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Roman domination excellent graphs: trees

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Abstract: A Roman dominating function (RDF) on a graph $G = (V, E)$ is a labeling $f : V \to \{0, 1, 2\}$ such that every vertex with label 0 has a neighbor with label 2. The weight of f is the value $f(V) = \Sigma_{v \in V} f(v)$ The Roman domination number, $\gamma_R(G)$, of G is the minimum weight of an RDF on G. An RDF of minimum weight is called a γ_R-function. A graph G is said to be γ_R-excellent if for each vertex $x \in V$ there is a γ_R-function h_x on G with $h_x(x) \neq 0$. We present a constructive characterization of γ_R-excellent trees using labelings. A graph G is said to be in class UVR if $\gamma(G-v) = \gamma(G)$ for each $v \in V$, where $\gamma(G)$ is the domination number of G. We show that each tree in UVR is γ_R-excellent.

Keywords: Roman domination number, excellent tree, coalescence

AMS Subject classification: 05C69, 05C05

1. Introduction and preliminaries

For basic notation and graph theory terminology not explicitly defined here, we in general follow Haynes et al. [9]. Specifically, let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A *spanning subgraph* for G is a subgraph of G which contains every vertex of G. In a graph G, for a subset $S \subseteq V(G)$ the *subgraph induced* by S is the graph $\langle S \rangle$ with vertex set S and edge set $\{xy \in E(G) \mid x, y \in S\}$. The complement $\overline{G}$ of G is the graph whose vertex set is $V(G)$ and whose edges are the pairs of nonadjacent vertices of G. We write K_n for the *complete graph* of order n and P_n for the *path* on n vertices. Let C_m denote the *cycle* of length m. For any vertex x of a graph G, $N_G(x)$ denotes the set of all neighbors of x in G, $N_G[x] = N_G(x) \cup \{x\}$ and the degree of x is $deg_G(x) = |N_G(x)|$. The *minimum* and *maximum* degrees of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a subset S of vertices, let

$N_G[S] = \cup_{v \in S} N_G[v]$. The *external private neighborhood* $epn(v, S)$ of $v \in S$ is defined by $epn(v, S) = \{u \in V(G) - S \mid N_G(u) \cap S = \{v\}\}$. A *leaf* is a vertex of degree one and a *support vertex* is a vertex adjacent to a leaf. If F and H are disjoint graphs, $v_F \in V(F)$ and $v_H \in V(H)$, then the *coalescence* $(F \cdot H)(v_F, v_H : v)$ of F and H via v_F and v_H, is the graph obtained from the union of F and H by identifying v_F and v_H in a vertex labeled v. If F and H are graphs with exactly one vertex in common, say x, then the *coalescence* $(F \cdot H)(x)$ of F and H via x is the union of F and H.

Let $\mathtt{Y}$ be a finite set of integers which has positive as well as non-positive elements. Denote by $P(\mathtt{Y})$ the collection of all subsets of $\mathtt{Y}$. Given a graph G, for a $\mathtt{Y}$-valued function $f : V(G) \to \mathtt{Y}$ and a subset S of $V(G)$ we define $f(S) = \Sigma_{v \in S} f(v)$. The *weight* of f is $f(V(G))$. A $\mathtt{Y}$-*valued Roman dominating function* on a graph G is a function $f : V(G) \to \mathtt{Y}$ satisfying the conditions: (a) $f(N_G[v]) \geq 1$ for each $v \in V(G)$, and (b) if $v \in V(G)$ and $f(v) \leq 0$, then there is $u_v \in N_G(v)$ with $f(u_v) = \max\{k \mid k \in \mathtt{Y}\}$. For a $\mathtt{Y}$-valued Roman dominating function f on a graph G, where $\mathtt{Y} = \{r_1, r_2, \dots, r_k\}$ and $r_1 < r_2 < \dots < r_k$, let $V^f_{r_i} = \{v \in V(G) \mid f(v) = r_i\}$ for $i = 1, .., k$. Since these k sets determine f, we can equivalently write $f = (V^f_{r_1}; V^f_{r_2}; \dots; V^f_{r_k})$. If f is $\mathtt{Y}$-valued Roman dominating function on a graph G and H is a subgraph of G, then we denote the restriction of f on H by $f|_H$. The $\mathtt{Y}$-*Roman domination number* of a graph G, denoted $\gamma_{\mathtt{Y}R}(G)$, is defined to be the minimum weight of a $\mathtt{Y}$-valued dominating function on G. As examples, let us mention: (a) the domination number $\gamma(G) \equiv \gamma_{\{0,1\}R}(G)$, (b) the minus domination number [6], where $\mathtt{Y} = \{-1, 0, 1\}$, (c) the signed domination number [5], where $\mathtt{Y} = \{-1, 1\}$, (d) the Roman domination number $\gamma_R(G) \equiv \gamma_{\{0,1,2\}R}(G)$ [4], and (e) the signed Roman domination number [1], where $\mathtt{Y} = \{-1, 1, 2\}$. A $\mathtt{Y}$-valued Roman dominating function f on G with weight $\gamma_{\mathtt{Y}R}(G)$ is called a $\gamma_{\mathtt{Y}R}$-*function* on G.

Now we introduce a new partition of a vertex set of a graph, which plays a key role in the paper. In determining this partition, all $\gamma_{\mathtt{Y}R}$-functions of a graph are necessary. For each $\mathtt{X} \in P(\mathtt{Y})$ we define the set $V^{\mathtt{X}}(G)$ as consisting of all $v \in V(G)$ with $\{f(v) \mid f \text{ is a } \gamma_{\mathtt{Y}R}\text{-function on } G\} = \mathtt{X}$. Then all members of the family $(V^{\mathtt{X}}(G))_{\mathtt{X} \in P(\mathtt{Y})}$ clearly form a partition of $V(G)$. We call this partition the $\gamma_{\mathtt{Y}R}$-*partition of* G.

Fricke et al. [7] in 2002 began the study of graphs, which are excellent with respect to various graph parameters. Let us concentrate here on the parameter $\gamma_{\mathtt{Y}R}$. A vertex $v \in V(G)$ is said to be (a) $\gamma_{\mathtt{Y}R}$-*good*, if $h(v) \geq 1$ for some $\gamma_{\mathtt{Y}R}$-function h on G, and (b) $\gamma_{\mathtt{Y}R}$-*bad* otherwise. A graph G is said to be $\gamma_{\mathtt{Y}R}$-*excellent* if all vertices of G are $\gamma_{\mathtt{Y}R}$-good. Any vertex-transitive graph is $\gamma_{\mathtt{Y}R}$-excellent. Note that when $\gamma_{\mathtt{Y}R} \equiv \gamma$, the set of all γ-good and the set of all γ-bad vertices of a graph G form the γ-partition of G. For further results on this topic see e.g. [2, 10–15].

In this paper we begin an investigation of $\gamma_{\mathtt{Y}R}$-excellent graphs in the case when $\mathtt{Y} = \{0, 1, 2\}$. In what follows we shall write γ_R instead of $\gamma_{\{0,1,2\}R}$, and we shall abbreviate a $\{0, 1, 2\}$-valued Roman dominating function to an RD-function. Let us describe all members of the γ_R-partition of any graph G (we write $V^i(G)$, $V^{ij}(G)$ and $V^{ijk}(G)$ instead of $V^{\{i\}}(G)$, $V^{\{i,j\}}(G)$ and $V^{\{i,j,k\}}(G)$, respectively).

(i) $V^i(G) = \{x \in V(G) \mid f(x) = i \text{ for each } \gamma_R\text{-function } f \text{ on } G\}, i = 1, 2, 3;$

(ii) $V^{012}(G) = \{x \in V(G) \mid$ there are γ_R-functions f_x, g_x, h_x on G with $f_x(x) = 0, g_x(x) = 1$ and $h_x(x) = 2\}$;

(iii) $V^{ij}(G) = \{x \in V(G) - V^{012}(G) \mid$ there are γ_R-functions f_x and g_x on G with $f_x(x) = i$ and $g_x(x) = j\}, 0 \leq i < j \leq 2$.

Clearly a graph G is γ_R-excellent if and only if $V^0(G) = \emptyset$.
It is often of interest to known how the value of a graph parameter is affected when a small change is made in a graph. In this connection, Hansberg, Jafari Rad and Volkmann studied in [8] changing and unchanging of the Roman domination number of a graph when a vertex is deleted, or an edge is added.

Lemma 1. ([8]) *Let v be a vertex of a graph G. Then $\gamma_R(G - v) < \gamma_R(G)$ if and only if there is a γ_R-function $f = (V_0^f; V_1^f; V_2^f)$ on G such that $v \in V_1^f$. If $\gamma_R(G - v) < \gamma_R(G)$ then $\gamma_R(G - v) = \gamma_R(G) - 1$.*

Lemma 1 implies that $V^1(G), V^{01}(G), V^{12}(G), V^{012}(G)$ form a partition of $V^-(G) = \{x \in V(G) \mid \gamma_R(G - x) + 1 = \gamma(G)\}$.

Lemma 2. ([8]) *Let x and y be non-adjacent vertices of a graph G. Then $\gamma_R(G) \geq \gamma_R(G + xy) \geq \gamma_R(G) - 1$. Moreover, $\gamma_R(G + xy) = \gamma_R(G) - 1$ if and only if there is a γ_R-function f on G such that $\{f(x), f(y)\} = \{1, 2\}$.*

The same authors defined the following two classes of graphs:

(i) $\mathcal{R}_{CVR}$ is the class of graphs G such that $\gamma_R(G - v) < \gamma_R(G)$ for all $v \in V(G)$.

(ii) $\mathcal{R}_{CEA}$ is the class of graphs G such that $\gamma_R(G + e) < \gamma_R(G)$ for all $e \in E(\overline{G})$.

Remark 1. By Lemmas 1 and 2 it easy follows that:

(i) each graph in $\mathcal{R}_{CVR} \cup \mathcal{R}_{CEA}$ is γ_R-excellent,

(ii) if G is a γ_R-excellent graph, $e \in E(\overline{G})$ and $\gamma_R(G) = \gamma_R(G + e)$, then $G + e$ is γ_R-excellent,

(iii) each graph (in particular each γ_R-excellent graph) is a spanning subgraph of a graph in $\mathcal{R}_{CEA}$ with the same Roman domination number.

Denote by $\mathtt{G}_{n,k}$ the family of all mutually non-isomorphic n-order γ_R-excellent connected graphs having the Roman domination number equal to k. With the family $\mathtt{G}_{n,k}$, we associate the poset $\mathbb{RE}_{n,k} = (\mathtt{G}_{n,k}, \prec)$ with the order $\prec$ given by $H_1 \prec H_2$ if and only if H_2 has a spanning subgraph which is isomorphic to H_1 (see [16] for terminology on posets). Remark 1 shows that all maximal elements of $\mathbb{RE}_{n,k}$ are in $\mathcal{R}_{CEA}$. Here we concentrate on the set of all minimal elements of $\mathbb{RE}_{n,k}$. Clearly a graph $H \in \mathtt{G}_{n,k}$ is a minimal element of $\mathbb{RE}_{n,k}$ if and only if for each $e \in E(H)$ at

least one of the following holds: (a) $H - e$ is not connected, (b) $\gamma_R(H) \neq \gamma_R(H - e)$, and (c) $H - e$ is not γ_R-excellent. All trees in $\mathtt{G}_{n,k}$ are obviously minimal elements of $\mathbb{RE}_{n,k}$.
The remainder of this paper is organized as follows. In Section 2, we formulate our main result, namely, a constructive characterization of γ_R-excellent trees. We present a proof of this result in Sections 3 and 4. Applications of our main result are given in Sections 5 and 6. We conclude in Section 7 with some open problems.
We end this section with the following useful result.

