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Reformulated F-index of graph operations

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Abstract: The first general Zagreb index is defined as $M_1^\lambda(G) = \sum_{v\in V(G)} d_G(v)^\lambda$ where $\lambda \in \mathbb{R} - \{0,1\}$. The case $\lambda = 3$, is called F-index. Similarly, reformulated first general Zagreb index is defined in terms of edge-drees as $EM_1^\lambda(G) = \sum_{e\in E(G)} d_G(e)^\lambda$ and the reformulated F-index is $RF(G) = \sum_{e\in E(G)} d_G(e)^3$. In this paper, we compute the reformulated F-index for some graph operations.

Keywords: First general Zagreb index, reformulated first general Zagreb index, F-index, reformulated F-index.

AMS Subject classification: 05C12, 05C07

1. Introduction

We follow Bondy and Murty [3] for terminology and notation not defined here and consider finite simple connected graphs only. In 1972, Gutman and Trinajstić [7] explored the study of total π-electron energy on the molecular structure and introduced a vertex degree-based graph invariants. These invariants are defined as

$$M_1(G) = \sum_{v\in V(G)} d_G(v)^2 = \sum_{uv\in E(G)} (d_G(u) + d_G(v))$$

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and

$$M_2(G) = \sum_{uv \in E(G)} (d_G(u).d_G(v)).$$

In the same paper, where first Zagreb index was introduced, Gutman and Trinajstić indicated that another term of the form $\sum_{v \in V(G)} d_G(v)^3$ influences the total φ-electron energy. But this remained unstudied by the researchers for a long time, except for a few occasions [8, 9, 12] until publication of an article by Furtula and Gutman in 2015 and so they named it forgotten topological index or F-index in short [6]. Thus F-index of a graph G is defined as

$$F(G) = \sum_{v \in V(G)} d_G(v)^3 = \sum_{uv \in E(G)} (d_G(u)^2 + d_G(v)^2).$$

In 2004, Miličević et al. [13] reformulated first Zagreb index in terms of edge degrees instead of vertex degrees, where the degree of an edge $e = uv$ is defined as $d_G(e) = d_G(u) + d_G(v) - 2$. Thus, the reformulated first Zagreb index of a graph G is defined as

$$EM_1(G) = \sum_{e \in E(G)} d_G(e)^2.$$

In 2004 and 2005, Li et al. [9, 12] introduced the concept of the *first general Zagreb index* $M_1^\lambda(G)$ of G as follows:

$$M_1^\lambda(G) = \sum_{v \in V(G)} d_G(v)^\lambda = \sum_{e=uv \in E(G)} d_G(u)^{\lambda-1} + d_G(v)^{\lambda-1} \qquad \text{for } \lambda \in \mathbb{R} - \{0,1\}.$$

The reformulated version of the general first Zagreb index is defined as

$$EM_1^\lambda(G) = \sum_{e \in E(G)} d_G(e)^\lambda \qquad \text{for } \lambda \in \mathbb{R} - \{0,1\}.$$

For the special case $\lambda = 3$, $RF(G)$ is called reformulated F-index. Graph operations play an important role in chemical graph theory. Some chemically important graphs can be obtained from some graphs by different graph operations, such as some nanotorus or Hamming graph, that is Cartesian product of complete graphs. Many authors computed some indices for some graph operations (see, for instance [2, 4, 5, 10, 11] and the references cited therein).
We [1] defined and studied *the general Zagreb coindices* for $\lambda \in \mathbb{R}$ as

$$\overline{M_1}^\lambda(G) = \sum_{uv \notin E(G)} (d_G(u)^{\lambda-1} + d_G(v)^{\lambda-1})$$

and

$$\overline{M_2}^\lambda(G) = \sum_{uv \notin E(G)} (d_G(u)^\lambda d_G(v)^\lambda).$$

In this paper, we compute the reformulated F-index for some graph operations.

2. The join of graphs

The join $G+H$ of graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$ is the graph union $G \cup H$ together with all the edges between $V(G)$ and $V(H)$. Obviously, $|V(G+H)| = |V(G)| + |V(H)|$ and $|E(G+H)| = |E(G)| + |E(H)| + |V(G)||V(H)|$.

