

## The locating-chromatic number for Halin graphs

I.A. Purwasih<sup>1</sup>, E.T. Baskoro<sup>1\*</sup>, H. Assiyatun<sup>1</sup>, D. Suprijanto<sup>1</sup> and  
M. Bača<sup>2</sup>

<sup>1</sup>Combinatorial Mathematics Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jalan Ganesa 10 Bandung 40132, Indonesia  
ira.apni.p@students.itb.ac.id and {ebaskoro, hilda, djoko}@math.itb.ac.id

<sup>2</sup>Department of Applied Mathematics and Informatics, Faculty of Mechanical Engineering, Technical University in Košice, Slovakia  
martin.baca@tuke.sk

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**Abstract:** Let  $G$  be a connected graph. Let  $f$  be a proper  $k$ -coloring of  $G$  and  $\Pi = \{R_1, R_2, \dots, R_k\}$  be an ordered partition of  $V(G)$  into color classes. For any vertex  $v$  of  $G$ , define the *color code*  $c_{\Pi}(v)$  of  $v$  with respect to  $\Pi$  to be a  $k$ -tuple  $(d(v, R_1), d(v, R_2), \dots, d(v, R_k))$ , where  $d(v, R_i) = \min\{d(v, x) \mid x \in R_i\}$ . If distinct vertices have distinct color codes, then we call  $f$  a *locating coloring* of  $G$ . The *locating-chromatic number* of  $G$  is the minimum number  $k$  such that  $G$  admits a locating coloring with  $k$  colors. In this paper, we determine a lower bound of the locating-chromatic number of Halin graphs. We also give the locating-chromatic number of a Halin graph of a double star.

**Keywords:** locating-chromatic number, Halin graph, double star.

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### 1. Introduction

Let  $G$  be a connected graph. Let  $f$  be a proper  $k$ -coloring of  $G$  and  $\Pi = \{R_1, R_2, \dots, R_k\}$  be an ordered partition of  $V(G)$  into color classes. For any vertex  $v$  of  $G$ , define the *color code*  $c_{\Pi}(v)$  of  $v$  with respect to  $\Pi$  to be a  $k$ -tuple  $(d(v, R_1), d(v, R_2), \dots, d(v, R_k))$ , where  $d(v, R_i) = \min\{d(v, x) \mid x \in R_i\}$ . If

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\* Corresponding Author

distinct vertices have distinct color codes, then we call  $f$  a *locating coloring* of  $G$ . The *locating-chromatic number* of  $G$ , denoted by  $\chi_L(G)$ , is the minimum number  $k$  such that  $G$  admits a locating coloring with  $k$  colors. Two distinct vertices  $x$  and  $y$  in  $G$  are *resolved* by a vertex  $z$  if  $d(x, z) \neq d(y, z)$ . Without loss of generality, if  $f(x) = t \neq f(y)$ , then the  $t^{\text{th}}$ -element of color code of  $x$  is 0 but the  $t^{\text{th}}$ -element of color code of  $y$  is at least 1. So, in this case, we only need to consider two distinct vertices in  $G$  which have same color.

The locating-chromatic number of graphs was initially studied by Chartrand *et al.* in [9]. They established some bounds on the locating-chromatic number of a graph and determined values of this parameter for some well known classes of graphs such as path, cycle, and double star. They also studied the locating-chromatic number of trees. However, the locating-chromatic number of all trees are not completely solved. Furthermore, regarding trees, the locating-chromatic number for the amalgamation of stars is given in [1], and the value for a firecracker is given in [4]. Baskoro and Asmiati [3] were able to characterize all trees with locating-chromatic number 3.

Chartrand *et al.* [10] characterized all non-tree graphs of order  $n$  with locating-chromatic number  $n - 1$ . Asmiati and Baskoro [2] gave a characterization of all non-tree graphs with locating-chromatic number 3. One well-known kind of non-tree graph is Halin graph.