Lemma 3. ([4]) *Let $f = (V_0^f; V_1^f; V_2^f)$ be any γ_R-function on a graph G. Then each component of a graph $\left\langle V_1^f \right\rangle$ has order at most 2 and no edge of G joins V_1^f and V_2^f.*

In most cases Lemmas 1, 2 and 3 will be used in the sequel without specific reference.

2. The main result

In this section, we present a constructive characterization of γ_R-excellent trees using labelings. We define a *labeling* of a tree T as a function $S : V(T) \to \{A, B, C, D\}$. A labeled tree is denoted by a pair (T, S). The label of a vertex v is also called its *status*, denoted $sta_T(v : S)$ or $sta_T(v)$ if the labeling S is clear from context. We denote the sets of vertices of status A, B, C and D by $S_A(T), S_B(T), S_C(T)$ and $S_D(T)$, respectively. In all figures in this paper we use $\bullet$ for a vertex of status A, $\blacktriangledown$ for a vertex of status B, $\blacklozenge$ for a vertex of status C, and $\circ$ for a vertex of status D. If H is a subgraph of T, then we denote the restriction of S on H by $S|_H$.

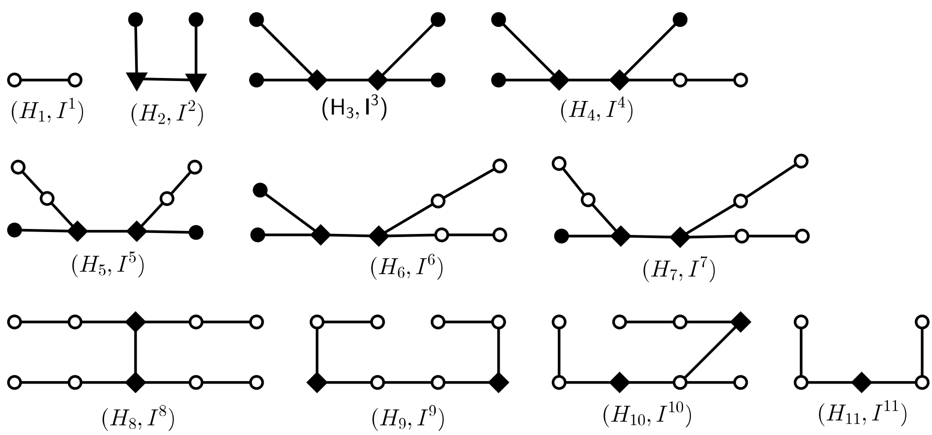

Figure 1. **All trees with $|L_B \cup L_C| \leq 2$.**

To state a characterization of γ_R-excellent trees, we introduce four types of operations. Let $\mathscr{T}$ be the family of labeled trees (T, S) that can be obtained from a sequence of labeled trees $\tau : (T^1, S^1), \dots, (T^j, S^j)$, $(j \geq 1)$, such that (T^1, S^1) is in $\{(H_1, I^1), .., (H_5, I^5)\}$ (see Figure 1) and $(T, S) = (T^j, S^j)$, and, if $j \geq 2$, (T^{i+1}, S^{i+1}) can be obtained recursively from (T^i, S^i) by one of the operations O_1, O_2, O_3 and O_4 listed below; in this case τ is said to be a $\mathscr{T}$*-sequence* of T. When the context is clear we shall write $T \in \mathscr{T}$ instead of $(T, S) \in \mathscr{T}$.

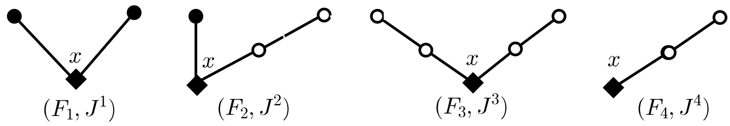

Figure 2. (F, J)**-graphs**

Operation O_1. The labeled tree (T^{i+1}, S^{i+1}) is obtained from (T^i, S^i) and $(F, J) \in \{(F_1, J^1), (F_2, J^2), (F_3, J^3)\}$ (see Figure 2) by adding the edge ux, where $u \in V(T_i)$, $x \in V(F)$ and $sta_{T^i}(u) = sta_F(x) = C$.

Operation O_2. The labeled tree (T^{i+1}, S^{i+1}) is obtained from (T^i, S^i) and (F_4, J^4) (see Figure 2) by adding the edge ux, where $u \in V(T^i)$, $x \in V(F_4)$, $sta_{T^i}(u) = D$, and $sta_{F_4}(x) = C$.

Operation O_3. The labeled tree (T^{i+1}, S^{i+1}) is obtained from (T^i, S^i) and (H_k, I^k), $k \in \{2, 3, \dots, 7\}$ (see Figure 1), in such a way that $T^{i+1} = (T^i \cdot H_k)(u, v : u)$, where $sta_{T^i}(u) = sta_{H_k}(v) = A$, and $sta_{T^{i+1}}(u) = A$.

Operation O_4. The labeled tree (T^{i+1}, S^{i+1}) is obtained from (T^i, S^i) and (H_k, I^k), $k \in \{3, 4, 6\}$ (see Figure 1), in such a way that $T^{i+1} = (T^i \cdot H_k)(u, v : u)$, where $sta_{T^i}(u) = D$, $sta_{H_k}(v) = A$, and $sta_{T^{i+1}}(u) = D$.

Remark that if $y \in V(T^i)$ and $i \leq k \leq j$, then $sta_{T^i}(y) = sta_{T^k}(y)$. Now we are prepared to state the main result.

Theorem 1. *Let T be a tree of order at least 2. Then T is γ_R-excellent if and only if there is a labeling $S : V(T) \to \{A, B, C, D\}$ such that (T, S) is in $\mathscr{T}$. Moreover, if $(T, S) \in \mathscr{T}$ then*

$(\mathcal{P}_1)$ $S_B(T) = \{x \in V^{02}(T) \mid deg(x) = 2$ and $|N(x) \cap V^{02}(T)| = 1\}$, $S_A(T) = V^{01}(T)$, $S_D(T) = V^{012}(T)$, and $S_C(T) = V^{02}(T) - S_B(T)$.

3. Preparation for the proof of Theorem 1

3.1. Coalescence

We shall concentrate on the coalescence of two graphs via a vertex in V^{01} and derive the properties which will be needed for the proof of our main result.

Proposition 1. *Let $G = (G_1 \cdot G_2)(x)$ be a connected graph and $x \in V^{01}(G)$. Then the following holds.*

(i) If f is a γ_R-function on G and $f(x) = 1$, then $f|_{G_i}$ is a γ_R-function on G_i, and $f|_{G_i - x}$ is a γ_R-function on $G_i - x$, $i = 1, 2$.

(ii) $\gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1$.

(iii) If h is a γ_R-function on G and $h(x) = 0$, then exactly one of the following holds:

(iii.1) $h|_{G_1}$ is a γ_R-function on G_1, $h|_{G_2 - x}$ is a γ_R-function on $G_2 - x$, and $h|_{G_2}$ is no RD-function on G_2;

(iii.2) $h|_{G_1 - x}$ is a γ_R-function on $G_1 - x$, $h|_{G_1}$ is no RD-function on G_1, and $h|_{G_2}$ is a γ_R-function on G_2.

(iv) Either $\{x\} = V^{01}(G_1) \cap V^{01}(G_2)$ or $\{x\} = V^{01}(G_i) \cap V^1(G_j)$, where $\{i, j\} = \{1, 2\}$.

Proof. (i) and (ii): Since $f(x) = 1$, $f|_{G_i}$ is an RD-function on G_i, and $f|_{G_i - x}$ is an RD-function on $G_i - x$, $i = 1, 2$. Assume g_1 is a γ_R-function on G_1 with $g_1(V(G_1)) < f|_{G_1}(V(G_1))$. Define an RD-function f' as follows: $f'(u) = g_1(u)$ for all $u \in V(G_1)$ and $f'(u) = f(u)$ when $u \in V(G_2 - x)$. Then $f'(V(G)) = g_1(V(G_1)) + f|_{G_2 - x}(V(G_2 - x)) < f(V(G))$, a contradiction. Thus, $f|_{G_i}$ is a γ_R-function on G_i, $i = 1, 2$. Now, Lemma 1 implies that $f|_{G_i - x}$ is a γ_R-function on $G_i - x$, $i = 1, 2$. Hence $\gamma_R(G) = f|_{G_1}(V(G_1)) + f|_{G_2}(V(G_2)) - f(x) = \gamma_R(G_1) + \gamma_R(G_2) - 1$.
(iii) First note that $h(x) = 0$ implies $h|_{G_i}$ is an RD-function on G_i for some $i \in \{1, 2\}$, say $i = 1$. If $h|_{G_2}$ is an RD-function on G_2 then $\gamma_R(G) = h(V(G)) \geq \gamma_R(G_1) + \gamma_R(G_2)$, a contradiction with (ii). Thus, $h|_{G_2 - x}$ is an RD-function on $G_2 - x$. Now we have $\gamma_R(G_1) + \gamma_R(G_2) - 1 = \gamma_R(G) = h(V(G)) = h|_{G_1}(V(G_1)) + h|_{G_2 - x}(V(G_2 - x)) \geq \gamma_R(G_1) + (\gamma_R(G_2) - 1)$. Hence $h|_{G_1}$ is a γ_R-function on G_1 and $h|_{G_2 - x}$ is a γ_R-function on $G_2 - x$.
(iv) Let f_1 be a γ_R-function on G_1. Assume first that $f_1(x) = 2$. Define an RD-function g on G as follows: $g(u) = f_1(u)$ when $u \in V(G_1)$ and $g(u) = f(u)$ when $u \in V(G_2 - x)$, where f is defined as in (i). The weight of g is $\gamma_R(G_1) + (\gamma_R(G_2) + 1) - 2 = \gamma_R(G)$. But $g(x) = 2$ and $x \in V^{01}(G)$, a contradiction. Thus $f_1(x) \neq 2$. Now by (i) we have $x \in V^1(G_i) \cup V^{01}(G_i)$, $i = 1, 2$, and by (iii), $x \in V^{01}(G_j)$ for some $j \in \{1, 2\}$. □

Proposition 2. *Let $G = (G_1 \cdot G_2)(x)$, where G_1 and G_2 are connected graphs and $\{x\} = V^{01}(G_1) \cap V^{01}(G_2)$.*

(i) If f_i is a γ_R-function on G_i with $f_i(x) = 1$, $i = 1, 2$, then the function $f : V(G) \to \{0, 1, 2\}$ with $f|_{G_i} = f_i$, $i = 1, 2$, is a γ_R-function on G.

(ii) $\gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1$.

(iii) Let $V_R = \{V^0, V^1, V^2, V^{01}, V^{02}, V^{12}, V^{012}\}$. Then for any $A \in V_R$, $A(G_1) \cup A(G_2) = A(G)$.

Proof. (i) and (ii): Note that f is an RD-function on G and $\gamma_R(G) \leq f(V(G)) = f_1(V(G_1)) + f_2(V(G_2)) - f(x) = \gamma_R(G_1) + \gamma_R(G_2) - 1$. Now let h be any γ_R-function on G.

Case 1: $h(x) \geq 1$. Then $h|_{G_i}$ is an RD-function on G_i, $i = 1, 2$. If $h(x) = 2$ then since $x \in V^{01}(G_1) \cap V^{01}(G_2)$, $h|_{G_i}$ is no γ_R-function on G_i, $i = 1, 2$. Hence $\gamma_R(G) \geq (\gamma_R(G_1) + 1) + (\gamma_R(G_2) + 1) - h(x) = \gamma_R(G_1) + \gamma_R(G_2)$, a contradiction. If $h(x) = 1$ then $\gamma_R(G) = h(V(G)) = h(V(G_1)) + h(V(G_2)) - h(x) \geq \gamma_R(G_1) + \gamma_R(G_2) - 1$. Thus $h(x) = 1$, $\gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1$ and f is a γ_R-function on G.

