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Signed total Roman k-domination in directed graphs

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Abstract: Let D be a finite and simple digraph with vertex set $V(D)$. A signed total Roman k-dominating function (STRkDF) on D is a function $f : V(D) \to \{-1, 1, 2\}$ satisfying the conditions that (i) $\sum_{x \in N^-(v)} f(x) \geq k$ for each $v \in V(D)$, where $N^-(v)$ consists of all vertices of D from which arcs go into v, and (ii) every vertex u for which $f(u) = -1$ has an inner neighbor v for which $f(v) = 2$. The weight of an STRkDF f is $\omega(f) = \sum_{v \in V(D)} f(v)$. The signed total Roman k-domination number $\gamma^k_{stR}(D)$ of D is the minimum weight of an STRkDF on D. In this paper we initiate the study of the signed total Roman k-domination number of digraphs, and we present different bounds on $\gamma^k_{stR}(D)$. In addition, we determine the signed total Roman k-domination number of some classes of digraphs. Some of our results are extensions of known properties of the signed total Roman k-domination number $\gamma^k_{stR}(G)$ of graphs G.

Keywords: Digraph, Signed total Roman k-dominating function, Signed total Roman k-domination.

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1. Introduction

Let D be a finite and simple digraph with vertex set $V(D)$ and arc set $A(D)$. The integers $n = n(D) = |V(D)|$ and $m = m(D) = |A(D)|$ are the order and the size of the digraph D, respectively. We write $d_D^+(v) = d^+(v)$ for the out-degree of a vertex v and $d_D^-(v) = d^-(v)$ for its in-degree. The minimum and maximum in-degree are $\delta^-(D) = \delta^-$ and $\Delta^-(D) = \Delta^-$ and the minimum and maximum out-degree are $\delta^+(D) = \delta^+$ and $\Delta^+(D) = \Delta^+$. The sets $N_D^+(v) = N^+(v) = \{u \mid (v,u) \in A(D)\}$ and $N_D^-(v) = N^-(v) = \{u \mid (u,v) \in A(D)\}$ are called the out-neighborhood and in-neighborhood of the vertex v. Likewise, $N_D^+[v] = N^+[v] = N^+(v) \cup \{v\}$ and $N_D^-[v] = N^-[v] = N^-(v) \cup \{v\}$. If $S \subseteq V(D)$, then $D[S]$ is the subdigraph induced by S. For an arc $(u,v) \in A(D)$, the vertex v is an out-neighbor of u and u is an in-neighbor of v, and we also say that u dominates v or v is dominated by u. The underlying graph of a digraph D is that graph obtained by replacing each arc (u,v) or symmetric pairs (u,v), (v,u) of arcs by the edge uv. A digraph D is connected if its underlying graph is connected. For a real-valued function $f : V(D) \to R$, the weight of f is $\omega(f) = \sum_{v \in V(D)} f(v)$, and for $S \subseteq V(D)$, we define $f(S) = \sum_{v \in S} f(v)$, so $\omega(f) = f(V(D))$. Consult [1, 2] for notation and terminology which are not defined here.

A signed total k-dominating function on a digraph D defined in [5] is a function $f : V(D) \to \{-1,1\}$ such that $\sum_{u \in N^-(v)} f(u) \geq k$ for every $v \in V(D)$.

A signed total Roman k-dominating function (STRkDF) on D defined is a function $f : V(D) \to \{-1,1,2\}$ such that $\sum_{u \in N^-(v)} f(u) \geq k$ for every $v \in V(D)$ and every vertex u for which $f(u) = -1$ has an in-neighbor v for which $f(v) = 2$. The weight of an STRkDF f on a digraph D is $\omega(f) = \sum_{v \in V(D)} f(v)$. The signed total Roman k-domination number $\gamma_{stR}^k(D)$ of D is the minimum weight of an STRkDF on D. A $\gamma_{stR}^k(D)$-function is a signed total Roman k-dominating function on D of weight $\gamma_{stR}^k(D)$. For an STRkDF f on D, let $V_i = V_i^f = \{v \in V(D) : f(v) = i\}$ for $i = -1,1,2$. An STRkDF $f : V(D) \to \{-1,1,2\}$ can be represented by the ordered partition (V_{-1}, V_1, V_2) of $V(D)$. In the special case where $k = 1$, the signed total Roman 1-domination number is the usual signed total Roman domination number [8].