Theorem 1. Let G_i be a graph of order n_i and size m_i for $i = 1, 2$. Then

$$\begin{aligned} RF(G_1+G_2) &= RF(G_1) + RF(G_2) + 7n_2F(G_1) + 7n_1F(G_2) + 12n_2M_2(G_1) + \\ &\quad 12n_1M_2(G_2) + 3(5n_2^2 - 10n_2 + 2m_2 + n_1n_2)M_1(G_1) + \\ &\quad 3(5n_1^2 - 10n_1 + 2m_1 + n_1n_2)M_1(G_2) + (8n_2^3 - 24n_2^2 + 24n_2)m_1 + \\ &\quad (8n_1^3 - 24n_1^2 + 24n_1)m_2 + n_1n_2(n_1+n_2-2)^3 + \\ &\quad 24m_1m_2(n_1+n_2-2) + (6m_1n_2 + 6n_1m_2)(n_1+n_2-2)^2. \end{aligned}$$

Proof. By definition,

$$RF(G_1+G_2) = \sum_{e=uv \in E(G_1+G_2)} (d_{G_1+G_2}(e))^3.$$

We partition the edge set of G_1+G_2 into three subsets $E_1 = E(G_1), E_2 = E(G_2)$ and $E_3 = \{e = uv \mid u \in V(G_1), v \in V(G_2)\}$. For any vertex $v \in V(G_1)$, we have $d_{G_1+G_2}(v) = d_{G_1}(v) + n_2$ and for each vertex $u \in V(G_2)$ we have $d_{G_1+G_2}(u) = d_{G_2}(u) + n_1$. It follows that

$$\begin{aligned} \sum_{uv \in E_1} (d_{G_1+G_2}(uv))^3 &= \sum_{uv \in E_1} (d_{G_1}(u) + d_{G_1}(v) - 2 + 2n_2)^3 \\ &= \sum_{uv \in E_1} (d_{G_1}(u) + d_{G_1}(v) - 2)^3 + \\ &\quad \sum_{uv \in E_1} 6n_2(d_{G_1}(u) + d_{G_1}(v) - 2)^2 + \\ &\quad \sum_{uv \in E_1} 12n_2^2(d_{G_1}(u) + d_{G_1}(v) - 2) + \sum_{uv \in E_1} 8n_2^3 \\ &= RF(G_1) + 6n_2F(G_1) + 12n_2M_2(G_1) + \\ &\quad (12n_2^2 - 24n_2)M_1(G_1) + 8m_1(n_2^3 - 3n_2^2 + 3n_2), \end{aligned} \tag{1}$$

and

$$\begin{aligned} \sum_{e \in E_2} (d_{G_1+G_2}(e))^3 =& RF(G_2) + 6n_1F(G_2) + 12n_1M_2(G_2) + (12n_1^2 - 24n_1)M_1(G_2) + \\ & 8m_2(n_1^3 - 3n_1^2 + 3n_1). \end{aligned} \tag{2}$$

If $e = uv \in E_3$ where $u \in V(G_1)$ and $v \in V(G_2)$, then we have

$$\begin{aligned}
\sum_{uv \in E_3} (d_{G_1+G_2}(uv))^3 &= \sum_{u\in V(G_1), v\in V(G_2)} (d_{G_1}(u) + d_{G_2}(v) + n_1 + n_2 - 2)^3 \\
&= \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} (d_{G_1}(u) + d_{G_2}(v))^3 + \\
&\quad \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} (n_1 + n_2 - 2)^3 + \\
&\quad \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} 3(d_{G_1}(u) + d_{G_2}(v))^2 (n_1 + n_2 - 2) + \\
&\quad \sum_{u\in V(G_1)} \sum_{v\in V(G_2)} 3(d_{G_1}(u) + d_{G_2}(v))(n_1 + n_2 - 2)^2 \\
&= n_2F(G_1) + n_1F(G_2) + 6m_2M_1(G_1) + 6m_1M_1(G_2) + \\
&\quad n_1n_2(n_1 + n_2 - 2)^3 + \\
&\quad 3n_2(n_1 + n_2 - 2)M_1(G_1) + 3n_1(n_1 + n_2 - 2)M_1(G_2) + \\
&\quad 24m_1m_2(n_1 + n_2 - 2) + \\
&\quad (6m_1n_2 + 6n_1m_2)(n_1 + n_2 - 2)^2. \qquad (3)
\end{aligned}$$