Case 2: $h(x) = 0$. Then at least one of $h|_{G_1}$ and $h|_{G_2}$ is an RD-function, say the first. If $h|_{G_2}$ is an RD-function on G_2 then $h(V(G)) \geq \gamma_R(G_1) + \gamma_R(G_2)$, a contradiction. Hence $h|_{G_2 - x}$ is a γ_R-function on $G_2 - x$. But then $\gamma_R(G) = h(V(G)) \geq \gamma_R(G_1) + \gamma_R(G_2 - x) \geq \gamma_R(G_1) + \gamma_R(G_2) - 1 \geq \gamma_R(G)$.

Thus, (i) and (ii) hold.

(iii): Let g_1 be a γ_R-function on G_1 with $g_1(x) = 0$, and g_2 a γ_R-function on $G_2 - x$. Then the RD-function g on G for which $g|_{G_1} = g_1$ and $g|_{G_2 - x} = g_2$ has weight $g_1(V(G_1)) + g_2(V(G_2 - x)) = \gamma_R(G_1) + \gamma_R(G_2 - x) = \gamma_R(G_1) + \gamma_R(G_2) - 1 = \gamma_R(G)$. Hence by (i), $x \in V^{01}(G) \cup V^{012}(G)$. However, by Case 1 it follows that $h(x) \neq 2$ for any γ_R-function h on G. Thus $x \in V^{01}(G)$.

Let $y \in V(G_1 - x)$, l_1 a γ_R-function on G_1, and h a γ_R-function on G. We shall prove that the following holds.

Claim 4.1 There are a γ_R-function l on G, and a γ_R-function h_1 on G_1 such that $l(y) = l_1(y)$ and $h_1(y) = h(y)$.

Define an RD-function l on G as $l|_{G_1} = l_1$ and $l|_{G_2 - x} = l_2$, where l_2 is a γ_R-function on $G_2 - x$. Since $l(V(G)) = \gamma_R(G_1) + \gamma_R(G_2 - x) = \gamma_R(G)$, l is a γ_R-function on G and $l(y) = l_1(y)$.

Assume now that there is no γ_R-function h_1 on G_1 with $h_1(y) = h(y)$. Proposition 1 implies that, $h|_{G_1 - x}$ is a γ_R-function on $G_1 - x$. But then the function $h' : V(G_1) \to \{0, 1, 2\}$ defined as $h'(u) = 1$ when $u = x$ and $h'(u) = h|_{G_1}(u)$ otherwise, is a γ_R-function on G_1 with $h'(y) = h|_{G_1}(y)$, a contradiction.

By Claim 4.1 and since $x \in V^{01}(G)$, $A(G_1) = A(G) \cap V(G_1)$ for any $A \in V_R$. By symmetry, $A(G_2) = A(G) \cap V(G_2)$. Therefore $A(G_1) \cup A(G_2) = A(G)$ for any $A \in V_R$. □

Lemma 4. *Let $G = (G_1 \cdot G_2)(x)$, where G_1 and G_2 are connected graphs and $\{x\} = V^{012}(G_1) \cap V^{01}(G_2)$. Then $\gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1$ and $x \in V^{012}(G)$.*

Proof. Let f_i be a γ_R-function on G_i with $f_i(x) = 1$, $i = 1, 2$. Then the function f defined as $f|_{G_i} = f_i$ is an RD-function on G_i, $i = 1, 2$. Hence $\gamma_R(G) \leq f(V(G)) = \gamma_R(G_1) + \gamma_R(G_2) - 1$. Let now h be any γ_R-function on G.

Case 1: $h(x) = 2$.

Since $x \in V^{012}(G_1) \cap V^{01}(G_2)$, $h|_{G_1}$ is a γ_R-function on G_1 and $h|_{G_2}$ is an RD-function on G_2 of weight more than $\gamma_R(G_2)$. Hence $\gamma_R(G) = h(V(G)) \geq \gamma_R(G_1) + (\gamma_R(G_2) + 1) - h(x)$. Thus $\gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1$.

Case 2: $h(x) = 1$.
Then obviously $h|_{G_1}$ and $h|_{G_2}$ are γ_R-functions. Hence $\gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1$.
Case 3: $h(x) = 0$.
Hence at least one of $h|_{G_1}$ and $h|_{G_2}$ is a γ_R-function. If both $h|_{G_1}$ and $h|_{G_2}$ are γ_R-functions, then $\gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2)$, a contradiction. Hence either $h|_{G_1}$ and $h|_{G_2 - x}$ are γ_R-functions, or $h|_{G_1 - x}$ and $h|_{G_2}$ are γ_R-functions. Since $\{x\} = V^{012}(G_1) \cap V^{01}(G_2)$, in both cases we have $\gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1$.
Thus, $\gamma_R(G) = \gamma_R(G_1) + \gamma_R(G_2) - 1$ and $x \in V^{012}(G)$. □

3.2. Three lemmas for trees

Lemma 5. *Let T be a γ_R-excellent tree of order at least 2. Then $V(T) = V^{01}(T) \cup V^{012}(T) \cup V^{02}(T)$.*

Proof. Let $x \in V(T)$, $y \in N(x)$ and f a γ_R-function on T. Suppose $x \in V^1(T)$. If $f(y) = 1$, then the RD-function g on T defined as $g(x) = 2$, $g(y) = 0$ and $g(u) = f(u)$ for all $u \in V(T) - \{x, y\}$ is a γ_R-function on T, a contradiction. But then $N(x) \subseteq V^0(T)$, which is impossible.
Suppose now $x \in V^2(T) \cup V^{12}(T)$. Hence x is not a leaf. Choose a γ_R-function h on T such that (a) $h(x) = 2$, and (b) $k = |epn[x, V_2^h]|$ to be as small as possible. Let $epn[x, V_2^h] = \{y_1, y_2, \dots, y_k\}$ and denote by T_i the connected component of $T - x$, which contains y_i. Hence $h(y_i) = 0$ for all $i \leq k$. Since T is γ_R-excellent, there is a γ_R-function f_k on T with $f_k(y_k) \neq 0$. Since $x \in V^2(T) \cup V^{12}(T)$, $f_k(x) \neq 0$. If $f_k(y_k) = 1$ then $f_k(x) = 1$, which easily implies $x \in V^{012}(T)$, a contradiction. Hence $f_k(y_k) = f_k(x) = 2$. Define a γ_R-function l on T as $l|_{T_k} = f_k|_{T_k}$ and $l(u) = h(u)$ for all $u \in V(T) - V(T_k)$. But $|epn[x, V_2^l]| < k$, a contradiction with the choice of h. Thus $V^1(T) \cup V^2(T) \cup V^{12}(T)$ is empty, and the required follows. □

Lemma 6. *Let T be a tree and $V^-(T)$ is not empty. Then each component of $\langle V^-(T) \rangle$ is either K_1 or K_2.*

Proof. Assume that $P : x_1, x_2, x_3$ is a path in T and $x_1, x_2, x_3 \in V^-(T)$. Then there is a γ_R-function f_i on T with $f_i(x_i) = 1$, $i = 1, 2, 3$ (by Lemma 1). Denote by T_j the connected component of $T - x_2x_j$ that contains x_j, $j = 1, 3$. Then $f_2|_{T_j}$ and $f_j|_{T_j}$ are γ_R-functions on T_j, $j = 1, 3$. Now define a γ_R-function h on T such that $h|_{T_j} = f_j|_{T_j}$, $j = 1, 3$, and $h(u) = f_2(u)$ when $u \in V(T) - (V(T_1) \cup V(T_3))$. But $h(x_1) = h(x_2) = h(x_3) = 1$, a contradiction. □

Lemma 7. *Let T be a γ_R-excellent tree of order at least 2.*

(i) If $x \in V^{012}(T)$, then x is adjacent to exactly one vertex in $V^{-}(T)$, say y_1, and $y_1 \in V^{012}(T)$.

(ii) Let $x \in V^{02}(T)$. If $deg(x) \geq 3$ then x has exactly 2 neighbors in $V^{-}(T)$. If $deg(x) = 2$ then either $N_T(x) \subseteq V^{012}(T)$ or there is a path u, x, y, z in T such that $u, z \in V^{01}(T)$, $y \in V^{02}(T)$ and $deg(y) = 2$.

(iii) $V^{01}(T)$ is either empty or independent.

Proof. Let $x \in V^{012}(T) \cup V^{02}(T)$ and $N(x) = \{y_1, y_2, \ldots, y_r\}$. If x is a leaf, then clearly $x, y_1 \in V^{012}(T)$. So, let $r \geq 2$. Denote by T_i the connected component of $T - x$ which contains y_i, $i \geq 1$. Choose a γ_R-function h on T such that (a) $h(x) = 2$, and (b) $k = |epn[x, V_2^h]|$ to be as small as possible. Let without loss of generality $epn[x, V_2^h] = \{y_1, y_2, \ldots, y_k\}$. By the definition of h it immediately follows that (c) $h|_{T_j}$ is a γ_R-function on T_j for all $j \geq k+1$, (d) for each $i \in \{1, .., k\}$, $h|_{T_i}$ is no RD-function on T_i, and (e) $h|_{T_i - y_i}$ is a γ_R-function on $T_i - y_i$, $i \in \{1, \ldots, k\}$. Hence $\gamma_R(T_i) \leq \gamma_R(T_i - y_i) + 1$ for all $i \in \{1, \ldots, k\}$. Assume that the equality does not hold for some $i \leq k$. Define an RD-function h_i on T as follows: $h_i(u) = h(u)$ when $u \in V(T) - V(T_i)$ and $h_i|_{T_i} = h'_i$, where h'_i is some γ_R-function on T_i. But then either h_i has weight less than $\gamma_R(T)$ or h_i is a γ_R-function on T with $epn[x, V_2^{h_i}] = epn[x, V_2^h] - \{y_i\}$. In both cases we have a contradiction. Thus $\gamma_R(T_i) = \gamma_R(T_i - y_i) + 1$ for all $i \in \{1, .., k\}$. Therefore $\gamma_R(T) = h(V(T)) = 2 + \Sigma_{i=1}^{k}(\gamma_R(T_i) - 1) + \Sigma_{j=k+1}^{r}\gamma_R(T_j) = 2 - k + \Sigma_{i=1}^{r}\gamma_R(T_i) = 2 - k + \gamma_R(T - x)$. Thus $\gamma_R(T) = 2 - k + \gamma_R(T - x)$.

(i) Since $\gamma_R(T - x) + 1 = \gamma_R(T)$, $k = 1$. We already know that $h|_{T_j}$ is a γ_R-function on T_j, $j \geq 2$. Assume that $y_j \in V^{012}(T) \cup V^{01}(T)$ for some $j \geq 2$. Then there is a γ_R-function l on T with $l(y_j) = 1$. Clearly $l|_{T_j}$ is a γ_R-function on T_j. Now define a γ_R-function h'' on T as follows: $h''(u) = h(u)$ when $u \in V(T) - V(T_j)$ and $h''|_{T_j} = l|_{T_j}$. But then $h''(x) = 2, h''(y_j) = 1$ and $xy_j \in E(G)$, which is impossible. Thus, $y_2, y_3, \ldots, y_r \in V^{02}(T)$. Define now γ_R-functions h_1 and h_2 on T as follows: $h_1(u) = h_2(u) = h(u)$ for all $u \in V(T) - \{x, y_1\}$, $h_1(x) = h_1(y_1) = 1$, $h_2(x) = 0$ and $h_2(y_1) = 2$. Thus $y_1 \in V^{012}(T)$.