Let  $T$  be a tree with no vertices of degree two and having at least four vertices. A *Halin graph*  $H(T)$  is a planar graph constructed from a plane embedding of tree  $T$  by connecting all the leaves of the tree (the vertices of degree 1) with a cycle that passes around the tree in the natural cyclic order defined by the embedding of the tree [12]. Halin graphs play a vital role in understanding the inherent complexity of the problem and a good candidate graph class between the class of trees and planar graphs. Therefore, there are many researchers, both from theoretical and application perspective, interested in studying about Halin graphs. The first result was given by Rudolf Halin [12]. He studied minimally  $n$ -connected graphs and introduced the Halin graphs as a class of minimally 3-vertex-connected graphs. Some researchers studied problems dealing with Halin graphs, including travelling salesman problem [11], hamiltonian properties [5], coloring [13].

Some researchers also considered on the locating-chromatic number of graphs produced by a graph operation, i.e., the join of graphs [7], Cartesian product [8], and corona product [6].

Based on the results given by Asmiati and Baskoro [2], we know that there is no Halin graph having locating-chromatic number 3. So, we can make a conclusion as in the following corollary.

**Corollary 1.** *Let  $T$  be a tree with no vertices of degree two and having at least 4*

vertices. Then,  $\chi_L(H(T)) \geq 4$ .

In [7], Behtoei and Anbarloei defined another new parameter which is useful in finding the locating-chromatic number of some graphs. Let  $f$  be a proper  $k$ -coloring of a graph  $G$ . If  $f(N_G(u)) \neq f(N_G(v))$  for each two distinct vertices  $u$  and  $v$  with the same color, then we say that  $f$  is a *neighbor locating coloring* of  $G$ . The *neighbor locating-chromatic number*  $\chi_{L2}(G)$  is the minimum positive integer  $k$  such that  $G$  has a neighbor locating  $k$ -coloring. They also gave the following theorem for the neighbor locating-chromatic number of paths.

**Theorem 1.** [7] For  $n \geq 2$ ,  $\chi_{L2}(P_n) = m$ , where  $m = \min\{k \mid k \in \mathbb{N}, n \leq \frac{1}{2}(k^3 - k^2)\}$ . In particular, there exists a neighbor locating  $m$ -coloring  $f$  of the path  $P_n = u_1u_2\dots u_n$  such that  $f(u_{n-1}) = 2$  and  $f(u_n) = 1$ . Additionally, for  $n \geq 9$ ,  $f(u_{n-2}) = m$ . Moreover, for  $n \geq 9$  and  $n \neq \frac{1}{2}(m^3 - m^2) - 1$ ,  $f(u_1) = 2$  and  $f(u_2) = 1$ .

For  $n \geq 2$ , let  $F_n$  be a fan on  $n+1$  vertices with the hub  $u$ , namely  $F_n = K_1 + P_n$  where  $K_1 = \{u\}$  and  $P_n$  is a path  $u_1u_2\dots u_n$ . By using the neighbor locating coloring of a path, we can construct a neighbor locating coloring of a fan. By Theorem 1, there exists a neighbor locating  $m$ -coloring  $f$  of the path  $P_n = u_1u_2\dots u_n$  where  $m = \chi_{L2}(P_n)$  such that  $f(u_{n-1}) = 2$ ,  $f(u_n) = 1$ . For  $n \geq 9$ ,  $f(u_{n-2}) = m$ . Moreover, for  $n \geq 9$  and  $n \neq \frac{1}{2}(m^3 - m^2) - 1$ ,  $f(u_1) = 2$ , and  $f(u_2) = 1$ . Now define a new coloring  $f'$  of  $F_n$  as  $f'(u_i) = f(u_i)$ , for  $1 \leq i \leq n$ , and  $f'(u) = m+1$ . Since  $f'(u) = m+1 \neq f'(u_i)$  for each  $1 \leq i \leq n$  then  $f'$  is a proper coloring of  $F_n$ . Note that  $E(F_n) = E(P_n) \cup \{u_iu \mid 1 \leq i \leq n\}$ . Hence, for each integer  $i$  with  $1 \leq i \leq n$ , we have  $f'(N_{F_n}(u_i)) = f(N_{P_n}(u_i)) \cup \{m+1\}$ . Therefore,  $f'$  is a neighbor locating  $(m+1)$ -coloring of  $F_n$  where  $f'(u_{n-1}) = 2$ ,  $f'(u_n) = 1$ ,  $f'(u) = m+1$ . For  $n \geq 9$ ,  $f'(u_{n-2}) = m$ . Moreover, for  $n \geq 9$  and  $n \neq \frac{1}{2}(m^3 - m^2) - 1$ ,  $f'(u_1) = 2$  and  $f'(u_2) = 1$ . Hence, we have the following observation as a combination of two results in [7].