The signed total Roman k-domination number exists when $\delta^-(D) \geq \frac{k}{2}$. However, for investigations of the signed total Roman k-domination number it is reasonable to claim that $\delta^-(D) \geq k$. Thus we assume throughout this paper that $\delta^-(D) \geq k$.

Let G be a finite and simple with vertex set $V(G)$, and let $N(v) = N_G(v)$ be the neighborhood of the vertex v. A signed total k-dominating function on a graph G defined in [9] is a function $f : V(G) \to \{-1,1\}$ such that $\sum_{u \in N(v)} f(u) \geq k$ for every $v \in V(G)$. The minimum cardinality of a signed total k-dominating function is the signed total k-domination number $\gamma^k_{st}(G)$. This parameter is studied by several authors, see for example [3, 4, 10].

A signed total Roman k-dominating function (STRkDF) on a graph G defined in [6] is a function $f : V(G) \to \{-1, 1, 2\}$ such that $\sum_{u\in N_G(v)} f(u) \geq k$ for every $v \in V(G)$, and every vertex u for which $f(u) = -1$ is adjacent to a vertex v for which $f(v) = 2$. The weight of an STRkDF f on a graph G is $\omega(f) = \sum_{v\in V(G)} f(v)$. The signed total Roman k-domination number $\gamma^k_{stR}(G)$ of G is the minimum weight of an STRkDF on G. The special case $k = 1$ was introduced in [7].

In this paper, we initiate the study of the signed total Roman k-domination number in digraphs. We present different sharp lower and upper bounds on $\gamma^k_{stR}(D)$. In addition, we also determine exact values of some classes of digraphs. Some of our results imply known properties of the signed total Roman k-domination number $\gamma^k_{stR}(G)$ of graphs G given in [6].

The associated digraph $D(G)$ of a graph G is the digraph obtained from G when each edge e of G is replaced by two oppositely oriented arcs with the same ends as e. Since $N^-_{D(G)}(v) = N_G(v)$ for each vertex $v \in V(G) = V(D(G))$, the following useful observation is valid.

Observation 1. *If $D(G)$ is the associated digraph of a graph G, then $\gamma^k_{stR}(D(G)) = \gamma^k_{stR}(G)$.*

Let K_n be the complete graph of order n. In [6], the author determines the signed total Roman k-domination number of complete graphs.

Proposition 1. *[6] If $n \geq k + 2$, then $\gamma^k_{stR}(K_n) = k + 2$.*

Assume that K^*_n, complete digraph of order n, is the associated digraph $D(K_n)$ of a graph K_n. Using Observation 1 and Proposition 1, we obtain the signed total Roman k-domination number of complete digraphs.

Corollary 1. *If $n \geq k + 2$, then $\gamma^k_{stR}(K^*_n) = k + 2$.*

Let $K_{p,p}$ be the complete bipartite graph of order $2p$. In [6], the author determines the signed total Roman k-domination number of complete bipartite graphs.

Proposition 2. *[6] If $k \geq 1$ and $p \geq k$, then $\gamma^k_{stR}(K_{p,p}) = 2k$, with exception of the case that $k = 1$ and $p = 3$, in which case $\gamma^1_{stR}(K_{3,3}) = 4$.*

Assume that $K^*_{p,p}$, complete bipartite digraph of order $2p$, is the associated digraph $D(K_{p,p})$ of a graph $K_{p,p}$. Using Observation 1 and Proposition 2, we obtain the signed total Roman k-domination number of complete bipartite digraphs.

Corollary 2. *If $k \geq 1$ and $p \geq k$, then $\gamma^k_{stR}(K^*_{p,p}) = 2k$, with exception of the case that $k = 1$ and $p = 3$, in which case $\gamma^1_{stR}(K^*_{3,3}) = 4$.*

2. Bounds on the signed total Roman k-domination number

In this section, we present some sharp bounds on the signed total Roman k-domination number. We start with some preliminary results.
For an integer $p \geq 1$, a subset S of vertices of a digraph D is called a *total p-dominating set* if every vertex $v \in V(D)$ has at least p in-neighbors in S.

