned a status, this status remains unchanged as the labeled tree $(T, S)$ is recursively constructed. By the above definitions we see that if $(T, S) \in R_{01,02}$, then $S_A(T) = S_B(T) = \emptyset$. Therefore it is naturally to consider a labeling $S$ as $S : V(T) \to \{C, D\}$.

From Theorem 1 we immediately obtain the following result.

**Corollary 7.** A tree $T$ is in $R_{02,012}$ if and only if there is a labeling $S : V(T) \to \{C, D\}$ such that $(T, S)$ is in $\mathcal{R}_{02,012}$. Moreover, if $(T, S) \in \mathcal{R}_{02,012}$ then $S_C(T) = V^{02}(T)$ and $S_D(T) = V^{012}(T)$.

As an immediate consequence of Corollary 1 we obtain:

**Corollary 8.** If $(T, S_1), (T, S_2) \in \mathcal{R}_{02,012}$ then $S_1 \equiv S_2$.

**Theorem 3.** [3] If $G$ is a connected graph of order $n \geq 3$, then $\gamma_R(G) \leq 4n/5$. The equality holds if and only if $G$ is $C_5$ or is obtained from $\frac{5}{2}P_5$ by adding a connected subgraph on the set of centers of the components of $\frac{5}{2}P_5$.

As a consequence of Theorem 3 and Corollary 7 we have:

**Corollary 9.** Let $G$ be a connected $n$-vertex graph with $n \geq 6$ and $\gamma_R(G) = 4n/5$. Then $G$ is in $R_{02,012}$ and $V^{012}(G)$ consists of all leaves and all support vertices. Moreover, if $G$ is a tree, then $G$ has a $\mathcal{R}$-sequence $\tau : (G^1, S^1), \ldots, (G^j, S^j)$, $(j \geq 1)$, such that $(G^1, S^1) = (F_3, J^3)$ (see Figure 2) and if $j \geq 2$, then $(G^{i+1}, S^{i+1})$ can be obtained recursively from $(G^i, S^i)$ by operation $O_9$.

A graph $G$ is said to be in class $UVR$ if $\gamma(G - v) = \gamma(G)$ for each $v \in V(G)$. Constructive characterizations of trees belonging to $UVR$ are given in [14] by Samodivkin, and independently in [11] by Haynes and Henning. We need the following result in [14] (reformulated in our present terminology).

**Theorem 4.** [14] A tree $T$ of order at least 5 is in $UVR$ if and only if there is a labeling $S : V(T) \to \{C, D\}$ such that $(T, S)$ is in $\mathcal{R}_{02,012}$. Moreover, if $(T, S) \in \mathcal{R}_{02,012}$ then $S_C(T)$ and $S_D(T)$ are the sets of all $\gamma$-bad and all $\gamma$-good vertices of $T$, respectively.

We end with our main result in this subsection.
**Theorem 5.** For any tree $T$ the following are equivalent:

(A4) $T$ is in $R_{02,012}$,  
(A5) $T$ is in $R_{012}$,  
(A6) $T$ is in $UVR$.

**Proof.** Corollary 7 and Theorem 4 together imply the required result. 

**7. Open problems and questions**

We conclude the paper by listing some interesting problems and directions for further research. Let first note that if $n \geq 3$ and $G_{n,k}$ is not empty, then $k \leq 4n/5$ (Theorem 3).

An element of $\mathbb{RE}_{n,k}$ is said to be *isolated*, whenever it is both maximal and minimal. In other words, a graph $H \in G_{n,k}$ is isolated in $\mathbb{RE}_{n,k}$ if and only if $H \in R_{CEA}$ and for each $e \in E(H)$ at least one of the following holds: (a) $H - e$ is not connected, (b) $\gamma_R(H) \neq \gamma_R(H - e)$, (c) $H - e$ is not $\gamma_R$-excellent.

**Example 1.**

(i) All $\gamma_R$-excellent graphs with the Roman domination number equals to 2 are $K_2$ and $K_n$, $n \geq 2$. If a graph $G \in R_{CEA}$ and $\gamma_R(G) = 2$, then $G$ is complete. $K_n$ is isolated in $\mathbb{RE}_{n,2}$, $n \geq 2$.

(ii) $[8]$ $K_2$, $H_7$ and $H_8$ (see Fig. 1) are the only trees in $R_{CEA}$.

(iii) If $\mathbb{RE}_{n,k}$ has a tree $T$ as an isolated element, then either $(n, k) = (2, 2)$ and $T = K_2$, or $(n, k) = (9, 7)$ and $T = H_7$, or $(n, k) = (10, 8)$ and $T = H_8$.

- Find results on the isolated elements of $\mathbb{RE}_{n,k}$.
- What is the maximum number of edges $m(G_{n,k})$ of a graph in $G_{n,k}$? Note that (a) $m(G_{n,2}) = n(n - 1)/2$, (b) $m(G_{n,3}) = n(n - 1)/2 - \lceil n/2 \rceil$.
- Find results on those minimal elements of $\mathbb{RE}_{n,k}$ that are not trees.

**Example 2.**

(a) A cycle $C_n$ is a minimal element of $\mathbb{RE}_{n,k}$ if and only if $n \equiv 0 \pmod{3}$ and $k = 2n/3$. (b) A graph $G$ obtained from the complete bipartite graph $K_{p,q}$, $p \geq q \geq 3$, by deleting an edge is a minimal element of $\mathbb{RE}_{p+q,4}$.

The height of a poset is the maximal number of elements of a chain.

- Find the height of $\mathbb{RE}_{n,k}$.

**Example 3.**

(a) It is easy to check that any longest chain in $\mathbb{RE}_{6,4}$ has as the first element $H_3$ (see Fig 1) and as the last element one of the two 3-regular 6-vertex graphs. Therefore the height of $\mathbb{RE}_{6,4}$ is 5.
(b) Let us consider the poset \( \mathbb{RE}_{5r,4r} \), \( r \geq 2 \). All its minimal elements are \( \gamma_R \)-excellent trees (by Theorem 3 and Corollary 9), which are characterized in Corollary 9. Moreover, the graph obtained from \( rP_5 \) by adding a complete graph on the set of centers of the components of \( rP_5 \) is the largest element of \( \mathbb{RE}_{5r,4r} \). Therefore the height of \( \mathbb{RE}_{5r,4r} \) is \( (r - 1)(r - 2)/2 + 1 \).

- Find results on \( \gamma_{YR} \)-excellent graphs at least when \( Y \) is one of \( \{-1, 0, 1\} \), \( \{-1, 1\} \) and \( \{-1, 1, 2\} \).

References