(ii) Since $\gamma_R(T - x) = \gamma_R(T)$, $k = 2$. Recall that $h|_{T_j}$ is a γ_R-function on T_j, $j \geq 3$, and $\gamma_R(T_i - y_i) = \gamma_R(T_i) - 1$ for $i = 1, 2$. Hence there is a γ_R-function f_i on T_i with $f_i(y_i) = 1$, $i = 1, 2$.
Suppose first that $r \geq 3$. As in the proof of (i), we obtain $y_3, .., y_r \in V^{02}(T)$. Hence there is a γ_R-function g on T such that $g(y_3) = 2$. By the choice of h, $g(x) = 0$. Then $g|_{T_i}$ is a γ_R-function on T_i, $i = 1, 2$. Define now a γ_R-function g' on T as $g'|_{T_i} = f_i$, $i = 1, 2$, and $g'(u) = g(u)$ when $u \in V(T) - (V(T_1) \cup V(T_2))$. Since $g'(y_1) = g'(y_2) = 1$, $y_1, y_2 \in V^{-}(T)$.
So, let $r = 2$ and let f be a γ_R-function on T with $f(x) = 0$. Then there is y_s such that $f(y_s) = 2$, say $s = 2$. Hence $y_2 \in V^{02}(T) \cup V^{012}(T)$ and $f|_{T_1}$ is a γ_R-function on T_1. Define the γ_R-function l on T as $l|_{T_1} = f_1$ and $l(u) = f(u)$ when $u \in V(T) - V(T_1)$. Since $l(y_1) = 1$, $y_1 \in V^{01}(T) \cup V^{012}(T)$.

Assume first that $y_1 \in V^{012}(T)$. Then there is a γ_R-function f' on T with $f'(y_1) = 2$. Since $x \in V^{02}(T)$ and $deg(x) = 2$, $f'(x) = 0$. Hence $f'|_{T_2}$ is a a γ_R-function on T_2. But then we can choose f' so that $f'|_{T_2} = f_2$. Thus $y_2 \in V^{012}(T)$.
So let $y_1 \in V^{01}(T)$ and suppose $y_2 \in V^{012}(T)$. Then there is a γ_R-function f'' on T with $f''(y_2) = 1$. Since $x \in V^{02}(T)$, $f''(x) = 0$ and $f''(y_1) = 2$, a contradiction. Thus, if $y_1 \in V^{01}(T)$ then $y_2 \in V^{02}(T)$.
Finally, let us consider a path y_1, x, y_2, z in T, where $y_1 \in V^{01}(T)$, $x, y_2 \in V^{02}(T)$ and $deg(x) = 2$. Assume to the contrary that $N(y_2) = \{z_1, z_2, \dots, z_s = x\}$ with $s \geq 3$. Denote by T_{z_p} the connected component of $T - y_2$ that contains z_p, $p = 1, 2, .., s$. By applying results proved above for $x \in V^{02}(T)$ with $deg(x) \geq 3$ to y_2, we obtain that (a) y_2 has exactly 2 neighbors in $V^-(T)$, say, without loss of generality, $z_1, z_2 \in V^-(T)$, and (b) $\gamma_R(T_{z_i} - z_i) = \gamma_R(T_{z_i}) - 1$, where $i = 1, 2$. Recall now that: $h(x) = 2$, $h|_{T_i}$ is no RD-function on T_i and $h|_{T_i - y_i}$ is a γ_R-function on $T_i - y_i$, $i = 1, 2$. Hence $h(y_1) = h(y_2) = 0$ and $h|_{T_{z_j}}$ is a γ_R-function on T_{z_j}, $j \leq s - 1$. Since $\gamma_R(T_{z_i} - z_i) = \gamma_R(T_{z_i}) - 1$, $i = 1, 2$, additionally we can choose h so that $h(z_1) = h(z_2) = 1$. But then the function h_1 defined as $h_1(u) = h(u)$ when $u \in V(T) - \{y_1, x, y_2, z_1, z_2\}$ and $h_1(y_1) = h_1(x) = 1$, $h_1(y_2) = 2$, $h_1(z_1) = h(z_2) = 0$ is a γ_R-function on T. Now $h_1(x) = 1$, $h_1(y_2) = 2$ and $xy_2 \in E(G)$ lead to a contradiction. Thus, $N(y_2) = \{x, z\}$. Suppose $z \notin V^{01}(T)$. Then there is a γ_R-function h_4 on T with $h_4(z) = 2$. If $h_4(y_2) = 2$, then $h_4(x) = 0$ and the function h_5 on T defined as $h_5(x) = h_5(y_2) = 1$ and $h_5(u) = h_4(u)$ otherwise, is a γ_R-function on T, a contradiction. Hence $h_4(y_2) = 0$ and since $y_1 \in V^{01}(T)$, $h_4(x) = 2$ and $h_4(y_1) = 0$. But then the function h_6 on T defined as $h_6(x) = h_6(y_1) = 1$ and $h_6(u) = h_4(u)$ otherwise, is a γ_R-function on T, a contradiction. Therefore $z \in V^{01}(T)$, and we are done.

(iii) Assume that $u_1, u_2 \in V^{01}(T)$ are adjacent. Let T_{u_i} be the component of $T - u_1u_2$ that contains u_i, $i = 1, 2$. Let g_i be a γ_R-function on T with $g_i(u_i) = 1$, $i = 1, 2$. Hence $g_i(T_{u_j})$ is a γ_R-function on T_{u_j}, $i, j = 1, 2$. Thus $\gamma_R(T) = \gamma_R(T_{u_1}) + \gamma_R(T_{u_2})$. Define now a γ_R-function g_3 on T as $g_3|_{T_i} = g_i|_{T_i}$, $i = 1, 2$. But then a function g_4 defined as $g_4(u) = g_3(u)$ when $u \in V(T) - \{u_1, u_2\}$, $g_4(u_1) = 2$ and $g_4(u_2) = 0$ is a γ_R-function on T, contradicting $u_1 \in V^{01}(T)$. Thus $V^{01}(T)$ is independent. □

4. Proof of the main result

Proof of Theorem 1. Let T be a γ_R-excellent tree. First, we shall prove the following statement.

($\mathcal{P}_2$) There is a labeling $L : V(T) \to \{A, B, C, D\}$ such that (a) $L_A(T)$ is either empty or independent, (b) each component of $\langle L_B(T) \rangle$ and $\langle L_D(T) \rangle$ is isomorphic to K_2, (c) each element of $L_B(T)$ has degree 2 and it is adjacent to exactly one vertex in $L_A(T)$, (d) each vertex v in $L_C(T)$ has exactly 2 neighbors in $L_A(T) \cup L_D(T)$, and if $deg(v) = 2$ then both neighbors of v are in $L_D(T)$.

By Lemma 5 we know that $V(T) = V^{01}(T) \cup V^{012}(T) \cup V^{02}(T)$. Define a labeling $L : V(T) \to \{A, B, C, D\}$ by $L_A(T) = V^{01}(T)$, $L_D(T) = V^{012}(T)$, $L_B(T) = \{x \in$

$V^{02}(T) \mid deg(x) = 2$ and $|N(x) \cap V^{02}(T)| = 1\}$, and $L_C(T) = V^{02}(T) - L_B(T)$. The validity of $(\mathcal{P}_2)$ immediately follows by Lemma 7.
Denote by $\mathscr{T}_1$ the family of all labeled, as in $(\mathcal{P}_2)$, trees T. We shall show that if $(T, L) \in \mathscr{T}_1$ then $(T, L) \in \mathscr{T}$.

(I) *Proof of* $(T, L) \in \mathscr{T}_1 \Rightarrow (T, L) \in \mathscr{T}$.

Let $(T, L) \in \mathscr{T}_1$. The following claim is immediate.

Claim 1.1

(i) Each leaf of T is in $L_A(T) \cup L_D(T)$.

(ii) If v is a support vertex of T, then v is adjacent to at most 2 leaves.

(iii) If u_1 and u_2 are leaves adjacent to the same support vertex, then $u_1, u_2 \in L_A(T)$.

We now proceed by induction on $k = |L_B \cup L_C|$. The base case, $k \leq 2$, is an immediate consequence of the following easy claim, the proof of which is omitted.

Claim 1.2 (see Fig.1)

(i) If $k = 0$ then $(T, L) = (H_1, I^1)$.

(ii) If $k = 1$ then (T, L) is obtained from (H_1, I_1) by operation O_2, i.e. $(T, L) = (H_{11}, I^{11})$.

(iii) If $k = 2$ then either (T, L) is (H_r, I^r) with $r \in \{2, 3, 4, 5\}$, or (T, L) is obtained from (H_{11}, I^{11}) by operation O_1 or by operation O_2 (see the graphs (H_s, I^s) where $s \in \{6, 7, 8, 9, 10\}$.

Let $k \geq 3$ and suppose that each tree $(H, L') \in \mathscr{T}_1$ with $|L'_B(H) \cup L'_C(H)| < k$ is in $\mathscr{T}$. Let now $(T, L) \in \mathscr{T}_1$ and $k = |L_B(T) \cup L_C(T)|$. To prove the required result, it suffices to show that T has a subtree, say U, such that $(U, L|_U) \in \mathscr{T}_1$, and (T, L) is obtained from $(U, L|_U)$ by one of operations O_1, O_2, O_3 and O_4. Consider any diametral path $P : x_1, x_2, \ldots, x_n$ in T. Clearly x_1 is a leaf. Denote by $x_i^1, x_i^2, ..$ all neighbors of x_i, which do not belong to P, $2 \leq i \leq n - 1$.

Case 1: $sta(x_1) = A$ and $sta(x_2) = B$.
Then $deg(x_1) = 1$, $deg(x_2) = deg(x_3) = 2$, $sta(x_3) = B$ and $sta(x_4) = A$. Thus T is obtained from $T - \{x_1, x_2, x_3\} \in \mathscr{T}_1$ and a copy of H_2 by operation O_3 (via x_4). □

Case 2: $sta(x_1) = A$ and $sta(x_2) = C$.
Hence $deg(x_2) \geq 3$. By the choice of P, $deg(x_2) = 3$, x_2^1 is a leaf, $sta(x_2^1) = A$, and $sta(x_3) = C$. If $deg(x_3) \geq 4$ then T is obtained from $T - \{x_2^1, x_1, x_2\} \in \mathscr{T}_1$ and a copy of F_1 by operation O_1. So, let $deg(x_3) = 3$. Assume first that $sta(x_4) = A$. Then either x_3^1 is a leaf of status A or x_3^1 is a support vertex, $deg(x_3^1) = 2$, and both x_3^1 and its leaf-neighbor have status D. Thus, T is obtained from $T - (N[x_2] \cup N[x_3^1]) \in \mathscr{T}_1$ and a copy of H_3 or H_4, respectively, by operation O_3 (via x_4). Finally let $sta(x_4) = D$. By the choice of P, either x_3^1 is a leaf of status A and then T is obtained from

$T-(N[x_2]\cup\{x_3^1\})\in\mathscr{T}_1$ and a copy of H_3 by operation O_4, or x_3^1 is a support vertex of degree 2 and both x_3^1 and its leaf-neighbor have status D, and then T is obtained from $T-\{x_2^1,x_1,x_2\}\in\mathscr{T}_1$ and a copy of F_1 by operation O_1. □

In what follows, let $sta(x_1)=D$. Hence $deg(x_2)=2$, $sta(x_2)=D$ and $sta(x_3)=C$. If $deg(x_3)=2$ then T is obtained from $T-N[x_2]\in\mathscr{T}_1$ and a copy of F_4 by operation O_2.