Proposition 3. *If $f = (V_{-1}, V_1, V_2)$ is an STRkDF on a digraph D of order n and minimum in-degree $\delta^-(D) \geq k$, then*

1. $|V_{-1}| + |V_1| + |V_2| = n$.
2. $\omega(f) = |V_1| + 2|V_2| - |V_{-1}|$.
3. *$V_1 \cup V_2$ is a total $\lceil \frac{2k}{3} \rceil$-dominating set of D.*

Proof. Since (1) and (2) are immediate, we only prove (3). Suppose to the contrary, that there exists a vertex v with at most $\lceil \frac{2k}{3} \rceil - 1$ in-neighbors in $V_1 \cup V_2$. Then v has at least

$$\delta^-(D) - (\lceil \frac{2k}{3} \rceil - 1) \geq k - (\lceil \frac{2k}{3} \rceil - 1),$$

in-neighbors in V_{-1}. It follows that

$$\begin{aligned} k \leq f(N^-(v)) &\leq 2(\lceil \frac{2k}{3} \rceil - 1) - (k - \lceil \frac{2k}{3} \rceil + 1) \\ &= 3\lceil \frac{2k}{3} \rceil - k - 3 \leq \frac{3(2k+2)}{3} - k - 3 = k - 1, \end{aligned}$$

which is a contradiction. Consequently, $V_1 \cup V_2$ is a total $\lceil \frac{2k}{3} \rceil$-dominating set of D. □

Theorem 1. *Let $k \geq 1$ be an integer, and let D be a digraph of order n with minimum in-degree $\delta^-(D) \geq k$. If $\Delta^+(D) = \Delta^+$ and $\delta^+(D) = \delta^+$, then*

1. $(2\Delta^+ - k)|V_2| + (\Delta^+ - k)|V_1| \geq (\delta^+ + k)|V_{-1}|$.
2. $(2\Delta^+ + \delta^+)|V_2| + (\Delta^+ + \delta^+)|V_1| \geq (\delta^+ + k)n$.
3. $(\Delta^+ + \delta^+)\omega(f) \geq (\delta^+ + 2k - \Delta^+)n + (\delta^+ - \Delta^+)|V_2|$.
4. $\omega(f) \geq \frac{(\delta^+ + 2k - 2\Delta^+)n}{(2\Delta^+ + \delta^+)} + |V_2|$.

Proof. (1) It follows from Proposition 3 (1) that

$$\begin{aligned} k(|V_{-1}| + |V_1| + |V_2|) = kn \leq \sum_{v \in V(D)} f(N^-(v)) &= \sum_{v \in V(D)} d^+(v) f(v) \\ &= \sum_{v \in V_2} 2d^+(v) + \sum_{v \in V_1} d^+(v) - \sum_{v \in V_{-1}} d^+(v) \\ &\leq 2\Delta^+|V_2| + \Delta^+|V_1| - \delta^+|V_{-1}|. \end{aligned}$$

This inequality chain yields to the desired bound in (1).
(2) Proposition 3 (1) implies that $|V_{-1}| = n - |V_1| - |V_2|$. Using this identity and Part (1) of Proposition 1, we arrive at (2).
(3) According to Proposition 3 and Part (2) of Proposition 1, we obtain Part (3) of Proposition 1 as follows

$$\begin{aligned} (\Delta^+ + \delta^+)\omega(f) &= (\Delta^+ + \delta^+)(2(|V_1| + |V_2|) - n + |V_2|) \\ &\geq 2(\delta^+ + k)n + 2(\Delta^+ + \delta^+)|V_2| - 2(2\Delta^+ + \delta^+)|V_2| \\ &\quad + (\Delta^+ + \delta^+)(|V_2| - n) \\ &= (\delta^+ + 2k - \Delta^+)n + (\delta^+ - \Delta^+)|V_2|. \end{aligned}$$

(4) The inequality chain in the proof of Part (1) and Proposition 3 (1) show that

$$\begin{aligned} kn &\leq 2\Delta^+|V_1 \cup V_2| - \delta^+|V_{-1}| \\ &= 2\Delta^+|V_1 \cup V_2| - \delta^+(n - |V_1 \cup V_2|) \\ &= (2\Delta^+ + \delta^+)|V_1 \cup V_2| - \delta^+ n, \end{aligned}$$

and thus

$$|V_1 \cup V_2| \geq \frac{n(\delta^+ + k)}{2\Delta^+ + \delta^+}.$$

Using this inequality and Proposition 3, we obtain

$$\begin{aligned}\omega(f) &= 2|V_1 \cup V_2| - n + |V_2| \\ &\geq \frac{2n(\delta^+ + k)}{2\Delta^+ + \delta^+} - n + |V_2| \\ &= \frac{n(\delta^+ + 2k - 2\Delta^+)}{2\Delta^+ + \delta^+} + |V_2|.\end{aligned}$$

This is the bound in Part (4), and the proof is complete. □

A digraph D is out-regular or r-out-regular if $\delta^+(D) = \Delta^+(D) = r$.