**Observation 1.** For  $n \geq 2$ , let  $F_n = K_1 + P_n$  where  $K_1 = \{u\}$  and  $P_n = u_1u_2\dots u_n$ . Let  $m = \min\{k \in \mathbb{N} \mid n \leq \frac{1}{2}(k^3 - k^2)\}$ . Then,  $\chi_L(F_n) = m+1$ . In particular, there exists a neighbor locating  $(m+1)$ -coloring  $f'$  of  $F_n$  such that  $f'(u_{n-1}) = 2$ ,  $f'(u_n) = 1$ ,  $f'(u) = m+1$ . Additionally, for  $n \geq 9$ ,  $f'(u_{n-2}) = m$ . Moreover, for  $n \geq 9$  and  $n \neq \frac{1}{2}(m^3 - m^2) - 1$ ,  $f'(u_1) = 2$  and  $f'(u_2) = 1$ .

By using the neighbor locating coloring concept, Behtoei and Anbarloei were able to obtain the locating-chromatic number of wheels in the following theorem.

**Theorem 2.** [7] For  $n \geq 3$ , let  $W_n$  be a wheel of order  $n + 1$ . Let  $m = \min\{k \in \mathbb{N} \mid n \leq \frac{1}{2}(k^3 - k^2)\}$ . Then,

$$\chi_L(W_n) = \begin{cases} 1 + \chi_L(C_n), & \text{if } n < 9; \\ 1 + m, & \text{if } n \geq 9 \text{ and } n \neq \frac{1}{2}(m^3 - m^2) - 1; \\ 2 + m, & \text{if } n \geq 9 \text{ and } n = \frac{1}{2}(m^3 - m^2) - 1. \end{cases}$$

Now, we will derive a lower bound of the locating-chromatic number of a Halin graph in general. In particular, we determine the locating-chromatic number of a Halin graph of a double star.

## 2. Main Results

Let  $m$  be an integer with  $m \geq 11$ . In this section, consider a Halin graph  $H$  of order  $m$ . Let  $F_n$  be a largest fan on  $n + 1$  vertices in  $H$ . Obviously,  $n < m$ . Let  $A := \{u, u_1, u_2, \dots, u_n\}$  be the vertex set of  $F_n$  and  $f$  be a locating coloring on  $H$ . Now, let  $B = \{u_3, \dots, u_{n-2}\}$ . By the definition of  $H$ , every two distinct vertices  $x$  and  $y$  in  $B$  will have the same distance to each vertex  $z$  in  $V(H) - B - \{u_2, u_{n-1}\}$ , since  $d(x, z) = d(x, u) + d(u, z) = 1 + d(u, z) = d(y, u) + d(u, z) = d(y, z)$ . This means that any two vertices in  $B$  can be resolved only by their neighbors. The other fact that we can observe is two distinct vertices  $x \in \{u_2, u_{n-1}\}$  and  $y \in A$  can be resolved by a vertex  $z \in V(H) - A$  if and only if  $z$  has a color different from all vertices in  $A$ . If  $x \in \{u_1, u_n\}$  and  $y \in A$ , then two neighbors of  $x$  or two neighbors of  $y$  are the only vertices which can resolve  $x$  and  $y$ . Therefore, we have the following observation.

**Observation 2.** For  $n \geq 10$ , let  $F_n$  be a largest fan of order  $n + 1$  in a Halin graph  $H$ . Let  $A := \{u, u_1, u_2, \dots, u_n\}$  be the vertex set of  $F_n$  and  $f$  be a locating coloring of  $H$ . Then, we have the following facts.

