Sufficient conditions for maximally edge-connected and super-edge-connected graphs

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Abstract: Let \(G\) be a connected graph with minimum degree \(\delta\) and edge-connectivity \(\lambda\). A graph is maximally edge-connected if \(\lambda = \delta\), and it is super-edge-connected if every minimum edge-cut is trivial; that is, if every minimum edge-cut consists of edges incident with a vertex of minimum degree. In this paper, we show that a connected graph or a connected triangle-free graph is maximally edge-connected or super-edge-connected if the number of edges is large enough.

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1. Terminology and introduction

Let \(G\) be a finite and simple graph with vertex set \(V = V(G)\) and edge set \(E = E(G)\). The order and size of \(G\) are defined by \(n = n(G) = |V(G)|\) and \(m = m(G) = |E(G)|\), respectively. If \(N(v) = N_G(v)\) is the neighborhood of
the vertex \( v \in V(G) \), then we denote by \( d(v) = |N(v)| \) the degree of \( v \) and by \( \delta = \delta(G) \) the minimum degree of the graph \( G \). For a subset \( X \subset V(G) \), use \( G[X] \) to denote the subgraph of \( G \) induced by \( X \). For two subsets \( X \) and \( Y \) of \( V(G) \) let \( [X,Y] \) be the set of edges with one endpoint in \( X \) and the other one in \( Y \). We write \( K_n \) for the complete graph of order \( n \) and \( K_{p,q} \) for the complete bipartite graph whose partition sets have cardinality \( p \) and \( q \), respectively. An edge-cut of a connected graph \( G \) is a set of edges whose removal disconnects \( G \). The edge connectivity \( \lambda = \lambda(G) \) of a connected graph \( G \) is defined as the minimum cardinality of an edge-cut over all edge-cuts of \( G \). An edge-cut \( S \) is a minimum edge-cut or a \( \lambda \)-cut if \( |S| = \lambda(G) \). The inequality \( \lambda(G) \leq \delta(G) \) is immediate. We call a connected graph maximally edge-connected, if \( \lambda(G) = \delta(G) \). In 1981, Bauer et al. [1] proposed the concept of super-edge connectedness. A graph is called super-edge-connected or super-\( \lambda \) if every minimum edge-cut is trivial; that is, if every minimum edge-cut consists of edges incident with a vertex of minimum degree. Thus every super-edge-connected graph is also maximally edge-connected.

Sufficient conditions for graphs to be maximally edge-connected or super-edge-connected were given by several authors, see for example the survey paper by Hellwig and Volkmann [3]. The starting point was an article by Chartrand [2] in 1966. He observed that if \( \delta \) is large enough, then the graph is maximally edge-connected. A similar condition for super-edge-connectivity was given by Kelmans [4] six years later. Over the years, these results have been strengthened many times and in many ways. However, there seems to be no result in the literature showing that graphs of sufficiently large size are maximally edge-connected. In this paper we fill this gap.

We show that a connected graph or a connected triangle-free graph \( G \) is maximally edge-connected or super-\( \lambda \) if the number of edges is large enough. For example, if \( m \geq \left( \frac{n-\delta-1}{2} \right) + \left( \frac{\delta+1}{2} \right) + \delta - 1 \), then \( G \) is maximally edge-connected, unless \( G \) is a graph obtained from \( K_{\delta+1} \cup K_{n-\delta-1} \) by adding \( \delta - 1 \) edges between \( K_{\delta+1} \) and \( K_{n-\delta-1} \).

2. Maximally edge-connected graphs

**Theorem 1.** Let \( G \) be a connected graph of order \( n \geq 2 \), size \( m \), minimum degree \( \delta \) and edge-connectivity \( \lambda \). If

\[
m \geq \left( \frac{n-\delta-1}{2} \right) + \left( \frac{\delta+1}{2} \right) + \delta - 1,
\]

then \( \lambda = \delta \), unless \( G \) is a graph obtained from \( K_{\delta+1} \cup K_{n-\delta-1} \) by adding \( \delta - 1 \) edges
between $K_{\delta+1}$ and $K_{n-\delta-1}$.

