Communications in Combinatorics and Optimization
Vol. 2 No. 2, 2017 pp.139-147
DOI: 10.22049/CCO.2017.25850.1047

Approximate solutions for time-varying shortest path problem

Gholamhassan Shirdel[1*] **and Hassan Rezapour**[1]

[1]Department of Mathematics, University of Qom, Qom, Iran
Shirdel81math@gmail.com, Hassan.Rezapour@gmail.com

Received: 3 January 2017; Accepted: 6 August 2017
Published Online: 11 August 2017

Communicated by ***Alireza Ghaffari-Hadigheh***

Abstract: Time-varying network optimization problem, which is NP-complete in the ordinary sense, are traditionally solved by specialized algorithms. This paper considers the time-varying shortest path problem, which can be optimally solved in $O\big(T(m+n)\big)$ time, where T is a given integer. For this problem with arbitrary waiting times, we propose an approximate algorithm, which can find an acceptable solution of the problem with $O\big(\frac{T(m+n)}{k}\big)$ time complexity such that it evaluates only a subset of the values for $t \in \{0, 1, \ldots, T\}$.

Keywords: Approximate solutions, time-varying optimization, network flows

AMS Subject classification: 90B10, 90C35

1. Introduction

Time-varying shortest path problem has in important applications, such as telecommunication, computer networks and transportation. This problem finds shortest path from a source vertex to a target vertex, such that the total cost of the path is minimized. Moreover, the total time of the path is at most T, where T is a given integer time horizon. The transit times, the transit costs along each arc and all other parameters of the network are considered as functions of the departure time along the arcs. This problem is called "time-varying shortest path" (TV-SP) problem. The waiting times at all vertices are the decision variables, then the TV-SP problem can be categorized as problems with arbitrary waiting times (TV-SP-AW), zero waiting times (TV-SP-ZW) and bounded waiting times (TV-SP-BW).

* *Corresponding Author*

The TV-SP problem was first proposed in discrete time by Cooke and Halsey [6]. They considered the transit time $b(i,j,t)$ and the transit cost $c(i,j,t)$ are needed to traverse the arc (i,j) with departure time t from vertex i. They proposed a discrete model, where $b(i,j,t) = c(i,j,t)$, with no waiting time at a vertex, i.e., the TV-SP-ZW problem.
The TV-SP problem has been widely studied by optimization researchers. Some notable works between 1966 and 2000 are [2, 3, 5–8, 10–12, 14–16, 21, 22]. Recently, the time-varying optimization problems have been studied in both continuous and discrete models by Ahuja et al. [1] and Cai et al. [4]. The TV-SP problem has been surveyed in their books [1, 4]. S.M Hashemi et al. [9] presented the shortest path problem from one node to all other nodes in a network, where the arc costs can vary with time. R. Koch and E. Nasrabadi [13] studied the shortest path problems with negative transit time in continuous model. They extend the work of Philpott [15] to the cost of negative transit time. Shirdel and Rezapour [16] studied a k-objective time-varying shortest path problem, which cannot be combined into a single aggregate objective. They considered the minimum cost flow problem in dynamic networks with uncertain costs and proposed a robust formulation for their problem [18]. The time-varying shortest path problem with the fuzzy transit costs and the speed-up on this problem was studied by Shirdel and Rezapour [17, 20]. Moreover, they presented the maximum capacity path problem in the time-varying network, where waiting time at a vertex is not allowed. [19].
In this paper, the TV-SP problem is considered, where Cai et al. [4] studied it in three situations TV-SP-AW, TV-SP-ZW and TV-SP-BW. Some important results of TV-SP-AW [4] are reviewed in section 2. The main contributions of this paper appear in sections 3, where we want to find approximate solutions of TV-SP-AW. The time complexity of the algorithm is dependent on the parameter T, thus the key idea is to calculate only a suitable subset of the values $t \in \{0, 1, \ldots, T\}$. In this section, we propose an approximate algorithm for TV-SP-AW that can be extended to TV-SP-ZW and TV-SP-BW in a similar way . In section 4, a numerical example is solved by the proposed algorithm.