1. Every two distinct vertices in  $\{u_3, \dots, u_{n-2}\}$  can only be resolved by their neighbors.
2. If  $x \in \{u_2, u_{n-1}\}$  and  $y \in A$  are resolved only by a vertex  $z \in V(H) - A$ , then  $z$  has different color with all vertices in  $A$ .
3. If  $x \in \{u_1, u_n\}$  and  $y \in A$  are resolved by a vertex  $z \in V(H) - A$ , and  $f(z) \in f(A)$ , then  $z$  is adjacent to  $x$ .

Now, we are ready to prove the following theorem.

**Theorem 3.** Let  $H$  be a Halin graph. If  $F_n$  is a largest fan in  $H$  for some  $n \geq 2$ , then  $\chi_L(H) \geq \chi_L(F_n)$ .

*Proof.* If  $n \leq 9$ , then it is obvious that  $\chi_L(H) \geq 4 \geq \chi_L(F_n)$  by the Corollary 1 and Observation 1.

Now, assume  $n \geq 10$ . Let  $A = \{u, u_1, u_2, \dots, u_n\}$  be the vertex set of the largest fan subgraph  $F_n$  of  $H$ , and  $u$  be the hub (the vertex of degree  $n$ ) of  $F_n$ . Let  $\chi_L(F_n) = t$ . By Observation 1, we have  $t = m + 1$  where  $m = \min\{k \in \mathbb{N} \mid n \leq \frac{1}{2}(k^3 - k^2)\}$ . Therefore,  $n \leq \frac{1}{2}(m^3 - m^2)$  but  $n > \frac{1}{2}((m-1)^3 - (m-1)^2)$ . So,  $n \geq \frac{1}{2}((t-2)^3 - (t-2)^2) + 1$ . Suppose on the contrary that  $\chi_L(H) = t - 1$ . Then, we have a locating  $(t - 1)$ -coloring on  $H$ , say  $f$ .

**Case 1.**  $|f(A)| = t - 1$ .

Since  $u$  is adjacent to all  $u_i$ , then  $f(u) \neq f(u_i)$  for all  $1 \leq i \leq n$ . So,  $|f(A')| = t - 2$  where  $A' = \{u_1, \dots, u_n\}$ . Since all colors are in  $f(A)$ , then by Observation 2 every two distinct vertices in  $A'$  can be only distinguished by their adjacent vertices which are not  $u$ . Hence, there are at most  $(t - 2)(t - 3)$  possible color codes of  $x \in A'$  having exactly two ordinates 1 and there are at most  $(t - 2) \binom{t - 3}{2}$  possible color codes of  $x \in A'$  having exactly three ordinates 1.

This means that there are at most  $(t - 2) \left( (t - 3) + \binom{t - 3}{2} \right) = \frac{1}{2}((t - 2)^3 - (t - 2)^2)$  distinct color codes. It contradicts the fact  $n \geq \frac{1}{2}((t - 2)^3 - (t - 2)^2) + 1$ .

**Case 2.**  $|f(A)| < t - 1$ .

Since  $u$  is adjacent to all  $u_i$ , then  $f(u) \neq f(u_i)$  for all  $1 \leq i \leq n$ . So,  $|f(A')| < t - 2$  where  $A' = \{u_1, \dots, u_n\}$ . Since all colors, except one color, are in  $f(A)$ , then by Observation 2 every two distinct vertices in  $A'' = \{u_3, u_4, \dots, u_{n-2}\}$  can be only distinguished by their adjacent vertices which are not  $u$ . Hence, there are at most  $(t - 3)(t - 4)$  possible color codes of  $x \in A''$  having exactly two ordinates 1 and there are at most  $(t - 3) \binom{t - 4}{2}$  possible color codes of  $x \in A'$  having exactly three ordinates 1. This means that there are at most  $(t - 3) \left( (t - 4) + \binom{t - 4}{2} \right) = \frac{1}{2}((t - 3)^3 - (t - 3)^2)$  distinct color codes. It contradicts the fact  $n - 4 \geq \frac{1}{2}((t - 2)^3 - (t - 2)^2) - 3$ .

Therefore,  $\chi_L(H) \geq t = \chi_L(F_n)$ .  $\square$

The lower bound in Theorem 3 is sharp for some wheels  $W_n$ , i.e. for  $n \geq 9$  and  $n \neq \frac{1}{2}(m^3 - m^2) - 1$  where  $m = \min\{k \in \mathbb{N} \mid n \leq \frac{1}{2}(k^3 - k^2)\}$ . Next, we will give the another graph that has the locating-chromatic number in this bound, i.e. a Halin graph of a double star.