We conclude from Equations (1), (2) and (3) that

$$\begin{aligned}
RF(G_1 + G_2) &= RF(G_1) + RF(G_2) + 7n_2F(G_1) + 7n_1F(G_2) + 12n_2M_2(G_1) + \\
&\quad 12n_1M_2(G_2) + 3(5n_2^2 - 10n_2 + 2m_2 + n_1n_2)M_1(G_1) + \\
&\quad 3(5n_1^2 - 10n_1 + 2m_1 + n_1n_2)M_1(G_2) + (8n_2^3 - 24n_2^2 + 24n_2)m_1 + \\
&\quad (8n_1^3 - 24n_1^2 + 24n_1)m_2 + n_1n_2(n_1 + n_2 - 2)^3 + \\
&\quad 24m_1m_2(n_1 + n_2 - 2) + (6m_1n_2 + 6n_1m_2)(n_1 + n_2 - 2)^2.
\end{aligned}$$

□

3. The corona product of graphs

The corona product $G \circ H$ of graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$ is the graph obtained by one copy of G and $|V(G)|$ copies of H and joining the i-th vertex of G to every vertex in i-th copy of H. Obviously, $|V(G \circ H)| = |V(G)| + |V(G)||V(H)|$ and $|E(G \circ H)| = |E(G)| + |V(G)||E(H)| + |V(G)||V(H)|$.

Theorem 2. If G_i is a graph of order n_i and of size m_i for $i = 1, 2$, then

$$\begin{aligned}
RF(G_1 \circ G_2) &= RF(G_1) + 7n_2F(G_1) + n_1F(G_2) + 12n_2M_2(G_1) + \\
&\quad (15n_2^2 - 27n_2 + 6m_2)M_1(G_1) + \\
&\quad n_1RF(G_2) + 6n_1EM_1(G_2) + 12n_1(M_1(G_2) - 16m_2) + (3n_1(n_2 - 1) + \\
&\quad 6m_1)M_1(G_2) + 24m_1m_2(n_2 - 1) + (6m_1n_2 + 6n_1m_2)(n_2 - 1)^2.
\end{aligned}$$

Proof. Suppose $V(G_1) = \{v_1, \ldots, v_{n_1}\}$ and $V(G_2) = \{u_1, \ldots, u_{n_2}\}$. Let $E_1 = E(G_1), E_2^i = E(G_2^i)$ and $E_3^i = \{v_i u_j \mid 1 \leq j \leq n_2\}$ for $1 \leq i \leq n_1$. Then $E(G_1 \circ G_2) = E_1 \cup (\cup_{i=1}^{n_1} E_2^i) \cup (\cup_{i=1}^{n_1} E_3^i)$ is a partition of $E(G_1 \circ G_2)$. Using an argument similar to that described in the proof of Theorem 1, we have

$$\begin{aligned}\sum_{e\in E_1}(d_{G_1\circ G_2}(uv))^3 =& RF(G_1) + 6n_2F(G_1) + 12n_2M_2(G_1)+\\ &(12n_2^2 - 24n_2)M_1(G_1) + 8m_1(n_2^3 - 3n_2^2 + 3n_2).\end{aligned} \tag{4}$$

For any edge $uv \in E_2^i$, we have $d_{G_1\circ G_2}(uv) = d_{G_2}(u) + d_{G_2}(v)$ and hence

$$\begin{aligned}\sum_{i=1}^{n_1}\sum_{uv\in E_2^i}(d_{G_1\circ G_2}(uv))^3 =& \sum_{i=1}^{n_1}\sum_{uv\in E_2^i}(d_{G_2}(u) + d_{G_2}(v) - 2 + 2)^3\\ =& \sum_{i=1}^{n_1}\sum_{uv\in E_2^i}(d_{G_2}(u) + d_{G_2}(v) - 2)^3+\\ &\sum_{i=1}^{n_1}\sum_{uv\in E_2^i}6(d_{G_2}(u) + d_{G_2}(v) - 2)^2+\\ &\sum_{i=1}^{n_1}\sum_{uv\in E_2^i}12(d_{G_2}(u) + d_{G_2}(v) - 2) + \sum_{i=1}^{n_1}\sum_{uv\in E_2^i}8\\ =& n_1RF(G_2) + 6n_1EM_1(G_2) + 12n_1(M_1(G_2) - 16m_2).\end{aligned} \tag{5}$$

