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On the signed Roman edge k-domination in graphs

A. Mahmoodi[1]

[1]Department of Mathematics, Payame Noor University, I.R. Iran
akmahmoodi@yahoo.com

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Abstract: Let $k \geq 1$ be an integer, and $G = (V, E)$ be a finite and simple graph. The closed neighborhood $N_G[e]$ of an edge e in a graph G is the set consisting of e and all edges having a common end-vertex with e. A signed Roman edge k-dominating function (SREkDF) on a graph G is a function $f : E \to \{-1, 1, 2\}$ satisfying the conditions that (i) for every edge e of G, $\sum_{x \in N[e]} f(x) \geq k$ and (ii) every edge e for which $f(e) = -1$ is adjacent to at least one edge e' for which $f(e') = 2$. The minimum of the values $\sum_{e \in E} f(e)$, taken over all signed Roman edge k-dominating functions f of G, is called the signed Roman edge k-domination number of G and is denoted by $\gamma'_{sRk}(G)$. In this paper we establish some new bounds on the signed Roman edge k-domination number.

Keywords: Signed Roman edge k-dominating function, Signed Roman edge k-domination number

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1. Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* and *size* of a graph G are denoted by $n = n(G)$ and $m = m(G)$, respectively. For every vertex $v \in V$, the *open neighborhood* $N_G(v)$ is the set $\{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $d_G(v) = d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The *complement* of a graph G is denoted by $\bar{G}$. We write K_n for the *complete graph* and C_n for a *cycle* of order n. Two edges e_1 and e_2 of G is said to be *adjacent* if there exists a vertex $v \in V(G)$ to which they are both incident. The *line graph* of a graph G, written $L(G)$, is the graph with the edges of G as its vertices, and $ee' \in E(L(G))$ when e and e' are adjacent in

G. The *open neighborhood* $N(e) = N_G(e)$ of an edge $e \in E(G)$ is the set of all edges adjacent to e. Its *closed neighborhood* is $N[e] = N_G[e] = N_G(e) \cup \{e\}$. If v is a vertex, then denote by $E(v)$ the set of edges incident with the vertex v. For a function $f : E(G) \to \{-1, 1, 2\}$ and a subset S of $E(G)$, we define $f(S) = \sum_{e \in S} f(e)$. Also for every vertex v, we define $f(v) = \sum_{e \in E(v)} f(e)$.
A signed Roman k-dominating function (SRkDF) on a graph G is a function $f : V \to \{-1, 1, 2\}$ satisfying the conditions that (i) $\sum_{x \in N[v]} f(x) \geq k$ for each vertex $v \in V$, and (ii) every vertex u for which $f(u) = -1$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an SRkDF is the sum of its function values over all edges. The signed Roman k-domination number of G, denoted $\gamma^k_{sR}(G)$, is the minimum weight of an SRkDF in G. The signed Roman k-domination number was introduced by Henning and Volkman in [5] and has been studied in [4]. The special case $k = 1$ was introduced and investigated in [2].
A signed Roman edge k-dominating function (SREkDF) on a graph G is a function $f : E \to \{-1, 1, 2\}$ satisfying the conditions that (i) for every edge e of G, $\sum_{x \in N[e]} f(x) \geq k$ and (ii) every edge e for which $f(e) = -1$ is adjacent with at least one edge e' for which $f(e') = 2$. The weight of an SREkDF is the sum of its function values over all edges. The signed Roman edge k-domination number of G, denoted $\gamma'_{sRk}(G)$, is the minimum weight of an SREkDF in G. For an edge e, we denote $f[e] = f(N[e]) = \sum_{x \in N[e]} f(x)$ for notational convenience. The signed Roman edge k-domination number was introduced by Asgharsharghi et al. in [3]. The special case $k = 1$ was introduced by Ahangar et al. [1].
A signed Roman edge k-dominating function $f : E \to \{-1, 1, 2\}$ can be represented by the ordered partition (E_{-1}, E_1, E_2) (or (E^f_{-1}, E^f_1, E^f_2) to refer to f) of E, where $E_i = \{e \in E \mid f(e) = i\}$. In this representation, its weight is $\omega(f) = |E_1| + 2|E_2| - |E_{-1}|$.
A signed Roman edge k-dominating function of weight $\gamma'_{sRk}(G)$ is called $\gamma'_{sRk}(G)$-function. Let f be an SREkDF of G. An edge $e \in E(G)$ is called a i edge if $f(e) = i$ for $i = -1, 1, 2$. The signed Roman edge k-domination number exists if $|N_G(e)| \geq \frac{k}{2} - 1$ for every edge $e \in E$. However, for investigations of the signed Roman edge k-domination number it is reasonable to claim that for every edge $e \in E$, $|N_G(e)| \geq k - 1$. Thus we assume throughout this paper that $\delta \geq k - 1$. Let uv be an edge of G. Then