**Proof.** Suppose to the contrary that $\lambda \leq \delta - 1$. Let $S$ be an arbitrary $\lambda$-cut, and let $X$ and $Y$ denote the vertex sets of the two components of $G - S$ with $|X| \leq |Y|$. Each vertex in $X$ is adjacent to at most $|X| - 1$ vertices of $X$, and exactly $\lambda$ edges join vertices of $X$ to vertices of $Y$. Thus

$$\delta |X| \leq \sum_{x \in X} d(x) \leq |X|(|X| - 1) + \lambda,$$

and so $(|X| - 1)(|X| - \delta) \geq \delta - \lambda > 0$, which means that $|X| \geq \delta + 1$. Thus $|Y| \geq |X| \geq \delta + 1$. Therefore we have $\delta + 1 \leq |X| \leq |Y| \leq n - \delta - 1$. Since there are no edges between $X$ and $Y$ in $G - S$, we obtain

$$m - \lambda = |E(G - S)| \leq \left(\begin{array}{c} n \\ 2 \end{array}\right) - |X| \cdot |Y|.$$

This bound together with $\delta + 1 \leq |X| \leq |Y| \leq n - \delta - 1$, $n \geq 2|X|$ and $|X| + |Y| = n$ lead to

$$m \leq \left(\begin{array}{c} n \\ 2 \end{array}\right) - (\delta + 1)(n - \delta - 1) + (\delta - 1) = \left(\begin{array}{c} n - \delta - 1 \\ 2 \end{array}\right) + \left(\begin{array}{c} \delta + 1 \\ 2 \end{array}\right) + \delta - 1.$$

Combining this with (1), we have $m = \left(\begin{array}{c} n - \delta - 1 \\ 2 \end{array}\right) + \left(\begin{array}{c} \delta + 1 \\ 2 \end{array}\right) + \delta - 1$, and the two inequalities in the proof above must be equalities. Therefore $\lambda = \delta - 1$, $|X| = \delta + 1$, $|Y| = n - \delta - 1$, $G[X] = K_{\delta+1}$ and $G[Y] = K_{n-\delta-1}$. This completes the proof. \qed

For the next result, we use the famous theorem of Mantel [5] and Turán [6].

**Theorem 2.** For any triangle-free graph $G$ of order $n$, we have $|E(G)| \leq \left\lfloor \frac{1}{4} n^2 \right\rfloor$, with equality if and only if $G = K_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor}$.

**Theorem 3.** Let $G$ be a connected triangle-free graph of order $n \geq 2$, size $m$, minimum degree $\delta$ and edge-connectivity $\lambda$. If

$$m \geq \left\lfloor \frac{n^2}{4} \right\rfloor - \delta(n - 2\delta) + \delta - 1,$$

then $\lambda = \delta$, unless $G$ is a triangle-free graph obtained from $K_{\delta,\delta} \cup K_{\lfloor n/2 \rfloor - \delta, \lceil n/2 \rceil - \delta}$ by adding $\delta - 1$ edges between $K_{\delta,\delta}$ and $K_{\lfloor n/2 \rfloor - \delta, \lceil n/2 \rceil - \delta}$.
Suppose to the contrary that \( \lambda \leq \delta - 1 \). Let \( S \) be an arbitrary \( \lambda \)-cut, and let \( X \) and \( Y \) denote the vertex sets of the two components of \( G - S \) with \( |X| \leq |Y| \). Using Theorem 2, we conclude that

\[
|E(G[X])| \leq \left\lfloor \frac{|X|^2}{4} \right\rfloor \quad \text{and} \quad |E(G[Y])| \leq \left\lfloor \frac{|Y|^2}{4} \right\rfloor,
\]