On the other hand, we have

$$\begin{aligned}\sum_{i=1}^{n_1}\sum_{uv\in E_3^i}(d_{G_1\circ G_2}(uv))^3 =& \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}(d_{G_1\circ G_2}(v_iu_j))^3\\ =& \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}(d_{G_1}(v_i) + d_{G_2}(u_j) + n_2 - 1)^3\\ =& \sum_{i=1}^{n_1}\sum_{j=1}^{n_2}(d_{G_1}(v_i) + d_{G_2}(u_j))^3 + \sum_{u\in V(G_1)}\sum_{v\in V(G_2)}(n_2-1)^3+\\ &\sum_{u\in V(G_1)}\sum_{v\in V(G_2)}3(d_{G_1}(v_i) + d_{G_2}(u_j))^2(n_2-1)+\\ &\sum_{u\in V(G_1)}\sum_{v\in V(G_2)}3(d_{G_1}(v_i) + d_{G_2}(u_j))(n_2-1)^2\\ =& n_2F(G_1) + n_1F(G_2) + 6m_2M_1(G_1) + 6m_1M_1(G_2)+\\ &n_1n_2(n_2-1)^3 + 3n_2(n_2-1)M_1(G_1)+\\ &3n_1(n_2-1)M_1(G_2) + 24m_1m_2(n_2-1)+\\ &(6m_1n_2 + 6n_1m_2)(n_2-1)^2.\end{aligned} \tag{6}$$

Summing up the Equalities (4), (5) and (6) we obtain,

$$\begin{aligned} RF(G_1 \circ G_2) &= RF(G_1) + 7n_2F(G_1) + n_1F(G_2) + 12n_2M_2(G_1) + \\ &(15n_2^2 - 27n_2 + 6m_2)M_1(G_1) + n_1RF(G_2) + 6n_1EM_1(G_2) + \\ &12n_1(M_1(G_2) - 16m_2) + (3n_1(n_2-1) + 6m_1)M_1(G_2) + \\ &24m_1m_2(n_2-1) + (6m_1n_2 + 6n_1m_2)(n_2-1)^2. \end{aligned}$$

□

4. The Cartesian product of graphs

The Cartesian product $G \times H$ of graphs G and H has the vertex set $V(G \times H) = V(G) \times V(H)$ and $(u, x)(v, y)$ is an edge of $G \times H$ if $uv \in E(G)$ and $x = y$, or $u = v$ and $xy \in E(H)$. Obviously, $|V(G \times H)| = |V(G)||V(H)|$ and $|E(G \times H)| = |E(G)||V(H)| + |V(G)||E(H)|$. If $G_1, G_2, \ldots, G_n$ are arbitrary graphs, then we denote $G_1 \times \cdots \times G_n$ by $\otimes_{i=1}^n G_i$.

Lemma 1. (M.H. Khalifeh et al. [11]) Let $G_1 = (V_1, E_1), \ldots, G_n = (V_n, E_n)$ be graphs and let $G = \otimes_{i=1}^n G_i$ and $V = V(\otimes_{i=1}^n G_i)$. Then

$$M_1(\otimes_{i=1}^n G_i) = |V| \sum_{i=1}^{n} \frac{M_1(G_i)}{|V_i|} + 4|V| \sum_{i \neq j, i,j=1}^{n} \frac{|E_i||E_j|}{|V_i||V_j|}.$$

In particular, $M_1(G^n) = n|V(G)|^{n-2}(M_1(G)|V(G)| + 4(n-1)|E(G)|^2)$.

Lemma 2. (M.H. Khalifeh et al. [11]) Let $G_1 = (V_1, E_1), \ldots, G_n = (V_n, E_n)$ be graphs and let $G = \otimes_{i=1}^n G_i$, $V = V(\otimes_{i=1}^n G_i)$ and $E = E(\otimes_{i=1}^n G_i)$. Then

$$\begin{aligned} M_2(\otimes_{i=1}^n G_i) &= |V| \sum_{i=1}^{n} \frac{M_2(G_i)}{|V_i|} + 3M_1(G_i)(\frac{|E|}{|V_i|} - \frac{|V||E_i|}{|V_i|^2}) + \\ &4|V| \sum_{i,j,k=1, i\neq j, i\neq k, j\neq k}^{n} \frac{|E_i||E_j||E_k|}{|V_i||V_j||V_k|}. \end{aligned}$$

In particular,

$$M_2(G^n) = n|V(G)|^{n-3}(|V(G)|^2M_2(G) + 3(n-1)|E(G)||V(G)|M_1(G) + 4(n-1)(n-2)|E(G)|^3).$$

Lemma 3. (De Dilanjan et al. []) Let $G_1 = (V_1, E_1), \ldots, G_n = (V_n, E_n)$ be graphs and let $G = \otimes_{i=1}^n G_i$, $V = V(\otimes_{i=1}^n G_i)$ and $E = E(\otimes_{i=1}^n G_i)$. Then

$$F(\otimes_{i=1}^n G_i) = |V| \sum_{i=1}^{n} \frac{M_1(G_i)}{|V_i|} + 4n \sum_{i,j=1, i\neq j}^{n} \frac{|E(G_i)|}{|V(G_i)|} . \frac{|E(G_j)|}{|V(G_j)|}.$$

Now we determine the reformulated F-index of the Cartesian product of graphs.