Corollary 3. *Let D be a digraph of order n with minimum in-degree $\delta^- \geq k$, minimum out-degree δ^+ and maximum out-degree Δ^+. Then*

$$\gamma^k_{stR}(D) \geq \left(\frac{2\delta^+ + 3k - 2\Delta^+}{2\Delta^+ + \delta^+}\right)n.$$

Proof. If D is an r-out-regular digraph, then result is an immediate consequence of Theorem 1 part (3). Let D be not out-regular digraph. Multiplying both sides of the inequality in Theorem 1 part (4) by $\Delta^+ - \delta^+$ and adding the resulting inequality to the inequality in Theorem 1 part (3), we obtain the desired lower bound. □

Corollary 3 and Observation 1 lead to the next known result.

Corollary 4. *[6] Let G be a graph of order n, minimum degree $\delta \geq k$ and maximum degree Δ. If $\Delta > \delta$, then*

$$\gamma^k_{stR}(G) \geq \left(\frac{2\delta + 3k - 2\Delta}{2\Delta + \delta}\right)n.$$

The special case $k = 1$ of Corollary 4 can be found in [7]. Example 12 in [6] demonstrates that Corollary 4 is sharp. This example together with Observation 1 shows that Corollary 3 is sharp too.

Proposition 4. *If D is a digraph of order n with minimum in-degree $\delta^- \geq k$, then $\gamma^k_{stR}(D) \leq n$.*

Proof. Define the function $f : V(D) \to \{-1, 1, 2\}$ by $f(v) = 1$ for each vertex $v \in V(D)$. Since $\delta^- \geq k$, the function f is an STRkDF on D of weight n and thus $\gamma^k_{stR}(D) \leq n$. □

A digraph D is r-regular if $\Delta^-(D) = \Delta^+(D) = \delta^-(D) = \delta^+(D) = r$.

Example 1. If D is a k-regular digraph of order n, then it follows from Corollary 3 that $\gamma^k_{stR}(D) \geq n$ and so $\gamma^k_{stR}(D) = n$, according to Proposition 4.

Example 1 demonstrates that Proposition 4 and Corollary 3 are both sharp. If $\delta^- \geq k+2$, then we can improve the bound in Proposition 4.

Theorem 2. *If D is a digraph of order n with minimum in-degree $\delta^- \geq k+2$, then*

$$\gamma^k_{stR}(D) \leq n + 1 - 2\lfloor \frac{\delta^- - k}{2} \rfloor.$$

Proof. Define $t = \lfloor \frac{\delta^- - k}{2} \rfloor$. Since

$$n \cdot \Delta^+ \geq \sum_{u \in V(D)} d^+(u) = \sum_{u \in V(D)} d^-(u) \geq n \cdot \delta^-,$$

we observe that $\Delta^+ \geq \delta^- \geq t$. Let $v \in V(D)$ be a vertex of maximum out-degree, and let $A = \{u_1, u_2, \ldots, u_t\}$ be a set of t out-neighbors of v. Define the function $f : V(D) \to \{-1, 1, 2\}$ by $f(v) = 2$, $f(u_i) = -1$ for $1 \leq i \leq t$ and $f(w) = 1$ for $w \in V(D) - (A \cup \{v\})$. Then

$$f(N(x)) \geq -t + (\delta^- - t) = \delta^- - 2t = \delta^- - 2\lfloor \frac{\delta^- - k}{2} \rfloor \geq k,$$

for each vertex $x \in V(D)$. Therefore f is an STRkDF on D of weight $2 - t + (n - t - 1) = n + 1 - 2t$ and thus $\gamma^k_{stR}(D) \leq n + 1 - 2t = n + 1 - 2\lfloor \frac{\delta^- - k}{2} \rfloor$. □

Corollary 5. *If D is a digraph of order n with minimum in-degree $\delta^- \geq k+2$, then $\gamma^k_{stR}(D) \leq n - 1$.*

Corollary 5 implies that $\gamma^k_{stR}(D) \leq n(D) - 1$ when $\delta^-(D) \geq k+2$. Example 1 shows that $\gamma^k_{stR}(D) = n(D)$ is possible when $\delta^-(D) = k$. By Corollary 1, we have $\gamma^{n-2}_{stR}(K^*_n) = n$ and hence $\gamma^k_{stR}(D) = n(D)$ is also possible for $\delta^-(D) = k+1$. Consequently, $\gamma^k_{stR}(D) \leq n(D) - 1$ is not valid in general when $k \leq \delta^-(D) \leq k+1$.
Let K^*_n be the complete digraph. If $n \geq k+3$ and $n - k - 1$ is even, then it follows from Corollary 1 that

$$\gamma^k_{stR}(K^*_n) = k + 2 = n + 1 - 2\left\lfloor \frac{\delta^-(K^*_n) - k}{2} \right\rfloor,$$

and therefore the bound given in Theorem 2 is sharp.