with equalities if and only if \( G[X] = K_{\lfloor |X|/2 \rfloor, \lfloor |X|/2 \rfloor} \) and \( G[Y] = K_{\lfloor |Y|/2 \rfloor, \lfloor |Y|/2 \rfloor} \). Note that \( 2|E(G[X])| = \sum_{x \in X} d(x) - \lambda \). Thus \( \delta |X| \leq \sum_{x \in X} d(x) \leq \frac{|X|^2}{2} + \lambda \), and so \( |X|^2 - 2\delta |X| + 2\lambda \geq 0 \). It follows that

\[
(|X| - 1)(|X| - (2\delta - 1)) \geq 2\delta - 2\lambda - 1 \geq 1 > 0,
\]

which means that \( |X| \geq 2\delta \). Thus \( |Y| \geq |X| \geq 2\delta \). Therefore we arrive at

\[
2\delta \leq |X| \leq \frac{n}{2} \leq |Y| \leq n - 2\delta.
\]

This together with \( |X| + |Y| = n \) and (3) lead to

\[
m = |E(G[X])| + |E(G[Y])| + \lambda
\leq \left\lfloor \frac{|X|^2}{4} \right\rfloor + \left\lfloor \frac{|Y|^2}{4} \right\rfloor + \delta - 1 = \left\lfloor \frac{|X|^2}{4} \right\rfloor + \left\lfloor \frac{(n - |X|)^2}{4} \right\rfloor + \delta - 1
\leq \left\lfloor \frac{|X|^2}{4} + \frac{(n - |X|)^2}{4} \right\rfloor + \delta - 1
\leq \left\lfloor \frac{n^2}{4} + \frac{1}{2}(|X|^2 - n|X|) \right\rfloor + \delta - 1 \quad \text{(as} \ 2\delta \leq |X| \leq \frac{n}{2})
\leq \left\lfloor \frac{n^2}{4} - \delta(n - 2\delta) \right\rfloor + \delta - 1
= \left\lfloor \frac{n^2}{4} - \delta(n - 2\delta) \right\rfloor + \delta - 1.
\]

Combining this with (2), we have \( m = \left\lfloor \frac{n^2}{4} \right\rfloor - \delta(n - 2\delta) + \delta - 1 \), and so \( \lambda = \delta - 1 \), \( |X| = 2\delta \), \( |Y| = n - 2\delta \), \( |E(G[X])| = \delta^2 \) and \( |E(G[Y])| = \left\lfloor \frac{n^2}{4} \right\rfloor - \delta n + \delta^2 \). Therefore, \( G[X] = K_{\delta, \delta} \), \( G[Y] = K_{\lfloor n/2 \rfloor - \delta, \lfloor n/2 \rfloor - \delta} \) and \( ||X,Y|| = \delta - 1 \). This completes the proof. \( \square \)
3. Super edge-connected graphs

**Theorem 4.** Let $G$ be a connected graph of order $n \geq 4$, size $m$, minimum degree $\delta$ and edge-connectivity $\lambda$. If

$$m \geq \left(\frac{n}{2}\right) - \delta(n - \delta) + \delta,$$

then $G$ is super-$\lambda$, unless $\delta \geq 2$ and $G$ is a graph obtained from $K_\delta \cup K_{n-\delta}$ by adding $\delta$ edges between $K_\delta$ and $K_{n-\delta}$ such that the minimum degree of $G$ is $\delta$.