2. Preliminary

Consider a time-varying network $G(V, A, b, c)$, where the sets of vertices and arcs are shown by V and A, respectively with a single source s. Let $b(i,j,t)$ and $c(i,j,t)$ be the transit time and the transit cost of an arc $(i,j) \in A$, respectively, where t is the departure time of vertex i. Both of b and c are dependent on the time $t = 0, 1, \ldots, T$ and T is a given integer time horizon. Maximum waiting time at vertex i for the period of the time t to $t+1$ is denoted by $w(i,t)$. Moreover, waiting cost $c(i,t)$ must be paid when waiting takes place at vertex i during the time period from t to $t+1$. Meanwhile, suppose that the waiting time at vertex i, $w(i)$, the waiting cost at vertex i from t to $t+1$, $c(i,t)$ and the transit costs $c(i,j,t)$ are integers, whereas the transit time $b(i,j,t)$ is a non-negative integer. Finally, let $|A| = m$ and $|V| = n$.

In the following, without loss of generality, we can assume a time-varying network $G(V, A, b, c)$ contains neither loops nor parallel arcs. Otherwise, let network $G(V, A, b, c)$ have a loop (i, i) with the transit time and the transit cost $b(i, i, t)$ and $c(i, i, t)$, respectively. Delete the loop (i, i) and introduce a new node i'. Then, insert two new arcs (i, i') and (i', i) and let: $b(i, i', t) = 0, c(i, i', t) = 0, b(i', i, t) = b(i, i, t)$ and $c(i', i, t) = c(i, i, t)$ for any time t. Similarly, assume the network has two parallel arcs (i, j) with $b_1(i, j, t), c_1(i, j, t), b_2(i, j, t)$ and $c_2(i, j, t)$. Remove one of them associated with $b_1(i, j, t)$ and $c_1(i, j, t)$, insert a new node i'', two new arcs (i, i'') and (i'', j) then set $b(i, i'', t) = 0, c(i, i'', t) = 0, b(i'', j, t) = b_1(i, j, t)$ and $c(i'', j, t) = c_1(i, j, t)$.

Definition 1. Let $P(i_1 - i_2 - \ldots - i_k)$ be a path from i_1 to i_k. The arrival time of a vertex i_r on P is denoted by $\alpha(i_r)$ such that $\alpha(i_1) = t_0 \geq 0$ (for the source vertex s, let $\alpha(S) = 0$), then we have:

$$\alpha(i_r) = \alpha(i_{r-1}) + w(i_{r-1}) + b\big(i_{r-1}, i_r, \tau(i_{r-1})\big) \qquad \text{for} \qquad r = 2, \ldots, k,$$

where $\tau(i_r)$ (departure time of a vertex i_r on P) is defined as follows:

$$\tau(i_r) = \alpha(i_r) + w(i_r) \qquad for \qquad r = 1, \ldots, k-1.$$

Definition 2. $P(i_1 - i_2 - \ldots - i_k)$ is said to be a dynamic path from i_1 to i_k, if all the $\alpha(i_r), w(i_r)$ and the $\tau(i_r)$ for all $1 \leq r \leq k$ on the path are specified. Furthermore, the time of P is defined as $\alpha(i_k) + w(i_k) - \alpha(i_1)$. A path is said to have time at most t, if its time is less than or equal to t. Specifically, a path is said to have time t, if its time equals t.

Definition 3. [4] Let $P(i_1 - i_2 - \ldots - i_k)$ be a dynamic path from i_1 to i_k. Let $\xi(i_1) = \sum_{t'=0}^{w(i_1)-1} c\big(i_1, t' + \alpha(i_1)\big)$ and define recursively:

$$\xi(i_r) = \xi(i_{r-1}) + c\big(i_{r-1}, i_r, \tau(i_{r-1})\big) + \sum_{t'=0}^{w(i_r)} c\big(i_r, t' + \alpha(i_r)\big),$$

for $r = 2, \ldots, k$. The cost (or length) of P, $\xi(P)$, is defined as $\xi(i_k)$.

Waiting times at vertices are decision variables in the time-varying shortest path problem. They can be arbitrary, zero or it can have an upper bound. The time-varying shortest path problem with arbitrary waiting times (TV-SP-AWT) is considered in this paper.

The cost of the TV-SP with arbitrary waiting time can be computed using the recursive formula provided in the following lemma. Consider $d_a(j, t)$ shows the cost of shortest path from source vertex s to vertex j of time exactly t, where waiting times at vertices are arbitrary.