Proposition 5. *If D is a digraph of order n with minimum in-degree $\delta^-(D) \geq k$, then $\gamma^k_{stR}(D) \geq k + \Delta^-(D) - n$.*

Proof. Let $v \in V(D)$ be a vertex of maximum in-degree, and f be a $\gamma^k_{stR}(D)$-function. Then the definitions imply

$$\begin{aligned}\gamma^k_{stR}(D) &= \sum_{u\in V(D)} f(u) = \sum_{u\in N^-(v)} f(u) + \sum_{u\in V(D)-N^-(v)} f(u)\\ &\geq k + \sum_{u\in V(D)-N^-(v)} f(u) \geq k - (n - \Delta^-(D)) = k + \Delta^-(D) - n,\end{aligned}$$

and the proof is complete. □

Example 2. Let $k \geq 2$ and $r \geq 1$ be integers such that $k \geq r$, and D be a digraph obtained from a complete digraph of order k with vertex set $V(K^*_k) = \{u_i \mid 1 \leq i \leq k\}$ by adding the set $\{v_j, w_t \mid 1 \leq j \leq k \text{ and } 1 \leq t \leq r\}$ of new vertices and the set

$$\{(u_i, v_j), (u_i, w_t), (w_t, v_j) \mid 1 \leq i \leq k,\ 1 \leq j \leq k \text{ and } 1 \leq t \leq r\},$$

of new arcs. It is easy to see that the function $f : V(D) \to \{-1, 1, 2\}$ defined by $f(u_i) = 2$ for $1 \leq i \leq k$ and $f(x) = -1$ otherwise, is an STRkDF of D and so $\gamma^k_{stR}(D) \leq k - r$. By Proposition 5, we have

$$\gamma^k_{stR}(D) \geq k + \Delta^-(D) - n = k + 2k - (r + 2k) = k - r.$$

Proposition 6. *If D is a digraph of order $n \geq k + 2$ with minimum in-degree $\delta^-(D) \geq k$, then $\gamma^k_{stR}(D) \geq k + 3 + \delta^-(D) - n$.*

Proof. Let f be a $\gamma^k_{stR}(D)$-function. If $f(u) = 1$ for all $u \in V(D)$, then $\gamma^k_{stR}(D) = n \geq k + 3 + \delta^-(D) - n$. Now assume that there exists a vertex w with $f(w) = -1$. Then w has an in-neighbor v with $f(v) = 2$, and it follows that

$$\begin{aligned}\gamma^k_{stR}(D) &= \sum_{u\in V(D)} f(u) = f(v) + \sum_{u\in N^-(v)} f(u) + \sum_{u\in V(D)-N^-[v]} f(u)\\ &\geq 2 + k + \sum_{u\in V(D)-N^-[v]} f(u) \geq 2 + k - (n - d^-(v) - 1)\\ &\geq k + 3 + \delta^-(D) - n,\end{aligned}$$

and the proof is complete. □

Corollary 1 shows that Proposition 6 is sharp.
Now we show that the signed total Roman k-domination of digraphs can be arbitrary small.

Theorem 3. *For any positive integer $t \geq 1$, there exists a digraph D such that*

$$\gamma^k_{stR}(D) = -t.$$

Proof. Let $k \geq 1$ be an integer and D be a digraph obtained from a complete digraph of order $k+1$ with vertex set $V(K^*_{k+1}) = \{u_i \mid 1 \leq i \leq k+1\}$ by adding the set $\{v_j \mid 1 \leq j \leq t+k+2\}$ of new vertices and the set

$$\{(u_i, v_j) \mid 1 \leq i \leq k \text{ and } 1 \leq j \leq t+k+2\},$$

of new arcs. It is easy to see that the function $f : V(D) \to \{-1, 1, 2\}$ defined by $f(u_1) = 2$, $f(u_i) = 1$ for $2 \leq i \leq k+1$ and $f(x) = -1$ otherwise, is an STRkDF of D of weight $-t$ and so $\gamma^k_{stR}(D) \leq -t$. By Proposition 6, we have

$$\gamma^k_{stR}(D) \geq k+3+\delta^-(D) - n = k+3+k-(t+2k+3) = -t.$$

This completes the proof. □

We call a set $S \subseteq V(D)$ a 2-packing of the digraph D if $N^-[u] \cap N^-[v] = \varnothing$ for any two distinct vertices of $u, v \in S$. The maximum cardinality of a 2-packing in D is the 2-packing number of D, denoted by $\rho(D)$.