Case 3: $deg(x_3)=3$ and $sta(x_4)\in\{A,D\}$.
In this case $sta(x_3^1)=C$, x_3^1 is a support vertex, $deg(x_3^1)=3$, and the leaf neighbors of x_3^1 have status A. Now (a) if $sta(x_4)=A$ then T is obtained from $T-(N[x_2]\cup N[x_3^1])\in\mathscr{T}_1$ and a copy of H_4 by operation O_3 (via x_4), and (b) if $sta(x_4)=D$ then T is obtained from $T-(N[x_2]\cup N[x_3^1])\in\mathscr{T}_1$ and a copy of H_4 by operation O_4 (via x_4). □

Case 4: $deg(x_3)=3$, $sta(x_4)=C$ and $sta(x_3^1)=A$.
Hence x_3^1 is a leaf. If $deg(x_4)=3$ and $sta(x_5)=sta(x_4^1)=D$, or $deg(x_4)\geq 4$, then T is obtained from $T-\{x_1,x_2,x_3,x_3^1\}\in\mathscr{T}_1$ and a copy of F_2 by operation O_1. So, let $deg(x_4)=3$ and the status of at least one of x_5 and x_4^1 is A. Assume first that $sta(x_4^1)=A$. Hence x_4^1 is a leaf (by the choice of P). If $sta(x_5)=A$ then T is obtained from a copy of H_4 and a tree in $\mathscr{T}_1$ by operation O_3 (via x_5). If $sta(x_5)=D$ then T is obtained from a copy of H_4 and a tree in $\mathscr{T}_1$ by operation O_4 (via x_5). Second, let $sta(x_4^1)=D$. Hence $sta(x_5)=A$, $deg(x_4^1)=2$ and the status of the leaf-neighbor of x_4^1 is D. But then T is obtained from a copy of H_5 and a tree in $\mathscr{T}_1$ by operation O_3 (via x_5). □

Case 5: $deg(x_3)=3$, $sta(x_4)=C$ and $sta(x_3^1)=D$.
Hence $deg(x_3^1)=2$, x_3^1 is a support vertex, and the leaf-neighbor of x_3^1 has status D. If $deg(x_4)\geq 4$ or $sta(x_5)=sta(x_4^1)=D$, then T is obtained from $T-N[\{x_2,x_3^1\}]\in\mathscr{T}_1$ and a copy of F_3 by operation O_1. So, let $deg(x_4)=3$ and at least one of x_5 and x_4^1 has status A. Assume $sta(x_4^1)=A$. Hence x_4^1 is a leaf. If $sta(x_5)=A$ then T is obtained from $T-N[\{x_2,x_3^1,x_4^1\}]\in\mathscr{T}_1$ and a copy of H_6 by operation O_3 (via x_5). If $sta(x_5)=D$ then T is obtained from $T-N[\{x_2,x_3^1,x_4^1\}]\in\mathscr{T}_1$ and a copy of H_6 by operation O_4 (via x_5). Now let $sta(x_4^1)=D$. Hence $sta(x_5)=A$ and then T is obtained from a copy of H_7 and a tree in $\mathscr{T}_1$ by operation O_3 (via x_5). □

Case 6: $deg(x_3)\geq 4$.
Hence x_3 has a neighbor, say y, such that $y\neq x_4$ and $sta(y)=C$. By the choice of P, y is a support vertex which is adjacent to exactly 2 leaves, say z_1 and z_2, and $sta(z_1)=sta(z_2)=A$. But then T is obtained from $T-\{y,z_1,z_2\}\in\mathscr{T}_1$ and a copy of F_1 by operation O_1.

By Claim 2.1, there are no other possibilities. □

(II) $(T,S)\in\mathscr{T}\Rightarrow(T,S)\in\mathscr{T}_1$. Obvious. □

It remains the following.

(III) *Proof of* $(T,S)\in\mathscr{T}\Rightarrow T$ *is* γ_R*-excellent and* $(\mathcal{P}_1)$ *holds.*

Let $(T,S) \in \mathscr{T}$. We know that $(T,S) \in \mathscr{T}_1$. We now proceed by induction on $k = |S_B \cup S_C|$. First let $k \leq 2$. By Claim 1.2, $T \in \mathscr{H} = \{H_1, .., H_{11}\}$. It is easy to see that all elements of $\mathscr{H}$ are γ_R-excellent graphs and $(\mathcal{P}_1)$ holds for each $T \in \mathscr{H}$. Let $k \geq 3$ and suppose that if $(H, S') \in \mathscr{T}$ and $|S'_B(H) \cup S'_C(H)| < k$, then H is γ_R-excellent and $(\mathcal{P}_1)$ holds with (T,S) replaced by (H,S'). So, let $(T,S) \in \mathscr{T}$ and $k = |S_B(T) \cup S_C(T)|$. Then there is a $\mathscr{T}$-sequence $\tau : (T^1, S^1), \ldots, (T^{j-1}, S^{j-1}), (T,S)$. By induction hypothesis, T^{j-1} is γ_R-excellent and $(\mathcal{P}_1)$ holds with (T,S) replaced by (T^{j-1}, S^{j-1}). We consider four possibilities depending on whether T is obtained from T^{j-1} by operation O_1, O_2, O_3 or O_4.

Case 7: T is obtained from $T^{j-1} \in \mathscr{T}$ and F_a by operation O_1, $a \in \{1,2,3\}$. Hence T is obtained after adding the edge ux to the union of T^{j-1} and F_a, where $sta_{T^{j-1}}(u) = sta_{F_a}(x) = C$ (see Fig. 2). First note that $\gamma_R(F_a) = a+1$, and F_2 and F_3 are γ_R-excellent graphs. Since $\gamma_R(F_a - x) = \gamma_R(F_a)$ and $u \in V^{02}(T^{j-1})$, Lemma 2 implies $\gamma_R(T) = \gamma_R(T^{j-1}) + \gamma_R(F_a)$. Hence for any γ_R-function g on T, the weight of $g|_{F_a}$ is not more than $\gamma_R(F_a)$. But then $g(x) \neq 1$ and either $g|_{F_a}$ is a γ_R-function on F_a or $g|_{F_a - x}$ is a γ_R-function on $F_a - x$. By inspection of all γ_R-functions on F_a and $F_a - x$, we obtain

(α_1) $S_A(T) \cap V(F_a) = V^{01}(T) \cap V(F_a)$, $S_B(T) \cap V(F_a) = \emptyset$, $\{x\} = S_C(T) \cap V(F_a) = V^{02}(T) \cap V(F_a)$, and $S_D(T) \cap V(F_a) = V^{012}(T) \cap V(F_a)$.

By the definition of operation O_1 it immediately follows

(α_2) $S_X(T) \cap V(T^{j-1}) = S_X^{j-1}(T^{j-1})$, for all $X \in \{A,B,C,D\}$.

Let f_1 be a γ_R-function on T^{j-1} and f_2 a γ_R-function on F_a. Then the RD-function f on T defined as $f|_{T^{j-1}} = f_1$ and $f|_{F_a} = f_2$ is a γ_R-function on T. Since f_1 was chosen arbitrarily, we have

(α_3) $V^{01}(T^{j-1}) \subseteq V^{01}(T) \cup V^{012}(T)$, $V^{02}(T^{j-1}) \subseteq V^{02}(T) \cup V^{012}(T)$, and $V^{012}(T^{j-1}) \subseteq V^{012}(T)$.

By (α_1) and (α_3) we conclude that T is γ_R-excellent.

Now we shall prove that

(α_4) $V^{01}(T) \cap V(T^{j-1}) = V^{01}(T^{j-1})$, $V^{02}(T) \cap V(T^{j-1}) = V^{02}(T^{j-1})$, and $V^{012}(T) \cap V(T^{j-1}) = V^{012}(T^{j-1})$.

Assume there is a vertex $z \in V^{02}(T^{j-1}) \cap V^{012}(T)$. By Lemma 7, z is adjacent to at most 2 elements of $V^-(T^{j-1})$. Now by (α_3) and since $\Delta(\langle V^-(T)\rangle) \leq 1$ (by Lemma 6), z is adjacent to exactly one element of $V^-(T^{j-1})$. But then Lemma 7 implies that there is a path z_1, z, z_2, z_3 in T^{j-1} such that $deg_{T^{j-1}}(z) = deg_{T^{j-1}}(z_2) = 2$, $z, z_2 \in V^{02}(T^{j-1})$ and $z_1, z_3 \in V^{01}(T^{j-1})$. Since $(\mathcal{P}_1)$ is true for T^{j-1}, $sta_{T^{j-1}}(z_1) = sta_{T^{j-1}}(z_3) = A$, and $sta_{T^{j-1}}(z) = sta_{T^{j-1}}(z_2) = B$. Clearly, at least one of z_1 and z_3 is a cut-vertex. Denote by Q the graph $\langle\{z_1, z, z_2, z_3\}\rangle$ and let the vertices of Q

are labeled as in T^{j-1}. Let U_s be the connected component of $T - \{z, z_2\}$, which contains z_s, $s = 1, 3$.
Assume first that T^1 is a subtree of $U \in \{U_1, U_3\}$. Then there is i such that T^i is obtained from T^{i-1} and Q by operation O_3. Hence T^{i-1} is a subtree of U. Recall that if $y \in V(T^r)$ and $r \leq s \leq j-1$, then $sta_{T^r}(y) = sta_{T^s}(y)$. Using this fact, we can choose τ so, that $T^{i-1} = U$. Therefore U is in $\mathscr{T}$. Suppose that neither z_1 nor z_3 is a leaf of T^{j-1}. Define $R^s = T^{i+s} - (V(T^{i-1}) \cup \{z, z_2\})$, $s = 1, 2, \ldots, j-1-i$. Since clearly R^1 is in $\{H_2, H_3, \ldots, H_7\}$, the sequence $R^1, R^2, \ldots, R^{j-1-i}$ is a $\mathscr{T}$-sequence of U', where $\{U, U'\} = \{U_1, U_2\}$. Thus, both U_1 and U_3 are in $\mathscr{T}$, and $sta_{U_1}(z_1) = A$. By the induction hypothesis, $z_1 \in V^{01}(U_1)$.
Suppose now that $u \in V(U_3)$. Consider the sequence of trees U_3, U_4, U_5, where U_4 is obtained from U_3 and Q by operation O_3 (via z_3), and U_5 is obtained from U_4 and F_a by operation O_1. Clearly U_5 is in $\mathscr{T}$, $sta_{U_5}(z_1) = A$ and by the induction hypothesis, $z_1 \in V^{01}(U_5)$. Since $T = (U_5 \cdot U_1)(z_1)$ and $\{z_1\} = V^{01}(U_1) \cap V^{01}(U_5)$, by Proposition 2 we have $z_1 \in V^{01}(T)$. But then Lemma 7 implies $z_2 \in V^{02}(T)$, a contradiction.
Now let $u \in V(U_1)$. Denote by U_2 the graph obtained from U_1 and F_a by operation O_3. Then U_2 is in $\mathscr{T}$, $sta_{U_2}(z_1) = A$, and by induction hypothesis, $z_1 \in V^{01}(U_2)$. Define also the graph U_6 as obtained from U_3 and Q by operation O_3, i.e. $U_6 = (U_3 \cdot Q)(z_3)$. Then U_6 is in $\mathscr{T}$, $sta_{U_6}(z_1) = A$ and by induction hypothesis, $z_1 \in V^{01}(U_6)$. Now by Proposition 2, $z_1 \in V^{01}(T)$, which leads to $z_2 \in V^{02}(T)$ (by Lemma 7), a contradiction.
Thus, in all cases we have a contradiction. Therefore $V^{02}(T^{j-1}) \subseteq V^{02}(T)$ when both z_1 and z_3 are cut-vertices. If z_1 or z_3 is a leaf, then, by similar arguments, we can obtain the same result.
Let now $T^1 \equiv Q$. Then T^2 is obtained from T^1 and H_k by operation O_3. Consider the sequence of trees $\tau_1 : T_1^1 = H_k, T^2, T^3, \ldots, T^{j-1}$. Clearly τ_1 is a $\mathscr{T}$-sequence of T^{j-1} and $T_1^1 \neq Q$. Therefore we are in the previous case. Thus, $V^{02}(T^{j-1}) = V(T^{j-1}) \cap V^{02}(T)$.
Assume now that there is a vertex $w \in V^{01}(T^{j-1}) \cap V^{012}(T)$. By Lemma 7(i) w has a neighbor in T, say w', such that $w' \in V^{012}(T)$. Since $w \not\equiv u$, $w' \in V(T^{j-1})$. But all neighbors of w in T^{j-1} are in $V^{02}(T^{j-1})$ (by Lemma 7 applied to T^{j-1} and w). Since $V^{02}(T^{j-1}) = V(T^{j-1}) \cap V^{02}(T)$, we obtain a contradiction.