Lemma 1. *[4] $d_a(s, 0) = 0$ and $d_a(j, 0) = \infty$ for all $j \neq S$. For $t > 0$, we have:*

$$d_a(j, t) = \min\Big\{d_a(j, t-1) + c(j, t-1), \min_{(i,j)\in A} \min_{u|u+b(i,j,u)=t} \big\{d_A(i, u) + c(i, j, u)\big\}\Big\}$$

Lemma 2. *[4] The optimal solution of TV-SP-AWT can be found in $O\big(T(m+n)\big)$ time.*

3. Finding approximate solutions for TV-SP

By finding approximate solutions, the time complexity of the algorithms could be improved as shown in this section. The time complexity of the algorithms depend on the parameter T, thus, the vital idea is to calculate only a suitable subset of the values $t \in \{0, 1, \ldots, T\}$. In the following, It is assumed that waiting at any vertex is not bounded and all waiting costs are equal to zero i.e. $w(i) = \infty$ and $c(i,t) = 0$ for all $i \in V$ and all time t. The approach is valid for others versions TV-SP problems with zero or bounded waiting times, too.
For a given network $N(V, A, b, c)$, a new problem is presented for finding the approximate solutions for TV-SP problem. This problem is the same as the TV-SP original problem with $t = 0, k, 2k, \ldots, k.\lfloor \frac{T}{k} \rfloor$ and $b'(i,j,t) = k.\lceil \frac{b(i,j,t)}{k} \rceil$ for $t = 0, k, 2k, \ldots, k.\lfloor \frac{T}{k} \rfloor$. An optimal solution P^0 for the TV-SP problem with approximate solutions (TV-SP-AWT*) can be found in time complexity $O\Big(\frac{T(m+n)}{k}\Big)$, which can be made sufficiently small if k is large.
The following lemma shows that how to ensure the solution P^0 be a suitable approximate solution when it is applied to the original problem. TV-SP-AWT* finds approximate solutions for TV-SP-AW problem.

Lemma 3. *If TV-SP-AWT* finds a solution P^0, then there is a path P in the original network N such that $d(P^0) = d(P)$.*

Proof. Let the approximate solution of TV-SP-AWT*, $P^0(s = x_0 - x_1 - \ldots - x_k)$ be a path from x_0 to x_k, where $\alpha^0(x_i)$, $\tau^0(x_i)$ and $w^0(x_i)$ are arrival time, departure time and waiting time at vertex (x_i) for $0 \leq i \leq k$, respectively. A path P can be constructed with the same topological structure as P^0 and let $\tau(x_i) = \tau^0(x_i)$ such that $0 \leq i \leq k$, for path P. Furthermore, let $\alpha(x_i) = \tau(x_{i-1}) + b\big(x_{i-1}, x_i, \tau(x_{i-1})\big)$. Consequently, $w(x_i) = \alpha^0(x_i) - \alpha(x_i) + w^0(x_i)$. Note that $\alpha^0(x_i) \geq \alpha(x_i)$, thus, the path P with all $\alpha(x_i)$, $\tau(x_i)$ and $w(x_i)$ is a feasible dynamic path in the original network, which can be traversed in the time T, Since $\tau(x_i) = \tau^0(x_i)$ for all $i \in V$, therefore $d(P^0) = d(P)$. □

Let $d'_a(j,t)$ be the cost of a shortest path from s to j of time exactly t for the problem TV-SP-AW*, where waiting at any vertex is not restricted. The following algorithm finds the approximate solutions of the problem.

Algorithm TV-SP-AW*
Begin
Initialize $d'_a(j,t) := 0$ and $d'_a(j,0) := \infty$ for $t = 0, k, 2k, \ldots, k.\lfloor \frac{T}{k} \rfloor$.
Let $b'(i,j,t) := k.\lceil \frac{b(i,j,t)}{k} \rceil$ for $t = 0, k, 2k, \ldots, k.\lfloor \frac{T}{k} \rfloor$ and each $(i,j) \in A$.
Sort all values $u + b'(i,j,u)$ for all $u = 0, k, 2k, \ldots, k.\lfloor \frac{T}{k} \rfloor$, $\forall (i,j) \in A$.
For $t = 0, k, 2k, \ldots, k.\lfloor \frac{T}{k} \rfloor$
For each $j \in V \setminus s$ do

$$d'_a(j,t) := \min_{(i,j) \in A} \min_{u | u + b'(i,j,u) = t} \Big\{ d'_a(i, t-k), \big\{ d'_a(i,u) + c(i,j,u) \big\} \Big\}$$

For every vertex j do:

$$d'_a(j) = \min_{0 \le k \le k\lfloor \frac{t}{k} \rfloor} d'_a(j,t)$$

End

Let $\delta_{min} = \min\{c(i,j,t)|(i,j) \in A, t \in T\}$ and $\delta_{max} = \max\{c(i,j,t)|(i,j) \in A, t \in T\}$. Further, let $\Delta = \delta_{max} - \delta_{min}$, moreover, the minimum arc costs are denoted by C_{min}. If the error of the solution P^0 is defined by E, then we have:

$$E = \frac{d(P^0) - d(P^*)}{d(P^*)}$$

where, P^* and P^0 are the optimal solutions of TV-SP-AWT and TV-SP-AWT*, respectively. The following theorem indicates that P^0 is a suitable approximate solution for TV-SP-AWT.