Theorem 4. *If D is a digraph of order n such that $\delta^-(D) \geq k$, then $\gamma^k_{stR}(D) \geq \rho(D)(\delta^-(D)+k) - n$.*

Proof. Let $\{v_1, v_2, \ldots, v_{\rho(D)}\}$ be a 2-packing of D, and f be a $\gamma^k_{stR}(D)$-function. If we define the set $A = \bigcup_{i=1}^{\rho(D)} N^-(v_i)$ then, since $\{v_1, v_2, \ldots, v_{\rho(D)}\}$ is a 2-packing of D, we have

$$|A| = \sum_{i=1}^{\rho(D)} d^-(v_i) \geq \delta^-(D) \cdot \rho(D).$$

It follows that

$$\begin{aligned}
\gamma^k_{stR}(D) &= \sum_{u \in V(D)} f(u) = \sum_{i=1}^{\rho(D)} f(N^-(v_i)) + \sum_{u \in V(D)-A} f(u) \\
&\geq k\rho(D) + \sum_{u \in V(D)-A} f(u) \geq k\rho(D) - n + |A| \\
&\geq k\rho(D) - n + \rho(D) \cdot \delta^-(D) = \rho(D)(\delta^- + k) - n.
\end{aligned}$$

□

Let n be an odd positive integer such $n = 2r + 1$ with a positive integer r. We define the circulant tournament $CT(n)$ with n vertices as follows. The vertex set of $CT(n)$ is $V(CT(n)) = \{u_0, u_1, ..., u_{n-1}\}$. For each i, the arcs are going from u_i to the vertices $u_{i+1}, u_{i+2}, \ldots, u_{i+r}$, where the indices are taken modulo n.
In [8], the author determines the signed total Roman domination number of circulant tournament $CT(n)$.

Proposition 7. *[8] Let $n = 2r+1$ with an integer $r \geq 1$. Then $\gamma_{stR}(CT(3)) = 3$, $\gamma_{stR}(CT(7)) = 5$ and $\gamma_{stR}(CT(n)) = 4$ for $n \geq 5$ with $n \neq 7$.*

We obtain the signed total Roman k-domination number of circulant tournament $CT(n)$ when $k \geq 2$.

Theorem 5. *Let $n = 2r + 1$ with an integer $r \geq k \geq 2$. Then $\gamma^k_{stR}(CT(n)) = n$ for $r = k$ and $\gamma^k_{stR}(CT(n)) = 2k + 2$ when $r > k$.*

Proof. According to Proposition 4, $\gamma^k_{stR}(CT(n)) \leq n$. First let $r = k$ and f be a $\gamma^k_{stR}(CT(n))$-function. If $f(u) = 1$ for each $u \in V(CT(n))$, then $\omega(f) = n$. Thus let $u \in V(CT(n))$ such that $f(u) = -1$. Therefore there exists a vertex, say u_r, such that $f(u_r) = 2$. Consider the sets $N^-(u_0) = \{u_{r+1}, u_{r+2}, \ldots, u_{2r}\}$ and $N^-(u_r) = \{u_0, u_1, \ldots, u_{r-1}\}$. Since f is an STRkDF on $CT(n)$, we deduce that

$$\omega(f) = f(N^-(u_0)) + f(N^-(u_r)) + f(u_r) \geq k + k + 2 = 2k + 2 > 2k + 1 = n,$$

which is a contradiction. Hence $\gamma^k_{stR}(CT(n)) = n = 2k + 1$ when $r = k$.
Now let $r > k$ and f be a $\gamma^k_{stR}(CT(n))$-function. If $f(u) = 1$ for each $u \in V(CT(n))$, then $\omega(f) = n > 2k + 2$ when $r > k$. Thus assume that $f(u) = -1$ for a vertex $u \in V(CT(n))$. Then there exists a vertex, say u_r, such that $f(u_r) = 2$. Consider the sets $N^-(u_0) = \{u_{r+1}, u_{r+2}, \ldots, u_{2r}\}$ and $N^-(u_r) = \{u_0, u_1, \ldots, u_{r-1}\}$. As f is an STRkDF on $CT(n)$, we deduce that

$$\omega(f) = f(N^-(u_0)) + f(N^-(u_r)) + f(u_r) \geq k + k + 2 = 2k + 2.$$

Consequently, $\gamma^k_{stR}(CT(n)) \geq 2k+2$ when $r > k$. Since $r > k \geq 2$, then $n \geq 7$. To prove the equality $\gamma^k_{stR}(CT(n)) = 2k + 2$ for $n \geq 7$ and $r > k$, we consider two cases.
Case 1. Let r be even. We consider the following subcases.