Thus (α_4) is true.
Now we are prepared to prove that $(\mathcal{P}_1)$ is valid. Using, in the chain of equalities below, consecutively (α_2), the induction hypothesis, (α_1) and (α_4), we obtain

$$S_A(T) = S_A^{j-1}(T^{j-1}) \cup (S_A(T) \cap V(F_a)) = V^{01}(T^{j-1}) \cup (V^{01}(T) \cap V(F_a)) = V^{01}(T),$$

and similarly, $S_D(T) = V^{012}(T)$. Since $u \notin S_B(T)$ and $S_B(T) \cap V(F_a) = \emptyset$, we have

$$\begin{aligned} S_B(T) &= S_B(T) \cap V(T^{j-1}) \overset{(\alpha_2)}{=} S_B^{j-1}(T^{j-1}) \\ &= \{t \in V^{02}(T^{j-1}) \mid deg_{T^{j-1}}(t) = 2 \text{ and } |N_{T^{j-1}}(t) \cap V^{02}(T^{j-1})| = 1\} \\ &\overset{(\alpha_4)}{=} \{t \in V^{02}(T) \cap V(T^{j-1}) \mid deg_T(t) = 2 \text{ and } |N_T(t) \cap V^{02}(T)| = 1\} \\ &= \{t \in V^{02}(T) \mid deg_T(t) = 2 \text{ and } |N_T(t) \cap V^{02}(T)| = 1\}. \end{aligned}$$

The last equality follows from $deg_T(x) > 2$ and $\{x\} = V^{02}(T) \cap V(F_a)$ (see (α_1)). Now the equality $S_C(T) = V^{02}(T) - S_B(T)$ is obvious. Thus, $(\mathcal{P}_1)$ holds and we are done.

Case 8: T is obtained from $T^{j-1} \in \mathscr{T}$ by operation O_2.
Clearly, $\gamma_R(F_4) = \gamma_R(F_4 - x) = 2$. By Lemma 2, $\gamma_R(T) = \gamma_R(T^{j-1}) + \gamma_R(H_4)$. Let f_1 be a γ_R-function on T^{j-1} and f_2 a γ_R-function on F_4. Then the function f defined as $f|_{T^{j-1}} = f_1$ and $f|_{F_4} = f_2$ is a γ_R-function on T. Therefore $V^{012}(T^{j-1}) \subseteq V^{012}(T)$, $V^{01}(T^{j-1}) \subseteq V^{01}(T) \cup V^{012}(T)$, and $V^{02}(T^{j-1}) \subseteq V^{02}(T) \cup V^{012}(T)$.
Assume that there is $y \in V^{0s}(T^{j-1}) \cap V^{012}(T)$, $s \in \{1, 2\}$, and let f' be a γ_R-function on T with $f'(y) = r \notin \{0, s\}$. If $f'|_{T^{j-1}}$ is an RD-function on T^{j-1}, then $f'|_{T^{j-1}}(V(T^{j-1})) > \gamma_R(T^{j-1})$ and $f'|_{F_4}(V(F_4)) \geq 2$. This leads to $f'(V(T)) > \gamma_R(T)$, a contradiction. Hence $f'|_{T^{j-1}}$ is no RD-function on T^{j-1} and $f'|_{T^{j-1}-u}$ is a γ_R-function on $T^{j-1} - u$. Define now an RD-function f'' on T^{j-1} as $f''|_{T^{j-1}-u} = f'|_{T^{j-1}-u}$ and $f''(u) = 1$. Since $u \in V^-(T^{j-1})$, f'' is a γ_R-function on T^{j-1} with $f''(y) = r \notin \{0, s\}$, a contradiction with $y \in V^{0s}(T^{j-1})$. Thus

(α_5) $V^{012}(T^{j-1}) = V^{012}(T) \cap V(T^{j-1})$, $V^{01}(T^{j-1}) = V^{01}(T) \cap V(T^{j-1})$, and $V^{02}(T^{j-1}) = V^{02}(T) \cap V(T^{j-1})$.

Let x, x_1, x_2 be a path in F_4, h_1 a γ_R-function on T^{j-1} with $h_1(u) = 2$, and h_2 a γ_R-function on $T^{j-1} - u$. Define γ_R-functions $g_1, .., g_4$ on T as follows:

- $g_1|_{T^{j-1}} = h_1$, $g_1(x) = g_1(x_2) = 0$ and $g_1(x_1) = 2$;
- $g_2|_{T^{j-1}} = h_1$, $g_2(x) = 0$ and $g_2(x_1) = g_2(x_2) = 1$;
- $g_3|_{T^{j-1}} = h_1$, $g_3(x) = g_3(x_1) = 0$ and $g_3(x_2) = 2$;
- $g_4|_{T^{j-1}-u} = h_2$, $g_4(u) = g_4(x_1) = 0$, $g(x) = 2$ and $g_4(x_2) = 1$.

This, (α_5) and Lemma 6 allows us to conclude that T is γ_R-excellent, $x_1, x_2 \in V^{012}(T)$ and $x \in V^{02}(T)$.
By induction hypothesis, $(\mathcal{P}_1)$ holds with (T, S) replaced by (T^{j-1}, S^{j-1}). Then Since $u \notin S_B(T)$ and $S_B(T) \cap V(F_4) = \emptyset$, we have

$$\begin{aligned} S_B(T) &= S_B^{j-1}(T^{j-1}) \\ &= \{t \in V^{02}(T^{j-1}) \mid deg_{T^{j-1}}(t) = 2 \text{ and } |N_{T^{j-1}}(t) \cap V^{02}(T^{j-1})| = 1\} \\ &= \{t \in V^{02}(T) \mid deg_T(t) = 2 \text{ and } |N_T(t) \cap V^{02}(T)| = 1\}. \end{aligned}$$

The last equality follows from $deg_T(x) > 2$ and $\{x\} = V^{02}(T) \cap V(F_4)$. Now the equality $S_C(T) = V^{02}(T) - S_B(T)$ is obvious. Thus, $(\mathcal{P}_1)$ is true.

Case 9: T is obtained from $T^{j-1} \in \mathscr{T}$ by operation O_3.
Let $T = (T^{j-1} \cdot H_k)(u, v : u)$, where $sta_{T^{j-1}}(u) = sta_{H_k}(v) = sta_T(u) = A$ and $k \in \{2, .., 7\}$. Hence $S_X(T) = S_X^{j-1}(T^{j-1}) \cup I_X^k(H_k)$, for any $X \in \{A, B, C, D\}$. We know that $(\mathcal{P}_1)$ holds with (T, S) replaced by any of (T^{j-1}, S^{j-1}) and (H_k, I^k). Hence $S_A(T) = S_A^{j-1}(T^{j-1}) \cup I_A^k(H_k) = V^{01}(T^{j-1}) \cup V^{01}(H_k)$. Now, by Proposition 2, applied to T^{j-1} and H_k, $S_A(T) = V^{01}(T)$. Similarly we obtain $S_D(T) = V^{012}(T)$. We also have

$$\begin{aligned}
S_B(T) &= S_B^{j-1}(T^{j-1}) \cup I_B^k(H_k) \\
&= \{t \in V^{02}(T^{j-1}) \mid deg_{T^{j-1}}(t) = 2 \text{ and } |N_{T^{j-1}}(t) \cap V^{02}(T^{j-1})| = 1\} \\
&\cup \{t \in V^{02}(H_k) \mid deg_{H_k}(t) = 2 \text{ and } |N_{H_k}(t) \cap V^{02}(H_k)| = 1\} \\
&= \{t \in V^{02}(T^{j-1}) \cup V^{02}(H_k) \mid deg_T(t) = 2 \text{ and } |N_T(t) \cap V^{02}(T)| = 1\},
\end{aligned}$$

as required, because $V^{02}(T^{j-1}) \cup V^{02}(H_k) = V^{02}(T)$ (by Proposition 2). Now the equality $S_C(T) = V^{02}(T) - S_B(T)$ is obvious.

Case 10: T is obtained from $T^{j-1} \in \mathscr{T}$ and $H_k \in \mathscr{T}$, $k \in \{3, 4, 6\}$, by operation O_4. By induction hypothesis and Lemma 4, we have $\gamma_R(T) = \gamma_R(T^{j-1}) + \gamma_R(H_k) - 1$ and $u \in V^{012}(T)$. Let f_1 be a γ_R-function on T^{j-1} and f_2 a γ_R-function on $H_k - v$. Then the function f defined as $f|_{T^{j-1}} = f_1$ and $f|_{H_k - v} = f_2$ is a γ_R-function on T. Therefore $V^{012}(T^{j-1}) \subseteq V^{012}(T)$, $V^{01}(T^{j-1}) \subseteq V^{01}(T) \cup V^{012}(T)$, and $V^{02}(T^{j-1}) \subseteq V^{02}(T) \cup V^{012}(T)$. Assume that there is $y \in V^{0s}(T^{j-1}) \cap V^{012}(T)$, $s \in \{1, 2\}$, and let f' be a γ_R-function on T with $f'(y) = r \notin \{0, s\}$. But then $f'|_{T^{j-1}}$ is no RD-function on T^{j-1}, $f'(u) = 0$, $f'|_{T^{j-1}-u}$ is a γ_R-function on $T^{j-1} - u$ and $f'|_{H_k}$ is a γ_R-function on H_k. Define now an RD-function g_1 on T^{j-1} as $g_1|_{T^{j-1}-u} = f'|_{T^{j-1}-u}$ and $g_1(u) = 1$. Since $g_1(V(T^{j-1})) = \gamma_R(T^{j-1} - u) + 1 = \gamma_R(T^{j-1})$, g_1 is a γ_R-function on T^{j-1}. But $g_1(y) = r \notin \{0, s\}$, a contradiction. Thus

(α_6) $V^{012}(T^{j-1}) = V^{012}(T) \cap V(T^{j-1})$, $V^{01}(T^{j-1}) = V^{01}(T) \cap V(T^{j-1})$, and $V^{02}(T^{j-1}) = V^{02}(T) \cap V(T^{j-1})$.

The next claim is obvious.

Claim 1.3 Let x be the neighbor of v in H_k, $k \in \{3, 4, 6\}$.Then $\gamma_R(H_3) = 4$, $\gamma_R(H_4) = 5$, $\gamma_R(H_6) = 6$, $\gamma_R(H_k - v) = \gamma_R(H_k - \{v, x\}) = \gamma_R(H_k)$, and $l(x) = 0$ for any γ_R-function l on $H_k - v$.

Let h be a γ_R-function on T. We know that $u \in V^{012}(T)$, $u \in V^{012}(T^{j-1})$, $v \in V^{01}(H_k)$, and $\gamma_R(T) = \gamma_R(T^{j-1}) + \gamma_R(H_k) - 1$. Then by Claim 1.3 we clearly have:

(a1) If $h(u) = 2$ then at least one of the following holds:

- (a1.1) $h|_{H_k - v}$ is a γ_R-function on $H_k - v$, and
- (a1.2) $h|_{H_k - \{v,x\}}$ is a γ_R-function on $H_k - \{v, x\}$.

(a2) If $h(u) = 1$ then $h|_{H_k - v}$ is a γ_R-function on $H_k - v$.

