On the signed Roman edge $k$-domination in graphs

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Abstract: Let $k \geq 1$ be an integer, and $G = (V, E)$ be a finite and simple graph. The closed neighborhood $N_G[e]$ of an edge $e$ in a graph $G$ is the set consisting of $e$ and all edges having a common end-vertex with $e$. A signed Roman edge $k$-dominating function (SREkDF) on a graph $G$ is a function $f : E \rightarrow \{-1, 1, 2\}$ satisfying the conditions that (i) for every edge $e$ of $G$, $\sum_{x \in N[e]} f(x) \geq k$ and (ii) every edge $e$ for which $f(e) = -1$ is adjacent to at least one edge $e'$ for which $f(e') = 2$. The minimum of the values $\sum_{e \in E} f(e)$, taken over all signed Roman edge $k$-dominating functions $f$ of $G$, is called the signed Roman edge $k$-domination number of $G$ and is denoted by $\gamma'_{sRk}(G)$. In this paper we establish some new bounds on the signed Roman edge $k$-domination number.

Keywords: Signed Roman edge $k$-dominating function, Signed Roman edge $k$-domination number

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1. Introduction

In this paper, $G$ is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order and size of a graph $G$ are denoted by $n = n(G)$ and $m = m(G)$, respectively. For every vertex $v \in V$, the open neighborhood $N_G(v)$ is the set $\{u \in V \mid uv \in E\}$ and the closed neighborhood of $v$ is the set $N[v] = N(v) \cup \{v\}$. The degree of a vertex $v \in V$ is $d_G(v) = d(v) = |N(v)|$. The minimum and maximum degree of a graph $G$ are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The complement of a graph $G$ is denoted by $\bar{G}$. We write $K_n$ for the complete graph and $C_n$ for a cycle of order $n$. Two edges $e_1$ and $e_2$ of $G$ is said to be adjacent if there exists a vertex $v \in V(G)$ to which they are both incident. The line graph of a graph $G$, written $L(G)$, is the graph with the edges of $G$ as its vertices, and $ee' \in E(L(G))$ when $e$ and $e'$ are adjacent in

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The open neighborhood $N(e) = N_G(e)$ of an edge $e \in E(G)$ is the set of all edges adjacent to $e$. Its closed neighborhood is $N[e] = N_G[e] = N_G(e) \cup \{e\}$. If $v$ is a vertex, then denote by $E(v)$ the set of edges incident with the vertex $v$. For a function $f : E(G) \rightarrow \{-1, 1, 2\}$ and a subset $S$ of $E(G)$, we define $f(S) = \sum_{e \in S} f(e)$. Also for every vertex $v$, we define $f(v) = \sum_{e \in E(v)} f(e)$.

A signed Roman $k$-dominating function (SR$k$DF) on a graph $G$ is a function $f : V \rightarrow \{-1, 1, 2\}$ satisfying the conditions that (i) $\sum_{x \in N[v]} f(x) \geq k$ for each vertex $v \in V$, and (ii) every vertex $u$ for which $f(u) = -1$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The weight of an SR$k$DF is the sum of its function values over all edges. The signed Roman $k$-domination number of $G$, denoted $\gamma_{SR}^k(G)$, is the minimum weight of an SR$k$DF in $G$. The signed Roman $k$-domination number was introduced by Henning and Volkmann in [5] and has been studied in [4]. The special case $k = 1$ was introduced and investigated in [2].

A signed Roman edge $k$-dominating function (SRE$k$DF) on a graph $G$ is a function $f : E \rightarrow \{-1, 1, 2\}$ satisfying the conditions that (i) for every edge $e$ of $G$, $\sum_{x \in N[e]} f(x) \geq k$ and (ii) every edge $e$ for which $f(e) = -1$ is adjacent with at least one edge $e'$ for which $f(e') = 2$. The weight of an SRE$k$DF is the sum of its function values over all edges. The signed Roman edge $k$-domination number of $G$, denoted $\gamma'_{SR}^k(G)$, is the minimum weight of an SRE$k$DF in $G$. For an edge $e$, we denote $f[e] = f(N[e]) = \sum_{x \in N[e]} f(x)$ for notational convenience. The signed Roman edge $k$-domination number was introduced by Asgharsharghi et al. in [3]. The special case $k = 1$ was introduced by Ahangar et al. [1].