Theorem 3. Let G_i be a graph of order n_i and size m_i for $i = 1, 2$. Then

$$\begin{aligned} RF(G_1 \times G_2) =& n_1 RF(G_2) + n_2 RF(G_1) + 12(m_2 EM_1(G_1) + m_1 EM_1(G_2))+ \\ & 24M_1(G_2)M_1(G_1) - 24(m_1 M_1(G_2) + m_2 M_1(G_1))+ \\ & 8(m_1 F(G_2) + m_2 F(G_1)). \end{aligned}$$

Proof. Suppose $V(G_1) = \{u_1, \ldots, u_{n_1}\}$ and $V(G_2) = \{v_1, \ldots, v_{n_2}\}$. Set $E_i = \{(u, v_i)(v, v_i) \mid uv \in E(G_1)\}$ for $1 \le i \le n_2$ and $L_j = \{(u_j, x)(u_j, y) \mid xy \in E(G_2)\}$ for $1 \le j \le n_1$. Clearly $(\cup_{i=1}^{n_2} E_i) \cup (\cup_{j=1}^{n_1} L_j)$ is a partition of $V(G_1 \times G_2)$. Since $d_{G_1 \times G_2}(u_i, v_j) = d_{G_1}(u_i) + d_{G_2}(v_j)$ for each i, j, we have

$$\begin{aligned} \sum_{i=1}^{n_2} \sum_{uv \in E(G_1)} (d_{G_1 \times G_2}((u, v_i)(v, v_i)))^3 =& \sum_{i=1}^{n_2} \sum_{uv \in E(G_1)} (d_{G_1}(u) + d_{G_1}(v) - 2 + 2d_{G_2}(v_i))^3 \\ =& \sum_{i=1}^{n_2} \sum_{uv \in E(G_1)} (d_{G_1}(u) + d_{G_1}(v) - 2)^3 + \\ & \sum_{i=1}^{n_2} \sum_{uv \in E(G_1)} 6(d_{G_1}(u) + d_{G_1}(v) - 2)^2 d_{G_2}(v_i) + \\ & \sum_{i=1}^{n_2} \sum_{uv \in E(G_1)} 12(d_{G_1}(u) + d_{G_1}(v) - 2) d_{G_2}^2(v_i) + \\ & \sum_{i=1}^{n_2} \sum_{uv \in E(G_1)} 8d_{G_2}^3(v_i) \\ =& \, n_2 RF(G_1) + 12m_2 EM_1(G_1) + \\ & 12M_1(G_2)(M_1(G_1) - 2m_1) + 8m_1 F(G_2). \end{aligned} \tag{7}$$

Similarly, we have

$$\begin{aligned} \sum_{j=1}^{n_1} \sum_{uv \in E(G_2)} (d_{G_1 \times G_2}((u_j, u)(u_j, v)))^3 =& n_1 RF(G_2) + 12m_1 EM_1(G_2) + \\ & 12M_1(G_1)(M_1(G_2) - 2m_2) + \\ & 8m_2 F(G_1). \end{aligned} \tag{8}$$

Summing up Equations (7) and (8), we obtain

$$\begin{aligned} RF(G_1 \times G_2) = & n_1 RF(G_2) + n_2 RF(G_1) + 12(m_2 EM_1(G_1) + m_1 EM_1(G_2)) + \\ & 24M_1(G_2)M_1(G_1) - 24(m_1 M_1(G_2) + m_2 M_1(G_1)) + 8(m_1 F(G_2) + \\ & m_2 F(G_1)). \end{aligned}$$

□

Example 1. *As regards, for natural numbers n, m, $M_1(C_n) = EM_1(C_n) = 4n$, $F(C_n) = RF(C_n) = 8n$, $M_1(P_m) = 4m-6$, $EM_1(P_m) = 4m-10$, $F(P_m) = 8m-14$ and $RF(P_m) = 8m-22$,*

1. $RF(C_n \times C_m) = 432mn$.
2. $RF(C_n \times P_m) = 432mn - 702n$.
3. $RF(P_n \times P_m) = 432mn - 582(m+n) + 1008$.