$$f[uv] = f(u) + f(v) - f(uv) \geq k. \tag{1}$$

In this note we continue the study of the signed Roman edge k-domination in graphs and present some (sharp) bounds for this parameter.
We make use of the following results in this paper.

Proposition A. *[3] For any r-regular graph G of size m, $(r \geq 1)$, $\gamma'_{sRk}(G) \geq \frac{km}{2r-1}$.*

2. Bounds on the signed Roman edge k-domination number

In this section we present new bounds on the signed Roman edge k-domination in graphs. We begin with a simple observation.

Observation 1. *Let G be a graph containing C_n as a subgraph. For any SREkDF f of G, $\sum_{v\in V(C_n)} f(v) \geq \frac{(k-1)n}{2}$.*

Proof. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of C_n. By (1), we have

$$\sum_{i=1}^{n} f(v_i) = \frac{1}{2}\sum_{i=1}^{n}(f(v_i) + f(v_{i+1})) \geq \frac{(k-1)n}{2}$$

where indices are modulo n. □

An *elementary graph* is a graph in which each component is a 1-regular graph or a 2-regular graph.

Theorem 2. For every graph G of order n, $\gamma'_{sRk}(G) \geq \frac{-(n-k+1)^2}{16}$.

Proof. Let H be an elementary subgraph of G with maximum number of vertices. Among this elementary subgraphs choose one such that the number of its K_2 components is as large as possible. By the choice of H, H has no even cycle. Let t be the number of vertices in $V(G) - V(H)$. If there is an edge uv in G for some $u, v \in V(G) - V(H)$, then by adding the edge uv to H we obtain an elementary subgraph of G with more vertices which is a contradiction. Hence $V(G) - V(H)$ is independent. We claim that for every vertex $v \in V(G) - V(H)$, $d(v) \leq \frac{n-t}{2}$.
If $V(G) - V(H) = \emptyset$, then there is nothing to prove. So let $V(G) - V(H) \neq \emptyset$ and $v \in V(G) - V(H)$. If v is adjacent to a vertex of an odd cycle $C = (v_1v_2 \ldots v_k)$ of H, say v_1, then $H' = (H - E(C)) \cup \{vv_1, v_2v_3. \ldots, v_{k-1}v_k\}$ is an elementary subgraph of G with more vertices than H which is a contradiction. Hence v is not adjacent to any vertex of an odd cycle of H. If v is adjacent to each vertex of a K_2 component of H such as uw, then $H' = H + \{uv, wv\}$ is an elementary subgraph of G with more vertices than H , a contradiction again. Thus v is adjacent to at most one vertex of each K_2 component of H. This implies that $d(v) \leq \frac{|V(H)|}{2} = \frac{n-t}{2}$. Let f be a $\gamma'_{sRk}(G)$-function. By (1) and Observation 1, we obtain $\sum_{v\in V(H)} f(v) \geq \frac{(k-1)(n-t)}{2}$. It follows that

$$\begin{aligned}
\gamma'_{sRk}(G) &= \sum_{e\in E(G)} f(e) \\
&= \frac{1}{2}\Big(\sum_{v\in V(H)} f(v) + \sum_{v\in V(G)-V(H)} f(v)\Big) \\
&\geq \frac{(k-1)(n-t)}{4} - \frac{t(n-t)}{4} \\
&\geq \frac{-(n-k+1)^2}{16},
\end{aligned}$$

and the proof is complete. □

Corollary 1. If G has a spanning elementary subgraph, then $\gamma'_{sRk}(G) \geq \frac{(k-1)n}{4}$.