Subcase 1.1. $k \equiv 0$ (mode 4).
Define the function $g : V(CT(n)) \to \{-1, 1, 2\}$ as follows

$$g(u_i) = \begin{cases} 2 \text{ if } & i = 0,\ 1 \le i \le \frac{k}{2} \text{ or } r+1 \le i \le r + \frac{k}{2}, \\ \text{-1 if } & \frac{k}{2}+1 \le i \le \frac{r}{2} + \frac{k}{4} \text{ or } r + \frac{k}{2} + 1 \le i \le \frac{3r}{2} + \frac{k}{4}, \\ 1 \text{ if } & \frac{r}{2} + \frac{k}{4} + 1 \le i \le r \text{ or } \frac{3r}{2} + \frac{k}{4} + 1 \le i \le 2r. \end{cases}$$

Obviously, g is an STRkDF on $CT(n)$ of weight $2k+2$ and thus $\gamma^k_{stR}(CT(n)) \le 2k+2$ when $k \equiv 0$ (mode 4).
Subcase 1.2. $k \equiv 1$ (mode 4).
Define the function $g : V(CT(n)) \to \{-1, 1, 2\}$ as follows

$$g(u_i) = \begin{cases} 2 \text{ if } & i = 0,\ 1 \le i \le \frac{k+1}{2} \text{ or } r+1 \le i \le r + \frac{k+1}{2}, \\ \text{-1 if } & \frac{k+1}{2}+1 \le i \le \frac{r}{2} + \frac{k+3}{4} \text{ or } r + \frac{k+1}{2} + 1 \le i \le \frac{3r}{2} + \frac{k+3}{4}, \\ 1 \text{ if } & \frac{r}{2} + \frac{k+3}{4} + 1 \le i \le r \text{ or } \frac{3r}{2} + \frac{k+3}{4} + 1 \le i \le 2r. \end{cases}$$

Obviously, g is an STRkDF on $CT(n)$ of weight $2k+2$ and thus $\gamma^k_{stR}(CT(n)) \le 2k+2$ when $k \equiv 1$ (mode 4).
Subcase 1.3. $k \equiv 2$ (mode 4).
Define the function $g : V(CT(n)) \to \{-1, 1, 2\}$ as follows

$$g(u_i) = \begin{cases} 2 \text{ if } & i = 0,\ 1 \le i \le \frac{k}{2}+1 \text{ or } r+1 \le i \le r + \frac{k}{2}+1, \\ \text{-1 if } & \frac{k}{2}+2 \le i \le \frac{r}{2} + \frac{k+2}{4}+1 \text{ or } r + \frac{k}{2} + 2 \le i \le \frac{3r}{2} + \frac{k+2}{4}+1, \\ 1 \text{ if } & \frac{r}{2} + \frac{k+2}{4} + 2 \le i \le r \text{ or } \frac{3r}{2} + \frac{k+2}{4} + 2 \le i \le 2r. \end{cases}$$

Obviously, g is an STRkDF on $CT(n)$ of weight $2k+2$ and thus $\gamma^k_{stR}(CT(n)) \le 2k+2$ when $k \equiv 2$ (mode 4).
Subcase 1.4. $k \equiv 3$ (mode 4).
Define the function $g : V(CT(n)) \to \{-1, 1, 2\}$ as follows

$$g(u_i) = \begin{cases} 2 \text{ if } & i = 0,\ 1 \le i \le \frac{k-1}{2} \text{ or } r+1 \le i \le r + \frac{k-1}{2}, \\ \text{-1 if } & \frac{k-1}{2}+1 \le i \le \frac{r}{2} + \frac{k+1}{4} - 1 \\ & \quad \text{or } r + \frac{k-1}{2} + 1 \le i \le \frac{3r}{2} + \frac{k+1}{4} - 1, \\ 1 \text{ if } & \frac{r}{2} + \frac{k+1}{4} \le i \le r \text{ or } \frac{3r}{2} + \frac{k+1}{4} \le i \le 2r. \end{cases}$$