Example 2. *Let $T = T[p, q]$ be the molecular graph of a nanotorus (Figure 1). Then $|V(T)| = pq$ and $|E(T)| = \frac{3}{2}pq$. Obviously, $M_1(T) = 9pq$, $EM_1(T) = 24pq$, $F(T) = 27pq$ and $RF(T) = 96pq$. Therefore for a* q-multi-walled nanotorus *$G = P_n \times T$ we have $RF(P_n \times T) = 1280npq - 1738pq$.*

Theorem 4. Let $G_1, G_2, \cdots, G_n$ be graphs with $V_i = V(G_i)$ and $E_i = E(G_i)$ for $1 \le i \le n$, and $V = V(\otimes_{i=1}^n G_i)$ and $E = E(\otimes_{i=1}^n G_i)$. Then

$$\begin{aligned} RF(\otimes_{i=1}^n G_i) &= \sum_{i=1}^{n} RF(G_i) \prod_{j=1, j\neq i}^{n} |V_j| + 12(|E_n| EM_1(\otimes_{i=1}^{n-1} G_i) + \\ &|E(\otimes_{i=1}^{n-1} G_i)| EM_1(G_n)) - 24(|E_n| M_1(\otimes_{i=1}^{n-1} G_i) + \\ &|E(\otimes_{i=1}^{n-1} G_i)| M_1(G_n)) + 8(|E_n| F(\otimes_{i=1}^{n-1} G_i) + |E(\otimes_{i=1}^{n-1} G_i)| F(G_n)) + \\ &24 M_1(G_n) M_1(\otimes_{i=1}^{n-1} G_i) + 24 \sum_{j=0}^{n-3} \prod_{i=0}^{j} |V_{n-i}| M_1(G_{n-i-1}) M_1(\otimes_{i=1}^{n-i-2} G_i) + \\ &\sum_{j=1}^{n-1} \prod_{i=1}^{j} |V_{n-i+1}| [|E_{n-i}| (12 EM_1(\otimes_{i=1}^{n-i-1} G_i) - 24 M_1(\otimes_{i=1}^{n-i-1} G_i)) + \\ &8F(\otimes_{i=1}^{n-i-1} G_i) + |E(\otimes_{i=1}^{n-i-1} G_i)| (12 EM_1(G_{n-i}) - \\ &24 M_1(G_{n-i}) + 8F(G_{n-i}))]. \end{aligned}$$

Proof. The proof is by induction on n. According on Theorem 3, the statement holds for $n = 2$. Clearly, $|E(\otimes_{i=1}^n G_i)| = |V| \sum_{i=1}^n \frac{|E_i|}{|V_i|}$ and $|V(\otimes_{i=1}^n G_i)| = \prod_{i=1}^n |V_i|$. One can see that for every graph G, $E(M_1(G)) = F(G) + 2M_2(G) + 4m - 4M_1(G)$. Now with applying the Theorem 3 for $\otimes_{i=1}^n G_i = \otimes_{i=1}^{n-1} G_i \times G_n$ and induction, the proof is completed. □

5. The composition product of graphs

The composition $G[H]$ of graphs G and H has the vertex set $V(G[H]) = V(G) \times V(H)$ and $(u, x)(v, y)$ is an edge of $G[H]$ if ($uv \in E(G)$) or ($xy \in E(H)$ and $u = v$). Obviously, $|V(G[H])| = |V(G)||V(H)|$ and $|E(G[H])| = |E(G)||V(H)|^2 + |E(H)||V(G)|$.

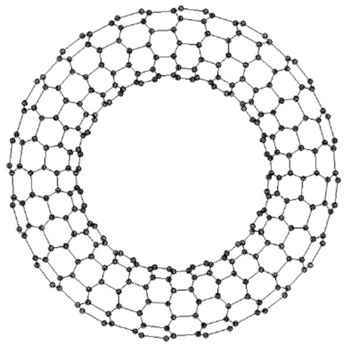

Figure 1. **The graph of a nanotorus**

Theorem 5. Let G_1 be a graph of order n_1 and size m_1 and let G_2 be a graph of order n_2 and size m_2. Then

$$\begin{aligned} RF(G_1[G_2]) &= RF(G_1)n_2^3 + m_1(RF(G_2) + R\overline{F}(G_2)) + 8m_1n_2^3 + \\ &\quad 4n_2m_1(EM_1(G_2) + E\overline{M_1}(G_2)) + 8n_2^2m_1(M_1(G_2) + \overline{M_1}(G_2) - 2) + \\ &\quad 2n_2^2EM_1(G_1)(M_1(G_2) + \overline{M_1}(G_2) - 2 + 2n_2) + \\ &\quad 2n_2(M_1(G_1) - 2m_1)(EM_1(G_2) + E\overline{M_1}(G_2) + 4n_2^2) + \\ &\quad 8m_2n_2^3F(G_1) + RF(G_2) + 12n_2^2M_1(G_1)(M_1(G_2) - 2m_2) + \\ &\quad 12n_2m_1EM_1(G_2). \end{aligned}$$