Theorem 3. Let $k \geq 1$ and let G be a graph with $\gamma'_{sRk}(G) < m$. Then

$$\gamma'_{sRk}(G) \geq 2\left\lceil \frac{k-1}{8} + \frac{1}{2}\sqrt{\frac{(k-1)^2}{16} + (k+1)m} \right\rceil + 1 - m.$$

Proof. Let $f = (E_{-1}, E_1, E_2)$ be a $\gamma'_{sRk}(G)$-function, and let $|E_1| = m_1$ and $|E_2| = m_2$. Then

$$\gamma'_{sRk}(G) = 2|E_2| + |E_1| - |E_{-1}| = 3m_2 + 2m_1 - m. \qquad (2)$$

Let $G^* = L(G)$ be the line graph of G and $E^*(E_1 \cup E_2, E_{-1}) = \{e = xy \in E(G^*) \mid x \in E_1 \cup E_2, y \in E_{-1}\}$. Clearly for each $y \in E_{-1}$, we have

$$|N[y] \cap (E_1 \cup E_2)| = |N_{G^*}[y] \cap (E_1 \cup E_2)| \geq \frac{k+1}{2}$$

and hence $|E^*(E_1 \cup E_2, E_{-1})| \geq \frac{k+1}{2}|E_{-1}| = \frac{k+1}{2}(m - (m_1 + m_2))$. So there exists at least one element $x \in E_1 \cup E_2$ such that x is adjacent to at least $\lceil \frac{\frac{k+1}{2}(m-(m_1+m_2))}{m_1+m_2} \rceil$ elements of E_{-1} in G. Since f is an SREkDF of G, we have

$$\begin{aligned} |N[x] \cap (E_1 \cup E_2)| &= |N_{G^*}[x] \cap (E_1 \cup E_2)| \\ &\geq \frac{k}{2} + \frac{1}{2}|N_{G^*}[x] \cap E_{-1}| \\ &\geq \frac{k}{2} + \frac{1}{2}\left\lceil \frac{\frac{k+1}{2}(m - (m_1 + m_2))}{m_1 + m_2} \right\rceil. \end{aligned}$$

Since $m_1 + m_2 = |E_1| + |E_2| \geq |N[x] \cap (E_1 \cup E_2)|$, we have

$$m_1 + m_2 \geq \frac{k}{2} + \frac{1}{2}\left\lceil \frac{\frac{k+1}{2}(m - (m_1 + m_2))}{m_1 + m_2} \right\rceil \geq \frac{k}{2} + \frac{k+1}{4}\frac{(m - (m_1 + m_2))}{m_1 + m_2}. \qquad (3)$$

It follows from (3) that $m_1 + m_2 \geq \frac{k-1}{8} + \frac{1}{2}\sqrt{\frac{(k-1)^2}{16} + (k+1)m}$ and hence $m_1 + m_2 \geq \lceil \frac{k-1}{8} + \frac{1}{2}\sqrt{\frac{(k-1)^2}{16} + (k+1)m} \rceil$. Since $m_2 \geq 1$, we have

$$2m_1 + 3m_2 \geq 2\left\lceil \frac{k-1}{8} + \frac{1}{2}\sqrt{\frac{(k-1)^2}{16} + (k+1)m} \right\rceil + 1.$$

Now, the result comes from (2) and the proof is complete. □

The bound in Theorem 3 is sharp for P_3 when $k = 1$. The proof of next result can be found in [3].

Proposition B. *Let $k \geq 2$ be an integer and T be a tree of order $n \geq k$. Then $\gamma'_{sRk}(T) \geq k$.*

Next we generalize the bound of Proposition B for general graphs.