Theorem 1. *The problem TV-SP-AWT* can find an approximate optimal solution P^0, where $E \le \varepsilon$.*

Proof. Let P^* and P^0 are optimal solutions of TV-SP-AWT and TV-SP-AWT*, respectively. Moreover, suppose that P' is the feasible solution, which has the same topological structure as P^0 with the departure time at the beginning vertex $t = 0, k, 2k, \ldots, k.\lfloor \frac{T}{k} \rfloor$, then :

$$d(P') \ge d(P^0) \ge d(P^*)$$

therefore,

$$E = \frac{d(P^0) - d(P^*)}{d(P^*)} \le \frac{d(P') - d(P^*)}{d(P^*)} \le \frac{nk\Delta}{d(P^*)} \le \frac{nk\Delta}{C_{min}}$$

if $k := [\frac{C_{min}\varepsilon}{n\Delta}]$ then $E \le \varepsilon$, i.e. the solution P^0 is the suitable approximate optimal solution for TV-SP-AWT, where $E \le \varepsilon$. □

Theorem 2. *Algorithm TV-SP-AWT* can be implemented in $O\big(\frac{T(m+n)}{k}\big)$.*

Proof. The initialization can be done in $O\big(\frac{Tn}{k}\big)$. The step 2 can be performed in $O\big(\frac{Tm}{k}\big)$ time for mentioned values sorting. The values $d'_a(j,t)$ are calculated in $O\big(\frac{Tm}{k}\big)$. Finally, it follows that the complexity of algorithm is bounded by $O\big(\frac{T(m+n)}{k}\big)$. □

4. Numerical Example

In this section, an example is examined to illustrate the Algorithm TV-SP-AW* and how to obtain approximate solutions of TV-SP-AW problem. Consider a time-varying network N as shown in Fig. 1, where if $M = \Big\{(1,2),(2,4),(4,7),(7,10)\Big\}$, let

$$\big(c(i,j,t), b(i,j,t)\big) = (92, 2) \text{ for } (i,j) \in M \text{ and } t \in \{0, 1, \ldots, 12\}.$$

Moreover, if $N = \Big\{(1,3),(3,6),(6,9)\Big\}$, then

$$\big(c(i,j,t), b(i,j,t)\big) = (95,1) \text{ for } (i,j) \in N \text{ and } t \in \{0,1,\ldots,12\}.$$

All other b and c are given in Table 1.

Table 1. The transit time $b(i,j,t)$ and the transit cost $c(i,j,t)$.

t	$(2,5)$	$(3,5)$	$(4,8)$	$(5,8)$	$(6,8)$	$(7,8)$	$(8,10)$	$(9,8)$	$(9,10)$
0	2,93	2,93	1,94	2,95	3,95	2,92	3,98	2,100	3,90
1	2,93	4,96	2,96	3,98	1,92	1,95	4,99	1,98	3,100
2	3,91	3,94	1,94	1,95	2,97	3,93	2,92	2,91	2,90
3	2,91	2,95	2,98	2,92	3,98	4,94	1,93	1,92	1,92
4	2,95	2,92	1,99	1,100	1,95	1,94	1,95	1,93	3,93
5	1,97	1,93	3,100	2,93	2,97	2,95	2,96	2,93	2,94
6	2,94	1,93	1,100	1,100	2,99	3,97	1,98	3,92	1,95
7	1,94	2,94	4,95	1,93	1,90	1,90	2,95	1,94	1,99
8	1,95	3,90	2,92	2,98	2,100	1,90	2,95	2,95	2,98
9	2,96	4,91	2,94	1,98	4,97	1,98	3,98	1,96	3,97
10	1,90	6,98	6,99	3,94	5,98	2,99	4,100	1,90	1,98
11	4,90	5,100	5,98	1,90	6,99	3,100	1,97	3,100	2,100
12	5,90	3,100	4,95	5,98	1,100	4,100	6,95	2,100	4,98