**Proof.** Suppose to the contrary that $G$ is not super-$\lambda$. Let $S$ be an arbitrary $\lambda$-cut such that each of the two components of $G - S$ has at least two vertices. Let $X$ and $Y$ denote the vertex sets of the two components of $G - S$ with $2 \leq |X| \leq |Y|$. Each vertex in $X$ is adjacent to at most $|X| - 1$ vertices of $X$, and exactly $\lambda$ edges join vertices of $X$ to vertices of $Y$. Thus

$$\delta|X| \leq \sum_{x \in X} d(x) \leq |X|(|X| - 1) + \lambda \leq |X|(|X| - 1) + \delta,$$

and so $(|X| - 1)(|X| - \delta) \geq 0$, which means that $|X| \geq \delta$. Thus $|Y| \geq |X| \geq \delta$. Therefore we have $\delta \leq |X| \leq |Y| \leq n - \delta$. Since there are no edges between $X$ and $Y$ in $G - S$, we obtain

$$m - \lambda = |E(G - S)| \leq \left(\frac{n}{2}\right) - |X| \cdot |Y|.$$

This bound together with $\delta \leq |X| \leq |Y| \leq n - \delta$ and $|X| + |Y| = n$ lead to

$$m \leq \left(\frac{n}{2}\right) - \delta(n - \delta) + \delta.$$

Combining this with (4), we have $m = \left(\frac{n}{2}\right) - \delta(n - \delta) + \delta$, and the two inequalities in the proof above must be equalities. Therefore, $\lambda = \delta$, $|X| = \delta \geq 2$, $|Y| = n - \delta$, $G[X] = K_\delta$ and $G[Y] = K_{n-\delta}$. This completes the proof. \qed

**Theorem 5.** Let $G$ be a connected triangle-free graph of order $n$, size $m$, minimum degree $\delta \geq 3$ and edge-connectivity $\lambda$. If

$$m \geq \left\lceil \frac{(n+1)^2}{4} \right\rceil - \delta(n - 2\delta + 1),$$

then $G$ is super-$\lambda$, unless $G$ is a triangle-free graph obtained from $K_{\delta,\delta-1} \cup K_{\lceil \frac{n+1}{2} \rceil - \delta, \lceil \frac{n+1}{2} \rceil - \delta}$ by adding $\delta$ edges between $K_{\delta,\delta-1}$ and $K_{\lceil \frac{n+1}{2} \rceil - \delta, \lceil \frac{n+1}{2} \rceil - \delta}$ such that the minimum degree of $G$ is $\delta$. 

Proof. Suppose to the contrary that $G$ is not super-$\lambda$. Let $S$ be an arbitrary $\lambda$-cut such that each of the two components of $G - S$ has at least two vertices. Let $X$ and $Y$ denote the vertex sets of the two components of $G - S$ with $2 \leq |X| \leq |Y|$. Using Theorem 2, we conclude that

$$|E(G[X])| \leq \left\lfloor \frac{|X|^2}{4} \right\rfloor \text{ and } |E(G[Y])| \leq \left\lfloor \frac{|Y|^2}{4} \right\rfloor,$$

(6)

with equalities if and only if $G[X] = K_{\lceil |X|/2 \rceil, \lceil |X|/2 \rceil}$ and $G[Y] = K_{\lceil |Y|/2 \rceil, \lceil |Y|/2 \rceil}$. Note that we have seen in the proof of Theorem 3 that $\delta X \leq \delta Y \leq \delta (n - \delta - 1)$.

If $|X| = 2$, then $\delta \leq 2$. Nevertheless, the hypothesis is $\delta \geq 3$, which means $|X| \geq 3$, and so $|X| \geq (2\delta - 1) - \left\lceil \frac{1}{n-1} \right\rceil \geq (2\delta - 1) - \frac{1}{2}$. Hence we have $|X| \geq 2\delta - 1$ and so $|Y| \geq |X| \geq 2\delta - 1$. Therefore $2\delta - 1 \leq |X| \leq \frac{n}{2} \leq |Y| \leq n - 2\delta + 1$. This together with $|X| + |Y| = n$ and (6) lead to