Theorem 4. Let $k \geq 2$ and G be a connected graph of order $n \geq 3$ and size m. Then

$$\gamma'_{sRk}(G) \geq k(n-m).$$

Proof. Let p be the number of cycles of G. The proof is by induction on p. The statement is true for $p = 0$ by Proposition B. Assume the statement is true for all simple connected graphs G for which the number of cycles is less than p, where $p \geq 1$. Let G be a simple connected graph with p cycles. Assume that f is a $\gamma'_{sRk}(G)$-function and let $e = uv$ be a non-cut edge. If $f(e) = -1$, then obviously $f|_{G-e}$ is an SREkDF for $G - e$ and by the induction hypothesis, we have

$$\begin{aligned}\gamma'_{sRk}(G) &= \omega(f)\\ &= \omega(f|_{G-e}) - 1\\ &\geq k(n-(m-1)) - 1\\ &\geq k(n-m).\end{aligned}$$

Thus, we may assume all non-cut edges are assigned 1 or 2 by f. Let $e = uv$ be a non-cut edge of G such that $f(e)$ is as small as possible. First let $f(u) \leq f(e)$. Then u has $f(e)$ neighbors $w_1, w_{f(e)}$ such that $f(uw_1) = f(uw_{f(e)}) = -1$. We deduce from $f(u) + f(v) - f(uv) \geq k$ and $f(u) + f(w_i) - f(uw_i) \geq k$ $(i = 1, f(e))$ that $f(v) \geq k$ and $\deg(w_1), \deg(w_{f(e)}) \geq 2$. Let G' be the graph obtained from G by removing $uw_1, uw_{f(e)}$ and uv and adding a pendant edge vv' and let $G_1, G_2, G_{f(e)+1}$ be the components of G' containing $w_1, w_{f(e)}$ and u, respectively. Define $g : E(G') \to \{-1, 1, 2\}$ by $g(vv') = f(uv)$ and $g(x) = f(x)$ for $x \in E(G') - \{vv'\}$. Clearly, g is an SREkDF for G' and the function $g_i = g|_{V(G_i)}$ is an SREkDF of G_i for $1 \leq i \leq f(e)+1$. It follows from the induction hypothesis that

$$\begin{aligned}\omega(f) &= \omega(g) - f(e)\\ &= \sum_{i=1}^{f(e)+1} \omega(g_i) - f(e)\\ &\geq \sum_{i=1}^{f(e)+1} k(n(G_i) - m(G_i)) - f(e)\\ &= k(n(G') - m(G')) - f(e)\\ &= k(n + 1 - (m - f(e))) - f(e)\\ &> k(n-m).\end{aligned}$$

Hence, we may assume without loss of generality that $f(u) \geq f(e)+1$. Similarly, we may assume that $f(v) \geq f(e)+1$. Here, we consider the following subcases.

Subcase 1.1. $f(u) \leq k$ and $f(v) \leq k$.
Let G' be the graph obtained from $G-\{e\}$ by adding $k-f(u)+1$ pendant edges uu_j at u and $k-f(v)+1$ pendant edges vv_i at v. Obviously, the function $g: E(G') \to \{-1,1,2\}$ defined by $g(uu_1)=g(vv_1)=f(e)$, $g(uu_j)=1$ for $2 \leq j \leq k-f(u)+1$, $g(vv_i)=1$ for $2 \leq i \leq k-f(v)+1$ and $g(a)=f(a)$ otherwise, is an SREkDF for G'. By the induction hypothesis and the fact $f(u)+f(v)-f(uv) \geq k$, we have

$$\begin{aligned}\omega(f) &= \omega(g)-2k+f(u)+f(v)-f(uv)\\ &\geq k(n(G')-m(G'))-k\\ &= k(n-(m-1))-k\\ &= k(n-m).\end{aligned}$$