Table 2. Calculation of shortest path.

t	1	2	3	4	5	6	7	8	9	10
0	0	∞	∞	∞	∞	∞	∞	∞	∞	∞
1	0	92	95	∞	∞	∞	∞	∞	∞	∞
2	0	92	95	∞	∞	190	∞	∞	∞	∞
3	0	92	95	∞	∞	190	∞	∞	285	∞
4	0	92	95	182	∞	190	∞	277	285	377
5	0	92	95	182	183	190	272	277	285	372
6	0	92	95	182	183	190	272	277	285	372
7	0	92	95	182	183	190	272	277	285	372
8	0	92	95	182	183	190	272	276	285	362
9	0	92	95	182	183	190	272	276	285	362
10	0	92	95	182	5	190	272	276	285	362
11	0	92	95	182	182	190	272	276	285	362
12	0	92	95	182	182	190	272	276	285	362
$d_a^*(j)$	0	92	95	182	182	190	272	272	285	362

The results in Table 2 can be obtained by applying Lemma 1. This table shows the cost of the shortest paths problem connecting the source vertex to other vertices.

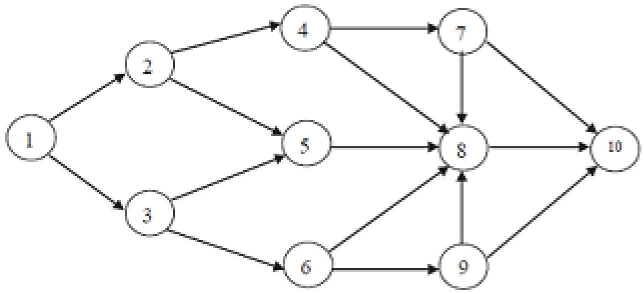

Fig 1. A numerical example

The approximate solutions are found using the algorithm TV-SP-AWT*, where $\varepsilon = 4$. In this example $\Delta = 10$, $C_{min} = 90$ and $n = 10$, therefore $k = 3$ and $b'(i,j,t) = 3\lceil \frac{b(i,j,t)}{3} \rceil$. Hence, if $M = \Big\{(1,2),(2,4),(4,7),(7,10)\Big\}$ then

$$\big(c(i,j,t), b'(i,j,t)\big) = (92,3) \text{ for } (i,j) \in M \text{ and } t \in \{0,3,6,9,12\}.$$

Moreover, if $N = \Big\{(1,3),(3,6),(6,9)\Big\}$ then

$$\big(c(i,j,t), b'(i,j,t)\big) = (95,3) \text{ for } (i,j) \in N \text{ and } t \in \{0,3,6,9,12\}.$$

Other transit times $b'(i,j,t)$ and transit costs $c(i,j,t)$ are listed in Table 3.

Table 3. The transit time $b'(i,j,t)$ and the transit cost $c(i,j,t)$.

t	(2,5)	(3,5)	(4,8)	(5,8)	(6,8)	(7,8)	(8,10)	(9,8)	(9,10)
0	3,93	3,93	3,94	3,95	3,95	3,92	3,98	3,100	3,90
3	3,91	3,95	3,98	3,92	3,98	6,94	3,93	3,92	3,92
6	3,94	3,93	3,100	3,100	3,99	3,97	3,98	3,92	3,95
9	3,96	6,91	3,94	3,98	6,97	3,98	3,98	3,98	3,97
12	6,90	3,100	6,95	6,98	3,100	6,100	6,95	3,100	6,98

Approximate solutions are reported in Table 4, by applying algorithm TV-SP-AWT*.

Table 4. Calculation of approximate solutions of shortest path.

t	1	2	3	4	5	6	7	8	9	10
$d_a'^*(j)$	0	92	95	182	182	190	272	273	285	362

We can compare each $d_a'^*(j)$ in Table 4 with $d_a^*(j)$ in the last row of Table 2. It contains one difference at vertex $j = 8$, where $d_a'^*(8) = 273$ while $d_a^*(8) = 272$, then we have: $E = \frac{1}{273} < 4$. Therefore, there is an approximate solution in Table 4 with an error bound 4, while the computation of the procedure was reduced.