Proof. Suppose $V(G_1) = \{u_1, \ldots, u_{n_1}\}$ and $V(G_2) = \{v_1, \ldots, v_{n_2}\}$. We partition the edges of $G_1[G_2]$ into two subsets E_1 and E_2, as follows:
Set $L = \{e = (u, x)(v, y) \mid uv \in E(G_1)\}$ and $E_i = \{(u_i, x)(u_i, y) \mid xy \in E(G_2)\}$ for $1 \leq i \leq n_1$. Clearly $(\cup_{i=1}^{n_1} E_i) \cup L$ is a partition of $V(G_1[G_2])$.
Let $e = (u, x)(v, y) \in L$. Then $d_{G_1[G_2]}(u, x) = n_2d_{G_1}(u) + d_{G_2}(x)$ and $d_{G_1[G_2]}(u, x)(v, y) = n_2(d_{G_1}(u) + d_{G_2}(v)) - 2 + d_{G_2}(x) + d_{G_2}(y)$. In this case we notice that xy can be adjacent in G_2 or not.

$$\begin{aligned}
\sum_{e=(u,x)(v,y)\ ,\ e\in L} d^3_{G_1[G_2]}(e) &= \sum_{\substack{e=(u,x)(v,y)\\ e\in L}} [n_2(d_{G_1}(u)+d_{G_1}(v)-2)-2+\\
&\quad d_{G_2}(x)+d_{G_2}(y)+2n_2]^3\\
&= \sum_{\substack{e=(u,x)(v,y)\\ e\in L}} [n_2^3(d_{G_1}(u)+d_{G_1}(v)-2)^3+\\
&\quad (d_{G_2}(x)+d_{G_2}(y)-2)^3+8n_2^3+\\
&\quad 4n_2(d_{G_2}(x)+d_{G_2}(y)-2)^2+8n_2^2(d_{G_2}(x)+d_{G_2}(y)-2)+\\
&\quad 2n_2^2(d_{G_1}(u)+d_{G_1}(v)-2)^2(d_{G_2}(x)+d_{G_2}(y)-2+2n_2)+\\
&\quad 2n_2(d_{G_1}(u)+d_{G_1}(v)-2)((d_{G_2}(x)+d_{G_2}(y)-2)^2+4n_2^2)]\\
&= RF(G_1)n_2^3+m_1(RF(G_2)+R\overline{F}(G_2))+8m_1n_2^3+\\
&\quad 4n_2m_1(EM_1(G_2)+E\overline{M_1}(G_2))+\\
&\quad 8n_2^2m_1(M_1(G_2)+\overline{M_1}(G_2)-2)+\\
&\quad 2n_2^2EM_1(G_1)(M_1(G_2)+\overline{M_1}(G_2)-2+2n_2)+\\
&\quad 2n_2(M_1(G_1)-2m_1)(EM_1(G_2)+E\overline{M_1}(G_2)+4n_2^2).
\end{aligned}$$

Let $e=(u_i,x)(u_i,y)\in E_i$. Then

$$d_{G_1[G_2]}(u_i,x)(u_i,y)=2n_2d_{G_1}(u_i)-2+d_{G_2}(x)+d_{G_2}(y)$$

and we have

$$\begin{aligned}
\sum_{e=(u,x)(u,y)\ ,\ e\in E_i} d^3_{G_1[G_2]}(e) &= \sum_{\substack{e=(u,x)(u,y)\\ e\in E_i}} [2n_2d_{G_1}(u)-2+d_{G_2}(x)+d_{G_2}(y)]^3\\
&= \sum_{\substack{e=(u,x)(u,y)\\ e\in E_i}} [8n_2^3d^3_{G_1}(u)+(d_{G_2}(x)+d_{G_2}(y)-2)^3+\\
&\quad 12n_2^2d^2_{G_1}(u)(d_{G_2}(x)+d_{G_2}(y)-2)+\\
&\quad 6n_2d_{G_1}(u)(d_{G_2}(x)+d_{G_2}(y)-2)^2\\
&= 8m_2n_2^3F(G_1)+RF(G_2)+\\
&\quad 12n_2^2M_1(G_1)(M_1(G_2)-2m_2)+12n_2m_1EM_1(G_2).
\end{aligned}$$