(a3) If $h(u) = 0$ then either $h|_{H_k}$ is a γ_R-function on H_k, or $h|_{H_k - v}$ is a γ_R-function on $H_k - v$.

Let l_1, l_2, l_3, l_4 and l_5 be γ_R-functions on $H_k, H_k - v$, $H_k - \{v, x\}$, $T^{j-1} - u$ and T^{j-1}, respectively, and let $l_5(u) = 2$. Define the functions h_1, h_2, and h_3 on T as follows: (i) $h_1|_{T^{j-1}} = l_5$, $h_1(x) = 0$ and $h_1|_{H_k - \{v,x\}} = l_3$, (ii) $h_2|_{T^{j-1}} = l_5$ and $h_1|_{H_k - v} = l_2$, and (iii) $h_3|_{T^{j-1} - u} = l_4$ and $h_3|_{H_k} = l_1$. Clearly h_1, h_2, and h_3 are γ_R-functions on T. After inspection of all γ_R-functions of $H_k, H_k - v$ and $H_k - \{v, x\}$, we conclude that $V^{01}(H_k) - \{v\} \subseteq V^{01}(T)$, $V^{02}(H_k) \subseteq V^{02}(T)$, and $V^{012}(H_k) \subseteq V^{012}(T)$. This and (α_6) imply

(α_7) $V^{012}(T) = V^{012}(T^{j-1}) \cup V^{012}(H_k)$, $V^{02}(T) = V^{02}(T^{j-1}) \cup V^{02}(H_k)$, and $V^{01}(T) = V^{01}(T^{j-1}) \cup (V^{01}(H_k) - \{v\})$.

Since $(\mathcal{P}_1)$ holds with T replaced by H_k or by T^{j-1} (by induction hypothesis), using (α_7) we obtain that $(\mathcal{P}_1)$ is satisfied. □

5. Corollaries

The next three results immediately follow by Theorem 1.

Corollary 1. *If $(T, S_1), (T, S_2) \in \mathscr{T}$ then $S_1 \equiv S_2$.*

If $(T, S) \in \mathscr{T}$ then we call S the *$\mathscr{T}$-labeling* of T.

Corollary 2. *Let T be a γ_R-excellent tree of order $n \geq 5$, and S the $\mathscr{T}$-labeling of T. Then $\frac{n}{5} \leq |V^{02}(T)| \leq \frac{2}{3}(n-1)$ and $\frac{4}{5}n \geq |V^-(T)| \geq \frac{1}{3}(n+2)$. Moreover,*

(i) $\frac{n}{5} = |V^{02}(T)|$ if and only if (T, S) has a $\mathscr{T}$-sequence $\tau : (T^1, S^1), \ldots, (T^j, S^j)$, such that $(T^1, S^1) = (F_3, J^3)$ and if $j \geq 2$, (T^{i+1}, S^{i+1}) can be obtained recursively from (T^i, S^i) and (F_3, J^3) by operation O_1.

(ii) $|V^{02}(T)| \leq \frac{2}{3}(n-1)$ if and only if (T, S) has a $\mathscr{T}$-sequence $\tau : (T^1, S^1), .., (T^j, S^j)$, such that $(T^1, S^1) = (H_2, I^2)$ and if $j \geq 2$, (T^{i+1}, S^{i+1}) can be obtained recursively from (T^i, S^i) and (H_2, I^2) by operation O_3.

Corollary 3. *Let G be an n-order γ_R-excellent connected graph of minimum size. Then either $G = K_3$ or $n \neq 3$ and G is a tree.*

6. Special cases

Let G be a graph and $\{a_1, .., a_k\} \subseteq \{0, 1, 2, 01, 02, 12, 012\}$. We say that G is a $\mathcal{R}_{a_1,..,a_k}$-graph if $V(G) = \cup_{i=1}^k V^{a_i}(G)$ and all $V^{a_1}(G), .., V^{a_k}(G)$ are nonempty. Now let T be a γ_R-excellent tree of order at least 2. By Theorem 1, we immediately conclude that $T \in \mathcal{R}_{012} \cup \mathcal{R}_{01,02} \cup \mathcal{R}_{02,012} \cup \mathcal{R}_{01,02,012}$. Moreover,

(i) $T \in \mathcal{R}_{012}$ if and only if $T = K_2$, and

(ii) $T \in \mathcal{R}_{01,02,012}$ if and only if none of $S_A(T), S_C(T)$ and $S_D(T)$ is empty, where S is the $\mathscr{T}$-labeling of T.

In this section, we turn our attention to the classes $\mathcal{R}_{01,02}$ and $\mathcal{R}_{02,012}$.

6.1. $\mathcal{R}_{01,02}$-graphs.

Here we give necessary and sufficient conditions for a tree to be in $\mathcal{R}_{01,02}$. We define a subfamily $\mathscr{T}_{01,02}$ of $\mathscr{T}$ as follows. A labeled tree $(T, S) \in \mathscr{T}_{01,02}$ if and only if (T, S) can be obtained from a sequence of labeled trees $\tau : (T^1, S^1), \ldots, (T^j, S^j)$, $(j \geq 1)$, such that (T^1, S^1) is in $\{(H_2, I^2), (H_3, I^3)\}$ (see Figure 1) and $(T, S) = (T^j, S^j)$, and, if $j \geq 2$, (T^{i+1}, S^{i+1}) can be obtained recursively from (T^i, S^i) by one of the operations O_5 and O_6 listed below; in this case τ is said to be a $\mathscr{T}_{01,02}$-*sequence* of T.

Operation O_5. The labeled tree (T^{i+1}, S^{i+1}) is obtained from (T^i, S^i) and (F_1, J^1) (see Figure 2) by adding the edge ux, where $u \in V(T_i)$, $x \in V(F_1)$ and $sta_{T^i}(u) = sta_{F_1}(x) = C$.

Operation O_6. The labeled tree (T^{i+1}, S^{i+1}) is obtained from (T^i, S^i) and (H_k, I^k), $k \in \{2, 3\}$ (see Figure 1), in such a way that $T^{i+1} = (T^i \cdot H_k)(u, v : u)$, where $sta_{T^i}(u) = sta_{H_k}(v) = A$, and $sta_{T^{i+1}}(u) = A$.

Remark that once a vertex is assigned a status, this status remains unchanged as the labeled tree (T, S) is recursively constructed. By the above definitions we see that $S_D(T)$ is empty when $(T, S) \in \mathscr{T}_{01,02}$. So, in this case, it is naturally to consider a labeling S as $S : V(T) \to \{A, B, C\}$. From Theorem 1 we immediately obtain the following result.

Corollary 4. *Let T be a tree of order at least 2. Then $T \in \mathcal{R}_{01,02}$ if and only if there is a labeling $S : V(T) \to \{A, B, C\}$ such that (T, S) is in $\mathscr{T}_{01,02}$. Moreover, if $(T, S) \in \mathscr{T}_{01,02}$ then*

($\mathcal{P}_3$) $S_B(T) = \{x \in V^{02}(T) \mid deg(x) = 2$ and $|N(x) \cap V^{02}(T)| = 1\}$, $S_A(T) = V^{01}(T)$, and $S_C(T) = V^{02}(T) - S_B(T)$.

As un immediate consequence of Corollary 1 we obtain:

Corollary 5. *If $(T, S_1), (T, S_2) \in \mathscr{T}_{01,02}$ then $S_1 \equiv S_2$.*

A graph G is called a 2-*corona* if each vertex of G is either a support vertex or a leaf, and each support vertex of G is adjacent to exactly 2 leaves. In a *labeled* 2-*corona* all leaves have status A and all support vertices have status C.

Proposition 3. *Every connected n-order graph H, $n \geq 2$, is an induced subgraph of a $\mathcal{R}_{01,02}$-graph with the domination number equals to $2|V(H)|$.*

Proof. Let a graph G be a 2-corona such that the induced subgraph by the set of all support vertices of G is isomorphic to H. Let x be a support vertex of G and y, z the leaf neighbors of x in G. Then clearly for any γ_R-function f on G, $f(x) + f(y) + f(z) \geq 2$, $f(y) \neq 2 \neq f(z)$ and $f(x) \neq 1$. Define RD-functions h and g on G as follows: (a) $h(u) = 2$ when u is a support vertex of G and $h(u) = 0$, otherwise, and (b) $g(v) = h(v)$ when $v \notin \{x, y, z\}$, and $g(x) = 0$, $g(y) = g(z) = 1$. Therefore $\gamma_R(G) = 2|V(H)|$ and G is in $\mathcal{R}_{01,02}$. □

Corollary 6. *There does not exist a forbidden subgraph characterization of the class of $\mathcal{R}_{01,02}$-graphs. There does not exist a forbidden subgraph characterization of the class of γ_R-excellent graphs.*

Let $\mathscr{T}'_{01,02}$ be the family of all labeled trees (T, L) that can be obtained from a sequence of labeled trees $\lambda : (T^1, L^1), \ldots, (T^j, L^j)$, $(j \geq 1)$, such that $(T, L) = (T^j, L^j)$, (T^1, L^1) is either (H_2, I^2) (see Figure 1) or a labeled 2-corona tree, and, if $j \geq 2$, (T^{i+1}, L^{i+1}) can be obtained recursively from (T^i, L^i) by one of the operations O_7 and O_8 listed below; in this case λ is said to be a $\mathscr{T}'_{01,02}$-*sequence* of T.

Operation O_7. The labeled tree (T^{i+1}, L^{i+1}) is obtained from (T^i, L^i) and (H_2, I^2), in such a way that $T^{i+1} = (T^i \cdot H_2)(u, v : u)$, where $sta_{T^i}(u) = sta_{H_2}(v) = A$, and $sta_{T^{i+1}}(u) = A$.

Operation O_8. The labeled tree (T^{i+1}, L^{i+1}) is obtained from (T^i, L^i) and a labeled 2-corona tree, say U_i, in such a way that $T^{i+1} = (T^i \cdot U_i)(u, v : u)$, where $sta_{T^i}(u) = sta_{U_i}(v) = A$, and $sta_{T^{i+1}}(u) = A$.

Again, once a vertex is assigned a status, this status remains unchanged as the 2-labeled tree T is recursively constructed.

Theorem 2. *For any tree T the following are equivalent.*

(A_1) T is in $\mathcal{R}_{01,02}$.

(A_2) There is a labeling $S : V(T) \to \{A, B, C\}$ such that (T, S) is in $\mathscr{T}_{01,02}$.

(A_3) There is a labeling $L : V(T) \to \{A, B, C\}$ such that (T, L) is in $\mathscr{T}'_{01,02}$.

Proof. $(A_1) \Leftrightarrow (A_2)$: By Corollary 4.

$(A_3) \Rightarrow (A_2)$:
Let a tree $(T, L) \in \mathscr{T}'_{01,02}$. It is clear that all $\mathscr{T}'_{01,02}$-sequences of (T, L) have the same number of elements. Denote this number by $r(T)$. We shall prove that $(T, L) \in \mathscr{T}'_{01,02} \Rightarrow (T, L) \in \mathscr{T}_{01,02}$. We proceed by induction on $r(T)$. If $r(T) = 1$ then either

(T,L) is a labeled 2-corona tree, or $(T,L) = (H_2, I^2)$. In both cases $(T,L) \in \mathscr{T}_{01,02}$. We need the following obvius claim.

Claim 2.1 If (T',L') is a labeled 2-corona tree, $w \in V(T')$ and $sta(w) = A$, then either (T',L') is (H_3, I^3) or there is a $\mathscr{T}$-sequence $\tau : (T^1,S^1),\dots,(T^l,S^l)$, $(l \geq 2)$, such that $(T^1,S^1) = (H_3,I^3)$, $w \in V(T^1)$, $(T^l,S^l) = (T',L')$, and (T^{i+1},S^{i+1}) can be obtained recursively from (T^i,S^i) and (F_1,J^1) by operation O_5.