A signed Roman edge $k$-dominating function $f : E \rightarrow \{-1, 1, 2\}$ can be represented by the ordered partition $(E_{-1}, E_1, E_2)$ (or $(E_{-1}^f, E_1^f, E_2^f)$ to refer to $f$) of $E$, where $E_i = \{e \in E \mid f(e) = i\}$. In this representation, its weight is $\omega(f) = |E_1| + 2|E_2| - |E_{-1}|$. A signed Roman edge $k$-dominating function of weight $\gamma'_{SR}^k(G)$ is called $\gamma'_{SR}^k(G)$-function. Let $f$ be an SRE$k$DF of $G$. An edge $e \in E(G)$ is called a $i$ edge if $f(e) = i$ for $i = -1, 1, 2$. The signed Roman edge $k$-domination number exists if $|N_G(e)| \geq \frac{k}{2} - 1$ for every edge $e \in E$. However, for investigations of the signed Roman edge $k$-domination number it is reasonable to claim that for every edge $e \in E$, $|N_G(e)| \geq k - 1$. Thus we assume throughout this paper that $\delta \geq k - 1$. Let $uv$ be an edge of $G$. Then

$$f[uv] = f(u) + f(v) - f(uv) \geq k. \quad (1)$$

In this note we continue the study of the signed Roman edge $k$-domination in graphs and present some (sharp) bounds for this parameter.

We make use of the following results in this paper.

**Proposition A.** [3] For any $r$-regular graph $G$ of size $m$, $(r \geq 1)$, $\gamma'_{SR}^k(G) \geq \frac{km}{2r-1}$.

## 2. Bounds on the signed Roman edge $k$-domination number

In this section we present new bounds on the signed Roman edge $k$-domination in graphs. We begin with a simple observation.
Observation 1. Let $G$ be a graph containing $C_n$ as a subgraph. For any SRE$k$DF $f$ of $G$, $\sum_{v \in V(C_n)} f(v) \geq \frac{(k-1)n}{2}$.

Proof. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of $C_n$. By (1), we have

$$\sum_{i=1}^{n} f(v_i) = \frac{1}{2} \sum_{i=1}^{n} (f(v_i) + f(v_{i+1})) \geq \frac{(k-1)n}{2}$$

where indices are modulo $n$. □

An elementary graph is a graph in which each component is a 1-regular graph or a 2-regular graph.

Theorem 2. For every graph $G$ of order $n$, $\gamma'_{sR_k}(G) \geq \frac{-(n-k+1)^2}{16}$.

Proof. Let $H$ be an elementary subgraph of $G$ with maximum number of vertices. Among this elementary subgraphs choose one such that the number of its $K_2$ components is as large as possible. By the choice of $H$, $H$ has no even cycle. Let $t$ be the number of vertices in $V(G) - V(H)$. If there is an edge $uv$ in $G$ for some $u, v \in V(G) - V(H)$, then by adding the edge $uv$ to $H$ we obtain an elementary subgraph of $G$ with more vertices which is a contradiction. Hence $V(G) - V(H)$ is independent. We claim that for every vertex $v \in V(G) - V(H)$, $d(v) \leq |V(H)| - t = n - t$.

If $V(G) - V(H) = \emptyset$, then there is nothing to prove. So let $V(G) - V(H) \neq \emptyset$ and $v \in V(G) - V(H)$. If $v$ is adjacent to a vertex of an odd cycle $C = (v_1 v_2 \ldots v_k)$ of $H$, say $v_1$, then $H' = (H - E(C)) \cup \{vv_1, v_2v_3, \ldots, v_{k-1}v_k\}$ is an elementary subgraph of $G$ with more vertices than $H$ which is a contradiction. Hence $v$ is not adjacent to any vertex of an odd cycle of $H$. If $v$ is adjacent to each vertex of a $K_2$ component of $H$ such as $uw$, then $H' = H + \{uv, wv\}$ is an elementary subgraph of $G$ with more vertices than $H$, a contradiction again. Thus $v$ is adjacent to at most one vertex of each $K_2$ component of $H$. This implies that $d(v) \leq |V(H)| - t = n - t$. Let $f$ be a $\gamma'_{sR_k}(G)$-function. By (1) and Observation 1, we obtain $\sum_{v \in V(H)} f(v) \geq \frac{(k-1)(n-t)}{2}$. It follows that

$$\gamma'_{sR_k}(G) = \sum_{e \in E(G)} f(e)$$

$$= \frac{1}{2} \left( \sum_{v \in V(H)} f(v) + \sum_{v \in V(G) - V(H)} f(v) \right)$$

$$\geq \frac{(k-1)(n-t)}{4} - \frac{t(n-t)}{4}$$

$$\geq \frac{-(n-k+1)^2}{16},$$

and the proof is complete. □
Corollary 1. If $G$ has a spanning elementary subgraph, then $\gamma_{sRk}(G) \geq \frac{(k-1)n}{4}$.