Let  $S_{a,b}$  be a double star with  $V(S_{a,b}) = \{u, u_1, u_2, \dots, u_a, v, v_1, v_2, \dots, v_b\}$  and  $E(S_{a,b}) = \{uu_i \mid 1 \leq i \leq a\} \cup \{vv_i \mid 1 \leq i \leq b\} \cup \{uv\}$ . Let  $W_n$  be a wheel of order  $n + 1$ .

**Theorem 4.** Let  $a, b$  be integers such that  $a \geq b$  and  $a + b \geq 10$ . Let  $H(S_{a,b})$  be a Halin graph of a double star  $S_{a,b}$  and  $m = \min\{k \in \mathbb{N} \mid a \leq \frac{1}{2}(k^3 - k^2)\}$ . Then,

$$\chi_L(H(S_{a,b})) = \begin{cases} m + 2, & \text{if } \chi_L(F_a) < \chi_L(W_{a+b}) \text{ and } \chi_L(F_a) = \chi_L(F_b); \\ m + 1, & \text{otherwise.} \end{cases}$$

*Proof.* Let  $V(H(S_{a,b})) = V(S_{a,b})$  and  $E(H(S_{a,b})) = E(S_{a,b}) \cup \{u_i u_{i+1} \mid 1 \leq i \leq a - 1\} \cup \{v_i v_{i+1} \mid 1 \leq i \leq b - 1\} \cup \{u_1 v_b\} \cup \{u_a v_1\}$ .

**Case 1.**  $\chi_L(F_a) = \chi_L(W_{a+b})$ .

Since  $a \geq b$  and  $a + b \geq 10$  and  $m = \min\{k \in \mathbb{N} \mid a \leq \frac{1}{2}(k^3 - k^2)\}$ , by Theorem 3,  $\chi_L(H(S_{a,b})) \geq \chi_L(F_a) = m + 1$ . Now, we need to construct a locating  $(m + 1)$ -coloring of  $H(S_{a,b})$ .

**Subcase 1.1.**  $\chi_L(F_a) > \chi_L(F_b)$ .

Let  $f$  be a locating  $(m + 1)$ -coloring of  $F_a$  and  $f_1$  be a locating coloring of  $F_b$ . Without loss of generality, by Observation 1 we can choose the colorings  $f$  and  $f_1$  such that  $f(u) = m + 1$ ,  $f_1(v_1) \neq f(u_{a-1})$  and  $f_1(v_b) \neq f(u_2)$ . Now, construct a locating coloring  $f_2$  of  $H(S_{a,b})$  such that

$$f_2(x) = \begin{cases} f(u_i), & \text{if } x = u_i \text{ for } 1 \leq i \leq a; \\ f_1(v_i), & \text{if } x = v_i \text{ for } 1 \leq i \leq b; \\ f(u), & \text{if } x = u; \\ f_1(v), & \text{if } x = v. \end{cases}$$

Let  $\Pi, \Pi_1$ , and  $\Pi_2$  be the resulting partitions by the colorings  $f, f_1$ , and  $f_2$ , respectively and  $R_i$  be the  $i^{\text{th}}$  color class induced by  $f_2$ . Let  $x, y \in V(H(S_{a,b}))$  with  $f_2(x) = f_2(y)$ . Since  $\chi_L(F_a) = \chi_L(W_{a+b})$ , by coloring  $f_2$ ,  $c_{\Pi_2}(u) = c_{\Pi}(u)$  and  $c_{\Pi_2}(u_i) = c_{\Pi}(u_i)$  for all  $1 \leq i \leq a$ . For each  $x \in \{v, v_1, v_2, \dots, v_b\}$ , the  $i^{\text{th}}$ -element of  $c_{\Pi_2}(x)$  is the same as the  $i^{\text{th}}$ -element of  $c_{\Pi_1}(x)$  for every  $i$ ,  $1 \leq i \leq \chi_L(F_b)$  and the  $(m + 1)^{\text{th}}$ -element of  $c_{\Pi_2}(x)$  is 1 if  $x = v$  and 2 otherwise. So, the remaining case to be considered is  $x = u_i$  and  $y = v_j$  for  $1 \leq i \leq a$  and  $1 \leq j \leq b$ . In this case, we have  $d(x, R_{m+1}) = 1 \neq 2 = d(y, R_{m+1})$ . So,  $f_2$  is a locating coloring on  $H(S_{a,b})$ .