Suppose now that each tree $(H,L_H) \in \mathscr{T}'_{01,02}$ with $r(H) < k$ is in $\mathscr{T}_{01,02}$, where $k \geq 2$. Let $\lambda : (T^1,L^1),\dots,(T^k,L^k)$, be a $\mathscr{T}'_{01,02}$-sequence of a labeled tree $(T,L) \in \mathscr{T}'_{01,02}$. By the induction hypothesis, (T^{k-1},L^{k-1}) is in $\mathscr{T}_{01,02}$. Let $\tau_1 : (U^1,S^1),\dots,(U^m,S^m)$ be a $\mathscr{T}$-sequence of (T^{k-1},L^{k-1}). Hence $U^m = T^{k-1}$ and $S^m = L^{k-1}$. If (T^k,L^k) is obtained from (T^{k-1},L^{k-1}) and (H_2,I^2) by operation O_7, then $(U^1,S^1),\dots,(U^m,S^m),(T^k,L^k) = (T,L)$ is a $\mathscr{T}$-sequence of (T,L). So, let (T^k,L^k) is obtained from (T^{k-1},L^{k-1}) and a labeled 2-corona tree, say (Q,L_q) by operation O_8. Hence T^{k-1} and Q have exactly one vertex in comman, say w, and $sta_{T^{k-1}}(w) = sta_Q(w) = sta_{T^k}(w) = A$. By Claim 2.1, $(Q,L_q) \in \mathscr{T}_{01,02}$ and it has a $\mathscr{T}_{01,02}$-sequence, say $(Q^1,L_q^1),\dots,(Q^s,L_q^s)$ such that $Q^s = Q$, $L_q = L_q^s$, and $w \in V(Q^1)$. Denote $W^{m+i} = \langle V(U^m) \cup V(Q^i) \rangle$ and let a labeling S^{m+i} be such that $S^{m+i}|_{U^m} = S^m$ and $S^{m+i}|_{Q^i} = L_q^i$. Then the sequence of labeled trees $(U^1,S^1),\dots,(U^m,S^m)$, $(W^{m+1},S^{m+1}),\dots,(W^{m+s},S^{m+s}) = (T,L)$ is a $\mathscr{T}_{01,02}$-sequence of (T,L).

$(A_2) \Rightarrow (A_3)$:
Let a labeled tree $(T,S) \in \mathcal{T}_{01,02}$. Then (T,S) has a $\mathscr{T}$-sequence $\tau : (T^1,S^1),\dots,(T^j,S^j) = (T,S)$, where $(T^1,S^1) \in \{(H_2,I^2),(H_3,I^3)\} \subset \mathscr{T}'_{01,02}$. We proceed by induction on $p(T) = \Sigma_{z \in \mathtt{C}(T)} deg_T(z)$, where $\mathtt{C}(T)$ is the set of all cut-vertices of T that belong to $S_A(T)$. Assume first $p(T) = 0$. If $j = 1$ then we are done. If $j \geq 2$ then $(T^1,S^1) = (H_3,I^3)$ and (T^{i+1},S^{i+1}) is obtained from (F_1,J^1) and (T^i,S^i) by operation O_5. Thus, (T,S) is a labeled 2-corona tree, which allow us to conclude that (T,S) is in $\mathscr{T}'_{01,02}$.
Suppose now that $p(T) = k \geq 1$ and for each labeled tree $(H,S_H) \in \mathscr{T}_{01,02}$ with $p(H) < k$ is fulfilled $(H,S_H) \in \mathscr{T}'_{01,02}$. Then there is a cut-vertex, say z, such that (a) $z \in S_A(T)$, (b) (T,S) is a coalescence of 2 graphs, say $(T',S|_{T'})$ and $(T'',S|_{T''})$, via z, and (c) no vertex in $S_A(T) \cap V(T'')$ is a cut-vertex of T''. Hence $(T',S|_{T'}) \in \mathscr{T}'_{01,02}$ (by induction hypothesis) and $(T'',S|_{T''})$ is either a labeled 2-corona tree or H_2. Thus (T,S) is in $\mathscr{T}'_{01,02}$. $\square$

6.2. $\mathcal{R}_{02,012}$-trees.

Our aim in this section is to present a characterization of $\mathcal{R}_{02,012}$-trees. For this purpose, we need the following definitions. Let $\mathscr{T}_{02,012} \subset \mathscr{T}$ be such that $(T,S) \in \mathscr{T}_{02,012}$ if and only if (T,S) can be obtained from a sequence of labeled trees $\tau : (T^1,S^1),\dots,(T^j,S^j)$, $(j \geq 1)$, such that $(T^1,S^1) = (F_3,J^3)\}$ (see Figure 2) and $(T,S) = (T^j,S^j)$, and, if $j \geq 2$, (T^{i+1},S^{i+1}) can be obtained recursively from (T^i,S^i) by one of the operations O_9 and O_{10} listed below.

Operation O_9. The labeled tree (T^{i+1}, S^{i+1}) is obtained from (T^i, S^i) and (F_3, J^3) by adding the edge ux, where $u \in V(T^i)$, $x \in V(F_3)$ and $sta_{T^i}(u) = sta_{F_3}(x) = C$.

Operation O_{10}. The labeled tree (T^{i+1}, S^{i+1}) is obtained from (T^i, S^i) and (F_4, J^4) (see Figure 2) by adding the edge ux, where $u \in V(T^i)$, $x \in V(F_4)$, $sta_{T^i}(u) = D$, and $sta_{F_4}(x) = C$.

Note that once a vertex is assigned a status, this status remains unchanged as the labeled tree (T, S) is recursively constructed. By the above definitions we see that if $(T, S) \in \mathcal{R}_{01,02}$, then $S_A(T) = S_B(T) = \emptyset$. Therefore it is naturally to consider a labeling S as $S : V(T) \to \{C, D\}$.
From Theorem 1 we immediately obtain the following result.

Corollary 7. *A tree T is in $\mathcal{R}_{02,012}$ if and only if there is a labeling $S : V(T) \to \{C, D\}$ such that (T, S) is in $\mathscr{T}_{02,012}$. Moreover, if $(T, S) \in \mathscr{T}_{02,012}$ then $S_C(T) = V^{02}(T)$ and $S_D(T) = V^{012}(T)$.*

As an immediate consequence of Corollary 1 we obtain:

Corollary 8. *If $(T, S_1), (T, S_2) \in \mathscr{T}_{02,012}$ then $S_1 \equiv S_2$.*

Theorem 3. *[3] If G is a connected graph of order $n \geq 3$, then $\gamma_R(G) \leq 4n/5$. The equality holds if and only if G is C_5 or is obtained from $\frac{n}{5}P_5$ by adding a connected subgraph on the set of centers of the components of $\frac{n}{5}P_5$.*

As a consequence of Theorem 3 and Corollary 7 we have:

Corollary 9. *Let G be a connected n-vertex graph with $n \geq 6$ and $\gamma_R(G) = 4n/5$. Then G is in $\mathcal{R}_{02,012}$ and $V^{012}(G)$ consists of all leaves and all support vertices. Moreover, if G is a tree, then G has a $\mathscr{T}$-sequence $\tau : (G^1, S^1), \ldots, (G^j, S^j)$, $(j \geq 1)$, such that $(G^1, S^1) = (F_3, J^3)$ (see Figure 2) and if $j \geq 2$, then (G^{i+1}, S^{i+1}) can be obtained recursively from (G^i, S^i) by operation O_9.*

A graph G is said to be in class UVR if $\gamma(G - v) = \gamma(G)$ for each $v \in V(G)$. Constructive characterizations of trees belonging to UVR are given in [14] by Samodivkin, and independently in [11] by Haynes and Henning. We need the following result in [14] (reformulated in our present terminology).

Theorem 4. *[14] A tree T of order at least 5 is in UVR if and only if there is a labeling $S : V(T) \to \{C, D\}$ such that (T, S) is in $\mathscr{T}_{02,012}$. Moreover, if $(T, S) \in \mathscr{T}_{02,012}$ then $S_C(T)$ and $S_D(T)$ are the sets of all γ-bad and all γ-good vertices of T, respectively.*

We end with our main result in this subsection.

Theorem 5. *For any tree T the following are equivalent:*

(A_4) *T is in $\mathcal{R}_{02,012}$,* (A_5) *T is in $\mathscr{T}_{02,012}$,* (A_6) *T is in UVR.*

Proof. Corollary 7 and Theorem 4 together imply the required result. $\square$

7. Open problems and questions

We conclude the paper by listing some interesting problems and directions for further research. Let first note that if $n \geq 3$ and $\mathtt{G}_{n,k}$ is not empty, then $k \leq 4n/5$ (Theorem 3).

An element of $\mathbb{RE}_{n,k}$ is said to be *isolated*, whenever it is both maximal and minimal. In other words, a graph $H \in \mathtt{G}_{n,k}$ is isolated in $\mathbb{RE}_{n,k}$ if and only if $H \in \mathcal{R}_{CEA}$ and for each $e \in E(H)$ at least one of the following holds: (a) $H - e$ is not connected, (b) $\gamma_R(H) \neq \gamma_R(H-e)$, (c) $H - e$ is not γ_R-excellent.

Example 1. (i) All γ_R-excellent graphs with the Roman domination number equals to 2 are $\overline{K_2}$ and K_n, $n \geq 2$. If a graph $G \in \mathcal{R}_{CEA}$ and $\gamma_R(G) = 2$, then G is complete. K_n is isolated in $\mathbb{RE}_{n,2}$, $n \geq 2$.

(ii) [8] K_2, H_7 and H_8 (see Fig. 1) are the only trees in $\mathcal{R}_{CEA}$.

(iii) If $\mathbb{RE}_{n,k}$ has a tree T as an isolated element, then either $(n,k) = (2,2)$ and $T = K_2$, or $(n,k) = (9,7)$ and $T = H_7$, or $(n,k) = (10,8)$ and $T = H_8$.

- Find results on the isolated elements of $\mathbb{RE}_{n,k}$.
- What is the maximum number of edges $m(\mathtt{G}_{n,k})$ of a graph in $\mathtt{G}_{n,k}$? Note that (a) $m(\mathtt{G}_{n,2}) = n(n-1)/2$, (b) $m(\mathtt{G}_{n,3}) = n(n-1)/2 - \lceil n/2 \rceil$.
- Find results on those minimal elements of $\mathbb{RE}_{n,k}$ that are not trees.

Example 2. (a) A cycle C_n is a minimal element of $\mathbb{RE}_{n,k}$ if and only if $n \equiv 0 \pmod 3$ and $k = 2n/3$. (b) A graph G obtained from the complete bipartite graph $K_{p,q}$, $p \geq q \geq 3$, by deleting an edge is a minimal element of $\mathbb{RE}_{p+q,4}$.

The height of a poset is the maximal number of elements of a chain.

- Find the height of $\mathbb{RE}_{n,k}$.

Example 3. (a) It is easy to check that any longest chain in $\mathbb{RE}_{6,4}$ has as the first element H_3 (see Fig 1) and as the last element one of the two 3-regular 6-vertex graphs. Therefore the height of $\mathbb{RE}_{6,4}$ is 5.

(b) Let us consider the poset $\mathbb{RE}_{5r,4r}$, $r \geq 2$. All its minimal elements are γ_R-excellent trees (by Theorem 3 and Corollary 9), which are characterized in Corollary 9. Moreover, the graph obtained from rP_5 by adding a complete graph on the set of centers of the components of rP_5 is the largest element of $\mathbb{RE}_{5r,4r}$. Therefore the height of $\mathbb{RE}_{5r,4r}$ is $(r-1)(r-2)/2+1$.

- Find results on $\gamma_{\mathtt{Y}R}$-excellent graphs at least when $\mathtt{Y}$ is one of $\{-1,0,1\}$, $\{-1,1\}$ and $\{-1,1,2\}$.

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