Theorem 3. Let $k \geq 1$ and let $G$ be a graph with $\gamma_{sRk}(G) < m$. Then

$$\gamma_{sRk}(G) \geq 2 \left\lceil \frac{k-1}{8} + \frac{1}{2} \sqrt{\frac{(k-1)^2}{16} + (k+1)m} \right\rceil + 1 - m.$$

Proof. Let $f = (E_1, E_2)$ be a $\gamma_{sRk}(G)$-function, and let $|E_1| = m_1$ and $|E_2| = m_2$. Then

$$\gamma_{sRk}(G) = 2|E_2| + |E_1| - |E_{-1}| = 3m_2 + 2m_1 - m. \quad (2)$$

Let $G^* = L(G)$ be the line graph of $G$ and $E^*(E_1 \cup E_2, E_{-1}) = \{ e = xy \in E(G^*) \mid x \in E_1 \cup E_2, y \in E_{-1} \}$. Clearly for each $y \in E_{-1}$, we have

$$|N[y] \cap (E_1 \cup E_2)| = |N_{G^*}[y] \cap (E_1 \cup E_2)| \geq \frac{k+1}{2}$$

and hence $|E^*(E_1 \cup E_2, E_{-1})| \geq \frac{k+1}{2}|E_{-1}| = \frac{k+1}{2}(m - (m_1 + m_2))$. So there exists at least one element $x \in E_1 \cup E_2$ such that $x$ is adjacent to at least $\left\lceil \frac{k+1}{2}(m - (m_1 + m_2)) \right\rceil$ elements of $E_{-1}$ in $G$. Since $f$ is an SRE$k$DF of $G$, we have

$$|N[x] \cap (E_1 \cup E_2)| = |N_{G^*}[x] \cap (E_1 \cup E_2)|$$

$$\geq \frac{k}{2} + \frac{1}{2} |N_{G^*}[x] \cap E_{-1}|$$

$$\geq \frac{k}{2} + \frac{1}{2} \left\lceil \frac{k+1}{2}(m - (m_1 + m_2)) \right\rceil.$$

Since $m_1 + m_2 = |E_1| + |E_2| \geq |N[x] \cap (E_1 \cup E_2)|$, we have

$$m_1 + m_2 \geq \frac{k}{2} + \frac{1}{2} \left\lceil \frac{k+1}{2}(m - (m_1 + m_2)) \right\rceil \geq \frac{k}{2} + \frac{k+1}{4} \frac{m - (m_1 + m_2)}{m_1 + m_2}. \quad (3)$$

It follows from (3) that $m_1 + m_2 \geq \frac{k-1}{8} + \frac{1}{2} \sqrt{\frac{(k-1)^2}{16} + (k+1)m}$ and hence $m_1 + m_2 \geq \left\lceil \frac{k-1}{8} + \frac{1}{2} \sqrt{\frac{(k-1)^2}{16} + (k+1)m} \right\rceil$. Since $m_2 \geq 1$, we have

$$2m_1 + 3m_2 \geq 2 \left\lceil \frac{k-1}{8} + \frac{1}{2} \sqrt{\frac{(k-1)^2}{16} + (k+1)m} \right\rceil + 1.$$

Now, the result comes from (2) and the proof is complete.}

The bound in Theorem 3 is sharp for $P_3$ when $k = 1$. The proof of next result can be found in [3].
Next we generalize the bound of Proposition B for general graphs.