$$m = |E(G[X])| + |E(G[Y])| + \lambda$$

$$\leq \left\lfloor \frac{|X|^2}{4} \right\rfloor + \left\lfloor \frac{|Y|^2}{4} \right\rfloor + \delta = \left\lfloor \frac{|X|^2}{4} \right\rfloor + \left\lceil \frac{(n - |X|)^2}{4} \right\rceil + \delta$$

$$\leq \left\lfloor \frac{|X|^2}{4} \right\rfloor + \left\lceil \frac{(n - |X|)^2}{4} \right\rceil + \delta$$

$$= \left\lfloor \frac{n^2}{4} + \frac{1}{2}(|X|^2 - n|X|) \right\rfloor + \delta \quad \text{(as } 2\delta - 1 \leq |X| \leq \frac{n}{2})$$

$$\leq \left\lfloor \frac{n^2}{4} - \frac{1}{2}(2\delta - 1)(n - 2\delta + 1) \right\rfloor + \delta$$

$$= \left\lfloor \frac{(n + 1)^2 + 1}{4} \right\rfloor - \delta(n - 2\delta + 1)$$

$$= \left\lfloor \frac{(n + 1)^2}{4} \right\rfloor - \delta(n - 2\delta + 1).$$

Combining this with (5), we have $m = \left\lfloor \frac{(n + 1)^2}{4} \right\rfloor - \delta(n - 2\delta + 1)$, and so $\lambda = \delta$, $|X| = 2\delta - 1$, $|Y| = n - 2\delta + 1$, $|E(G[X])| = \delta(\delta - 1)$ and $|E(G[Y])| = \left\lceil \frac{(n + 1)^2}{4} \right\rceil - \delta(n + 1) + \delta^2$. Therefore $G[X] = K_{\delta, \delta}$, $G[Y] = K_{\left\lfloor \frac{n}{4} \right\rfloor - \delta, \left\lfloor \frac{n + 1}{4} \right\rfloor - \delta}$ and $|[X, Y]| = \delta$. \qed
Theorem 6. Let $G$ be a connected triangle-free graph of order $n$, size $m$, minimum degree $\delta \leq 2$ and edge-connectivity $\lambda$. If

$$m \geq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + \delta + 1,$$  \hspace{1cm} (7)

then $G$ is super-$\lambda$, unless $G$ is a triangle-free graph obtained from $K_2 \cup K_{\left\lfloor \frac{n-2}{2} \right\rfloor, \left\lceil \frac{n-2}{2} \right\rceil}$ by adding $\delta$ edge(s) between $K_2$ and $K_{\left\lfloor \frac{n-2}{2} \right\rfloor, \left\lceil \frac{n-2}{2} \right\rceil}$ such that the minimum degree of $G$ is $\delta$.

Proof. In the proof of Theorem 5, $|X|^2 - 2\delta |X| + 2\delta \geq 0$. If $\delta \leq 2$, then $|X| \geq 2$ is enough to guarantee $|X|^2 - 2\delta |X| + 2\delta \geq 0$. Thus, with the same proceeding of the proof of Theorem 5, it is easy to get

$$m \leq \left\lfloor \frac{n^2}{4} + \frac{1}{2}(|X|^2 - n|X|) \right\rfloor + \lambda \quad (\text{as } 2 \leq |X| \leq \frac{n}{2})$$

$$\leq \left\lfloor \frac{n^2}{4} - \frac{1}{2}(2n - 4) \right\rfloor + \delta$$

$$= \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + \delta + 1.$$

Combining this with (7), we have $m = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + \delta + 1$, and so $\lambda = \delta$, $|X| = 2$, $|Y| = n - 2$, $|E(G[X])| = 1$ and $|E(G[Y])| = \left\lfloor \frac{(n-2)^2}{4} \right\rfloor$. Therefore $G[X] = K_2$, $G[Y] = K_{\left\lfloor \frac{n-2}{2} \right\rfloor, \left\lceil \frac{n-2}{2} \right\rceil}$ and $|[X,Y]| = \delta$. \hfill \Box

References