**Subcase 1.2.**  $\chi_L(F_a) = \chi_L(F_b)$ .

Let  $V(W_{a+b}) = \{u; u_1, u_2, \dots, u_a, v_1, v_2, \dots, v_b\}$  where  $u$  is the hub of  $W_{a+b}$ . Let  $f$  be a locating  $(m + 1)$ -coloring of  $W_{a+b}$  such that  $f(u) = m + 1$ . Let  $A = \{v_i \in V(W_{a+b}) \mid f(v_i) = m\}$ . Now, construct a locating coloring  $f_1$  for  $H(S_{a,b})$  such that

$$f_1(x) = \begin{cases} f(u_i), & \text{if } x = u_i \text{ for } 1 \leq i \leq a; \\ f(v_i), & \text{if } x = v_i \text{ for } 1 \leq i \leq b \text{ and } x \notin A; \\ m, & \text{if } x = v; \\ m + 1, & \text{otherwise.} \end{cases}$$

Let  $\Pi$  and  $\Pi_1$  be the resulting partitions by the colorings  $f$  and  $f_1$ , respectively. By the coloring  $f_1$ , vertex  $v$  becomes a dominant vertex. Now, consider vertex  $v_i$  with  $f_1(v_i) \neq m$ . For each  $j \in \{1, 2, \dots, m-1\}$ , the  $j^{\text{th}}$ -element of the color code of  $v_i$  is the same as before. The  $m^{\text{th}}$ -element of the color code of  $v_i$  under coloring  $f_1$  becomes the  $(m+1)^{\text{th}}$ -element of the color code of  $v_i$  under coloring  $f$ , and vice versa. Then all color codes of  $u_i$  and  $v_i$  are different. So,  $f_1$  is a locating coloring on  $H(S_{a,b})$ .

**Case 2.**  $\chi_L(F_a) < \chi_L(W_{a+b})$ .

**Subcase 2.1.**  $\chi_L(F_a) > \chi_L(F_b)$ .

Since  $a \geq b$  and  $a+b \geq 10$  and  $m = \min\{k \in \mathbb{N} \mid a \leq \frac{1}{2}(k^3 - k^2)\}$ , by Theorem 3,  $\chi_L(H(S_{a,b})) \geq \chi_L(F_a) = m+1$ . Now, we need to construct a locating  $(m+1)$ -coloring of  $H(S_{a,b})$ .

Let  $f$  be a locating  $(m+1)$ -coloring of  $F_a$  and  $f_1$  be a locating coloring of  $F_b$ . Without loss of generality, by Observation 1 we can choose the colorings  $f$  and  $f_1$  such that  $f(u) = m+1$ ,  $f_1(v_1) \neq f(u_{a-1})$  and  $f_1(v_b) \neq f(u_2)$ . Now, construct a locating coloring  $f_2$  of  $H(S_{a,b})$  such that

$$f_2(x) = \begin{cases} f(u_i), & \text{if } x = u_i \text{ for } 1 \leq i \leq a; \\ f_1(v_i), & \text{if } x = v_i \text{ for } 1 \leq i \leq b; \\ f(u), & \text{if } x = u; \\ f_1(v), & \text{if } x = v. \end{cases}$$