**Theorem 4.** Let \( k \geq 2 \) and \( G \) be a connected graph of order \( n \geq 3 \) and size \( m \). Then

\[
\gamma'_{sRk}(G) \geq k(n - m).
\]

**Proof.** Let \( p \) be the number of cycles of \( G \). The proof is by induction on \( p \). The statement is true for \( p = 0 \) by Proposition B. Assume the statement is true for all simple connected graphs \( G \) for which the number of cycles is less than \( p \), where \( p \geq 1 \). Let \( G \) be a simple connected graph with \( p \) cycles. Assume that \( f \) is a \( \gamma'_{sRk}(G) \)-function and let \( e = uv \) be a non-cut edge. If \( f(e) = -1 \), then obviously \( f|_{G-e} \) is an SREkDF for \( G - e \) and by the induction hypothesis, we have

\[
\gamma'_{sRk}(G) = \omega(f) = \omega(f|_{G-e}) - 1 \geq k(n - (m - 1)) - 1 \geq k(n - m).
\]

Thus, we may assume all non-cut edges are assigned 1 or 2 by \( f \). Let \( e = uw \) be a non-cut edge of \( G \) such that \( f(e) \) is as small as possible. First let \( f(u) \leq f(e) \). Then \( u \) has \( f(e) \) neighbors \( w_1, w_{f(e)} \) such that \( f(uw_1) = f(uw_{f(e)}) = -1 \). We deduce from \( f(u) + f(v) - f(uv) \geq k \) and \( f(u) + f(w_i) - f(uw_i) \geq k \) \((i = 1, f(e))\) that \( f(v) \geq k \) and \( \deg(w_1), \deg(w_{f(e)}) \geq 2 \). Let \( G' \) be the graph obtained from \( G \) by removing \( uw_1, uw_{f(e)} \) and \( uv \) and adding a pendant edge \( vv' \) and let \( G_1, G_2, G_{f(e)+1} \) be the components of \( G' \) containing \( w_1, w_{f(e)} \) and \( u \), respectively. Define \( g : E(G') \to \{-1, 1, 2\} \) by \( g(vv') = f(uv) \) and \( g(x) = f(x) \) for \( x \in E(G') - \{vv'\} \). Clearly, \( g \) is an SREkDF for \( G' \) and the function \( g_i = g|_{V(G_i)} \) is an SREkDF of \( G_i \) for \( 1 \leq i \leq f(e) + 1 \). It follows from the induction hypothesis that

\[
\omega(f) = \omega(g) - f(e) = \sum_{i=1}^{f(e)+1} \omega(g_i) - f(e) \geq \sum_{i=1}^{f(e)+1} k(n(G_i) - m(G_i)) - f(e) = k(n(G') - m(G')) - f(e) = k(n + 1 - (m - f(e))) - f(e) > k(n - m).
\]
Hence, we may assume without loss of generality that \( f(u) \geq f(e) + 1 \). Similarly, we may assume that \( f(v) \geq f(e) + 1 \). Here, we consider the following subcases.

**Subcase 1.1.** \( f(u) \leq k \) and \( f(v) \leq k \).

Let \( G' \) be the graph obtained from \( G - \{ e \} \) by adding \( k - f(u) + 1 \) pendant edges \( uu_j \) at \( u \) and \( k - f(v) + 1 \) pendant edges \( vv_i \) at \( v \). Obviously, the function \( g : E(G') \rightarrow \{-1, 1, 2\} \) defined by \( g(uu_1) = g(vv_1) = f(e) \), \( g(uu_j) = 1 \) for \( 2 \leq j \leq k - f(u) + 1 \), \( g(vv_i) = 1 \) for \( 2 \leq i \leq k - f(v) + 1 \) and \( g(a) = f(a) \) otherwise, is an SRE\( k \)DF for \( G' \). By the induction hypothesis and the fact \( f(u) + f(v) - f(uv) \geq k \), we have

\[
\omega(f) = \omega(g) - 2k + f(u) + f(v) - f(uv) \\
\geq k(n(G') - m(G')) - k \\
= k(n - (m - 1)) - k \\
= k(n - m).
\]