Therefore

$$\begin{aligned}
RF(G_1[G_2]) &= \sum_{e\in L}(d_{G_1[G_2]}(e))^3 + \sum_{e\in E_i}(d_{G_1[G_2]}(e))^3\\
&= RF(G_1)n_2^3 + m_1(RF(G_2) + R\overline{F}(G_2)) + 8m_1n_2^3 +\\
&\quad 4n_2m_1(EM_1(G_2) + E\overline{M_1}(G_2)) + 8n_2^2m_1(M_1(G_2) + \overline{M_1}(G_2) - 2) +\\
&\quad 2n_2^2EM_1(G_1)(M_1(G_2) + \overline{M_1}(G_2) - 2 + 2n_2) +\\
&\quad 2n_2(M_1(G_1) - 2m_1)(EM_1(G_2) + E\overline{M_1}(G_2) + 4n_2^2) +\\
&\quad 8m_2n_2^3F(G_1) + RF(G_2) + 12n_2^2M_1(G_1)(M_1(G_2) - 2m_2) +\\
&\quad 12n_2m_1EM_1(G_2).
\end{aligned}$$

□

6. The tensor product of graphs

The Tensor Product $G \otimes H$ of graphs G and H has the vertex set $V(G\otimes H) = V(G)\times V(H)$ and $(u,x)(v,y)$ is an edge of $G\otimes H$ if $uv \in E(G)$ and $xy \in E(H)$. Obviously, $|V(G\otimes H)| = |V(G)||V(H)|$, $|E(G\otimes H)| = 2|E(G)||E(H)|$ and $d_{G\otimes H}(u,x) = d_G(u).d_H(x)$.

Theorem 6. If G_i is a graph of order n_i and size m_i for $i = 1, 2$, then

$$\begin{aligned}
RF(G_1\otimes G_2) &= M_1^4(G_1\otimes G_2) - 6F(G_1\otimes G_2) + 12M_1(G_1\otimes G_2) +\\
&\quad 3M_2(G_1\otimes G_2)M_1(G_1\otimes G_2) - 12M_2(G_1\otimes G_2) - 8m_1m_2.
\end{aligned}$$

Proof. For any edge $e = (u,x)(v,y) \in E(G_1\otimes G_2)$, we have

$$d_{G_1\otimes G_2}(e) = d_{G_1}(u).d_{G_2}(x) + d_{G_1}(v).d_{G_2}(y) - 2.$$

By definition, we have

$$\begin{aligned}
RF(G_1\otimes G_2) &= \sum_{uv\in E(G_1)}\sum_{xy\in E(G_2)}(d_{G_1\otimes G_2}((u,x)(v,y)))^3\\
&= \sum_{uv\in E(G_1)}\sum_{xy\in E(G_2)}(d_{G_1}(u).d_{G_2}(x) + d_{G_1}(v).d_{G_2}(y) - 2)^3\\
&= \sum_{uv\in E(G_1)}\sum_{xy\in E(G_2)}[d_{G_1}^3(u)d_{G_2}^3(x) + d_{G_1}^3(v)d_{G_2}^3(y) - 6d_{G_1}^2(v)d_{G_2}^2(y) +\\
&\quad 12d_{G_1}(v)d_{G_2}(y) - 8 + 3d_{G_1}^2(u)d_{G_2}^2(x)d_{G_1}(v)d_{G_2}(y) -\\
&\quad 6d_{G_1}^2(u)d_{G_2}^2(x) + 3d_{G_1}(u)d_{G_2}(x)d_{G_1}^2(v)d_{G_2}^2(y) -\\
&\quad 12d_{G_1}(u)d_{G_2}(x)d_{G_1}(v)d_{G_2}(y) + 12d_{G_1}(u)d_{G_2}(x)]\\
&= M_1^4(G_1\otimes G_2) - 6F(G_1\otimes G_2) + 12M_1(G_1\otimes G_2) +\\
&\quad 3M_2(G_1\otimes G_2)M_1(G_1\otimes G_2) - 12M_2(G_1\otimes G_2) - 8m_1m_2.
\end{aligned}$$

□

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