5. Conclusion

In this paper we studied the time-varying shortest path problem with arbitrary waiting times at vertices, where each arc has a transit time. In this problem, the cost for traversing an arc as well as the cost for waiting at a vertex vary over time. TV-SP problem is a NP-complement with $O\big(T(m+n)\big)$ time complexity. Our attention was restricted to find approximate solutions of TV-SP problem. An algorithm with time complexity $O\big(\frac{T(m+n)}{k}\big)$ was proposed and a numerical example was solved, i.e we showed that the time complexity of the approximate solutions TV-SP-AW is reducible from $O\big(T(m+n)\big)$ to $O\big(\frac{T(m+n)}{k}\big)$. Finding approximate solutions of TV-SP problem may appear as a subproblem in the process of solving other time-varying network optimization problems and plays an important role in time-varying optimization problems.

Acknowledgements

The authors wish to thank the anonymous referees and the editors for a very careful reading of our manuscript and their useful comments.

References

[1] R.K. Ahuja, J.B. Orlin, S. Pallottino, and M.G. Scutellà, *Dynamic shortest paths minimizing travel times and costs*, Networks **41** (2003), no. 4, 197–205.

[2] Y. Aneja and K. Nair, *The constrained shortest path problem*, Naval Res. Logist. **25** (1978), no. 3, 549–555.

[3] X. Cai, T. Kloks, and C.K. Wong, *Time-varying shortest path problems with constraints*, Networks **29** (1997), no. 3, 141–149.

[4] X. Cai, D. Sha, and C.K. Wong, *Time-varying network optimization*, Springer New York, 2007.

[5] I. Chabini, *Discrete dynamic shortest path problems in transportation applications: complexity and algorithms with optimal run time*, Transportation Research Record **1645** (1998), 170–175.

[6] L. Cooke and E. Halsey, *Shortest route through a network with time-dependent internodal transit times*, J. Math. Anal. Appl. **14** (1966), no. 3, 492–498.

[7] J. Halpern, *Shortest route with time-dependent length of edges and limited delay possibilities in nodes*, Math. Methods Oper. Res. **21** (1977), no. 3, 117–124.

[8] G.Y. Handler and I. Zang, *A dual algorithm for the constrained shortest path proglem*, Networks **10** (1980), no. 4, 293–309.

[9] S.M. Hashemi, Sh. Mokarami, and E. Nasrabadi, *Dynamic shortest path problems with time-varying costs*, Optim. Lett. **4** (2010), no. 1, 147–156.

[10] M.M.D. Hassan, *Network reduction for the acyclic constrained shortest path problem*, European J. Oper. Res. **63**, no. 1.

[11] I. Ioachim, S. Gelinas, F. Soumis, and J. Desrosiers, *A dynamic programming algorithm for the shortest path problem with time windows and linear node costs*, Networks **31** (1998), no. 3, 193–204.

[12] H.C. Joksch, *The shortest route probelm with constraints*, J. Math. Anal. Appl. **14** (1966), no. 2, 191–197.

[13] R. Koch and E. Nasrabadi, *Continuos-time dynamic shortest path problems with negative transit times*, SIAM J. Control Optim. **52** (2014), no. 4, 2449–2481.

[14] A. Orda and R. Rom, *Shortest-path and minimum-delay algorithms in networks with time-dependent edge length*, Journal of the ACM **37** (1990), no. 3, 607–625.

[15] A.B. Philpott, *Continuous-time shortest path problems and linear programming*, SIAM J. Control Optim. **32** (1994), no. 2, 538–552.

[16] H. Rezapour and Gh. Shirdel, *K-objective time-varying shortest path problem with zero waiting times at vertices*, Trends in Applied Sciences Research **10** (2015), no. 5, 278–285.

[17] ———, *The arbitrary waiting times in the time-aarying shortest path with triangular fuzzy costs*, Adv. Appl. Math. Sci. **15** (2016), no. 2, 75–85.

[18] ———, *Dynamic network flows with uncertain costs belonging to interval*, Appl. Appl. Math. **11** (2016), no. 1, 285 – 299.

[19] ———, *Time-varying maximum capacity path problem with zero waiting times and fuzzy capacities*, SpringerPlus **5** (2016), 981–990.

[20] ———, *The time-varying shortest path problem with fuzzy transit costs and speedup*, Acta Univ. Sapientiae Math. **8** (2016), no. 1, 166–173.

[21] C.C. Skiscim and B.L. Golden, *Solving k-shortest and constrained shortest path problems efficiently*, Ann. Oper. Res. **20** (1989), no. 1, 249–282.

[22] A. Ziliaskopoulos and H. Mahmassani, *A time-dependent shortest path algorithm for real-time intelligent vehicle highway system applications*, Transportation Research Record **1408** (1993), 94–104.