**Subcase 1.2.** \( f(u) \leq k \) and \( f(v) \geq k + 1 \). Let \( G' \) be the graph obtained from \( G - \{ e \} \) by adding \( k - f(u) + 1 \) pendant edges \( uu_j \) at \( u \) and one pendant edge \( vv' \) at \( v \). Obviously, the function \( g : E(G') \rightarrow \{-1, 1, 2\} \) defined by \( g(vv') = g(uu_1) = f(uv) \), \( g(uu_i) = 1 \) for \( 2 \leq i \leq k - f(u) + 1 \) and \( g(a) = f(a) \) otherwise is an SRE\( k \)DF for \( G' \). By the induction hypothesis, we have

\[
\omega(f) = \omega(g) - k + f(u) - f(uv) \\
\geq k(n(G') - m(G')) - k + f(u) - f(uv) \\
= k(n + k - f(u) + 2 - (m - 1 + k - f(u) + 2)) - k + f(u) - f(uv) \\
= k(n - m) + f(u) - f(uv) \\
> k(n - m).
\]

**Subcase 1.3.** \( f(u) \geq k + 1 \) and \( f(v) \geq k + 1 \).

Let \( G' \) be the graph obtained from \( G - \{ e \} \) by adding two new pendant edges \( vv' \) and \( uu' \). Define \( g : E(G') \rightarrow \{-1, 1, 2\} \) by \( g(vv') = g(uu') = f(uv) \) and \( g(a) = f(a) \) otherwise. Clearly, \( g \) is an SRE\( k \)DF for \( G' \) and by the induction hypothesis, we obtain

\[
\omega(f) = \omega(g) - f(uv) \geq k(n(G') - m(G')) - f(uv) = k(n+2-(m+1))-f(uv) \geq k(n-m).
\]

This completes the proof. \( \square \)

As an application of Theorem A, we will prove the following Nordhaus- Gaddum type result for regular graphs.

**Theorem 5.** If \( G \) is an \( r \)-regular graph of order \( n \) such that \( k \leq r \leq \frac{n-1}{2} \) and \( n-r-1 \geq k \), then

\[
\gamma_{sRk}(G) + \gamma'_{sRk}(\overline{G}) \geq \frac{kn}{n-2}.
\]
If \( n \) is even, then
\[
\gamma'_{sRk}(G) + \gamma'_{sRk}(\overline{G}) \geq \frac{kn}{n-1}.
\]

**Proof.** Since \( G \) is \( r \)-regular, the complement \( \overline{G} \) is \((n-r-1)\)-regular. It follows from Proposition A that
\[
\gamma'_{sRk}(G) + \gamma'_{sRk}(\overline{G}) \geq \frac{kn}{2} \left( \frac{r}{2r-1} + \frac{n-r-1}{2n-2r-3} \right).
\]
Since \( r \leq \frac{n-1}{2} \), we have
\[
\gamma'_{sRk}(G) + \gamma'_{sRk}(\overline{G}) \geq \frac{kn}{2} \left( \frac{1}{2r-1} + \frac{1}{2n-2r-3} \right).
\]
The conditions \( r \geq k \) and \( n-r-1 \geq k \) imply that \( k \leq r \leq n-k-1 \). As the function \( f(x) = \frac{1}{2x-1} + \frac{1}{2n-2x-3} \) has its minimum for \( x = \frac{n-1}{2} \) when \( k \leq x \leq n-k-1 \), we obtain
\[
\gamma'_{sRk}(G) + \gamma'_{sRk}(\overline{G}) \geq \frac{kn}{2} \left( \frac{1}{2r-1} + \frac{1}{2n-2r-3} \right)
\]
\[
\geq \frac{kn}{2} \left( \frac{1}{n-2} + \frac{1}{n-2} \right)
\]
\[
= \frac{kn}{n-2}.
\]
and this is the desired bound. If \( n \) is even, then the function \( f \) has its minimum for \( r = x = \frac{n-2}{2} \) or \( r = x = \frac{n}{2} \), since \( r \) is an integer. We derive from the assumption \( r \leq \frac{n-1}{2} \) that \( r = \frac{n-2}{2} \) and so
\[
\gamma'_{sRk}(G) + \gamma'_{sRk}(\overline{G}) \geq \frac{kn}{2} \left( \frac{1}{2r-1} + \frac{1}{2n-2r-3} \right)
\]
\[
\geq \frac{kn}{2} \left( \frac{1}{n-3} + \frac{1}{n-1} \right)
\]
\[
\geq \frac{kn}{2} \left( \frac{1}{n-1} + \frac{1}{n-1} \right)
\]
\[
= \frac{kn}{n-1}.
\]
This completes the proof. \( \square \)

We conclude this paper with an open problem.

**Problem 1.** Characterize all connected graphs \( G \) attaining the bound in Theorem 4.
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