Subcase 1.2. $f(u) \leq k$ and $f(v) \geq k+1$. Let G' be the graph obtained from $G-\{e\}$ by adding $k-f(u)+1$ pendant edges uu_j at u and one pendant edge vv' at v. Obviously, the function $g: E(G') \to \{-1,1,2\}$ defined by $g(vv')=g(uu_1)=f(uv), g(uu_i)=1$ for $2 \leq i \leq k-f(u)+1$ and $g(a)=f(a)$ otherwise is an SREkDF for G'. By the induction hypothesis, we have

$$\begin{aligned}\omega(f) &= \omega(g)-k+f(u)-f(uv)\\ &\geq k(n(G')-m(G'))-k+f(u)-f(uv)\\ &= k(n+k-f(u)+2-(m-1+k-f(u)+2))-k+f(u)-f(uv)\\ &= k(n-m)+f(u)-f(uv)\\ &> k(n-m).\end{aligned}$$

Subcase 1.3. $f(u) \geq k+1$ and $f(v) \geq k+1$.
Let G' be the graph obtained from $G-\{e\}$ by adding two new pendant edges vv' and uu'. Define $g: E(G') \to \{-1,1,2\}$ by $g(vv')=g(uu')=f(uv)$ and $g(a)=f(a)$ otherwise. Clearly, g is an SREkDF for G' and by the induction hypothesis, we obtain

$$\omega(f)=\omega(g)-f(uv) \geq k(n(G')-m(G'))-f(uv)=k(n+2-(m+1))-f(uv) \geq k(n-m).$$

This completes the proof. □

As an application of Theorem A, we will prove the following Nordhaus- Gaddum type result for regular graphs.

Theorem 5. If G is an r-regular graph of order n such that $k \leq r \leq \frac{n-1}{2}$ and $n-r-1 \geq k$, then

$$\gamma'_{sRk}(G)+\gamma'_{sRk}(\overline{G}) \geq \frac{krn}{n-2}.$$

If n is even, then

$$\gamma'_{sRk}(G) + \gamma'_{sRk}(\overline{G}) \geq \frac{krn}{n-1}.$$

Proof. Since G is r-regular, the complement $\overline{G}$ is $(n-r-1)$-regular. It follows from Proposition A that

$$\gamma'_{sRk}(G) + \gamma'_{sRk}(\overline{G}) \geq \frac{kn}{2}(\frac{r}{2r-1} + \frac{n-r-1}{2n-2r-3}).$$

Since $r \leq \frac{n-1}{2}$, we have

$$\gamma'_{sRk}(G) + \gamma'_{sRk}(\overline{G}) \geq \frac{krn}{2}(\frac{1}{2r-1} + \frac{1}{2n-2r-3}).$$

The conditions $r \geq k$ and $n-r-1 \geq k$ imply that $k \leq r \leq n-k-1$. As the function $f(x) = \frac{1}{2x-1} + \frac{1}{2n-2x-3}$ has its minimum for $x = \frac{n-1}{2}$ when $k \leq x \leq n-k-1$, we obtain

$$\begin{aligned}\gamma'_{sRk}(G) + \gamma'_{sRk}(\overline{G}) &\geq \frac{krn}{2}(\frac{1}{2r-1} + \frac{1}{2n-2r-3}) \\ &\geq \frac{krn}{2}(\frac{1}{n-2} + \frac{1}{n-2}) \\ &= \frac{krn}{n-2},\end{aligned}$$

and this is the desired bound. If n is even, then the function f has its minimum for $r = x = \frac{n-2}{2}$ or $r = x = \frac{n}{2}$, since r is an integer. We derive from the assumption $r \leq \frac{n-1}{2}$ that $r = \frac{n-2}{2}$ and so

$$\begin{aligned}\gamma'_{sRk}(G) + \gamma'_{sRk}(\overline{G}) &\geq \frac{krn}{2}(\frac{1}{2r-1} + \frac{1}{2n-2r-3}) \\ &\geq \frac{krn}{2}(\frac{1}{n-3} + \frac{1}{n-1}) \\ &\geq \frac{krn}{2}(\frac{1}{n-1} + \frac{1}{n-1}) \\ &= \frac{krn}{n-1}.\end{aligned}$$

This completes the proof. □

We conclude this paper with an open problem.

Problem 1. Characterize all connected graphs G attaining the bound in Theorem 4.

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