Let  $\Pi, \Pi_1$ , and  $\Pi_2$  be the resulting partitions by the colorings  $f, f_1$ , and  $f_2$ , respectively and  $R_i$  be the  $i^{\text{th}}$  color class induced by  $f_2$ . Let  $x, y \in V(H(S_{a,b}))$  with  $f_2(x) = f_2(y)$ . Since  $\chi_L(F_a) > \chi_L(F_b)$ , by coloring  $f_2$ ,  $c_{\Pi_2}(u) = c_{\Pi}(u)$  and  $c_{\Pi_2}(u_i) = c_{\Pi}(u_i)$  for  $1 \leq i \leq a$ . For each  $x \in \{v, v_1, v_2, \dots, v_b\}$ , the  $i^{\text{th}}$ -element of  $c_{\Pi_2}(x)$  is the same as the  $i^{\text{th}}$ -element of  $c_{\Pi_1}(x)$  for every  $i$ ,  $1 \leq i \leq \chi_L(F_b)$  and the  $(m+1)^{\text{th}}$ -element of  $c_{\Pi_2}(x)$  is 1 if  $x = v$  and 2 otherwise. So, the remaining case to be considered is  $x = u_i$  and  $y = v_j$  for  $1 \leq i \leq a$  and  $1 \leq j \leq b$ . In this case, we have  $d(x, R_{m+1}) = 1 \neq 2 = d(y, R_{m+1})$ . So,  $f_2$  is a locating coloring on  $H(S_{a,b})$ .

**Subcase 2.2.**  $\chi_L(F_a) = \chi_L(F_b)$ .

For the lower bound, suppose that  $\chi_L(H(S_{a,b})) = m+1$ . Then we have a locating  $(m+1)$ -coloring  $f$  of  $H(S_{a,b})$  such that  $f(u) = m+1$  and  $f(v) = m$ . Let  $A = \{v_i \in V(H(S_{a,b})) \mid f(v_i) = m+1\}$ . Now, construct a coloring  $f_1$  on  $W_{a+b}$  such that

$$f_1(x) = \begin{cases} f(u_i), & \text{if } x = u_i \text{ for } 1 \leq i \leq a; \\ f(v_i), & \text{if } x = v_i \text{ for } 1 \leq i \leq b \text{ and } x \notin A; \\ m+1, & \text{if } x = u; \\ m, & \text{otherwise.} \end{cases}$$

It is easy to check that  $f_1$  is a locating  $(m + 1)$ -coloring on  $W_{a+b}$ . This is a contradiction with the fact that  $\chi_L(F_a) < \chi_L(W_{a+b})$ .

For the upper bound, let  $f$  be a locating  $(m + 1)$ -coloring of  $F_a$  and  $f_1$  be a locating  $(m + 1)$ -coloring of  $F_b$ . Without loss of generality, by Observation 1 we can choose the colorings  $f$  and  $f_1$  such that  $f(u) = m + 1$ ,  $f_1(v_1) \neq f(u_{a-1})$  and  $f_1(v_b) \neq f(u_2)$ . Now, construct a locating coloring  $f_2$  of  $H(S_{a,b})$  such that

$$f_2(x) = \begin{cases} f(u_i), & \text{if } x = u_i \text{ for } 1 \leq i \leq a; \\ f_1(v_i), & \text{if } x = v_i \text{ for } 1 \leq i \leq b; \\ f(u), & \text{if } x = u; \\ m + 2, & \text{if } x = v. \end{cases}$$

Let  $\Pi, \Pi_1$ , and  $\Pi_2$  be the resulting partitions by the colorings  $f, f_1$ , and  $f_2$ , respectively and  $R_i$  be a  $i^{\text{th}}$  color class induced by  $f_2$ . Let  $x, y \in V(H(S_{a,b}))$  with  $f_2(x) = f_2(y)$ . By coloring  $f_2$ , for every vertex in  $V(H(S_{a,b}))$ , the  $i^{\text{th}}$ -element of  $c_{\Pi_2}(x)$  is same as the  $i^{\text{th}}$ -element of  $c_{\Pi}(x)$  or  $c_{\Pi_1}(x)$ , except on  $i = m + 2$ . So, if  $x = u_i$  and  $y = u_j$  or  $x = v_i$  and  $y = v_j$ , then  $c_{\Pi_2}(x) \neq c_{\Pi_2}(y)$ . Therefore, the remaining case to be considered is  $x = u_i$  and  $y = v_j$  for  $1 \leq i \leq a$  and  $1 \leq j \leq b$ . In this case, we have  $d(x, R_{m+2}) = 2 \neq 1 = d(y, R_{m+2})$ . So,  $f_2$  is a locating coloring on  $H(S_{a,b})$ .  $\square$

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