Obviously, g is an STRkDF on $CT(n)$ of weight $2k+2$ and thus $\gamma^k_{stR}(CT(n)) \le 2k+2$ when $k \equiv 3$ (mode 4). Then $\gamma^k_{stR}(CT(n)) = 2k+2$ when r is even.
Case 2. Let r be odd. We consider the following subcases.
Subcase 2.1. $k \equiv 0$ (mode 4).
Define the function $g : V(CT(n)) \to \{-1, 1, 2\}$ as follows

$$g(u_i) = \begin{cases} 2 \text{ if } & i = 0,\ 1 \le i \le \frac{k}{2}-1 \text{ or } +1 \le i \le r + \frac{k}{2}-1, \\ \text{-1 if } & \frac{k}{2} \le i \le \frac{r-1}{2} + \frac{k}{4} - 1 \text{ or } r + \frac{k}{2} \le i \le \frac{3r-1}{2} + \frac{k}{4} - 1, \\ 1 \text{ if } & \frac{r-1}{2} + \frac{k}{4} \le i \le r \text{ or } \frac{3r-1}{2} + \frac{k}{4} \le i \le 2r. \end{cases}$$

Obviously, g is an STRkDF on $CT(n)$ of weight $2k+2$ and thus $\gamma^k_{stR}(CT(n)) \leq 2k+2$ when $k \equiv 0$ (mode 4).

Subcase 2.2. $k \equiv 1$ (mode 4).

Define the function $g : V(CT(n)) \to \{-1, 1, 2\}$ as follows

$$g(u_i) = \begin{cases} 2 & \text{if } \ i = 0,\ 1 \leq i \leq \frac{k-1}{2} \text{ or } r+1 \leq i \leq r + \frac{k-1}{2}, \\ \text{-1} & \text{if } \ \frac{k-1}{2} + 1 \leq i \leq \frac{r-1}{2} + \frac{k-1}{4} \text{ or } r + \frac{k-1}{2} + 1 \leq i \leq \frac{3r-1}{2} + \frac{k-1}{4}, \\ 1 & \text{if } \ \frac{r-1}{2} + \frac{k-1}{4} + 1 \leq i \leq r \text{ or } \frac{3r-1}{2} + \frac{k-1}{4} + 1 \leq i \leq 2r. \end{cases}$$

Obviously, g is an STRkDF on $CT(n)$ of weight $2k+2$ and thus $\gamma^k_{stR}(CT(n)) \leq 2k+2$ when $k \equiv 1$ (mode 4).

Subcase 2.3. $k \equiv 2$ (mode 4).

Define the function $g : V(CT(n)) \to \{-1, 1, 2\}$ as follows

$$g(u_i) = \begin{cases} 2 & \text{if } \ i = 0,\ 1 \leq i \leq \frac{k}{2} \text{ or } r+1 \leq i \leq r + \frac{k}{2}, \\ \text{-1} & \text{if } \ \frac{k}{2} + 1 \leq i \leq \frac{r-1}{2} + \frac{k+2}{4} \text{ or } r + \frac{k}{2} + 1 \leq i \leq \frac{3r-1}{2} + \frac{k+2}{4}, \\ 1 & \text{if } \ \frac{r-1}{2} + \frac{k+2}{4} + 1 \leq i \leq r \text{ or } \frac{3r-1}{2} + \frac{k+2}{4} + 1 \leq i \leq 2r. \end{cases}$$

Obviously, g is an STRkDF on $CT(n)$ of weight $2k+2$ and thus $\gamma^k_{stR}(CT(n)) \leq 2k+2$ when $k \equiv 2$ (mode 4).

Subcase 2.4. $k \equiv 3$ (mode 4).

Define the function $g : V(CT(n)) \to \{-1, 1, 2\}$ as follows

$$g(u_i) = \begin{cases} 2 & \text{if } \ i = 0,\ 1 \leq i \leq \frac{k+1}{2} \text{ or } r+1 \leq i \leq r + \frac{k+1}{2}, \\ \text{-1} & \text{if } \ \frac{k+1}{2} + 1 \leq i \leq \frac{r-1}{2} + \frac{k+1}{4} + 1 \\ & \quad \text{or } r + \frac{k+1}{2} + 1 \leq i \leq \frac{3r-1}{2} + \frac{k+1}{4} + 1, \\ 1 & \text{if } \ \frac{r-1}{2} + \frac{k+1}{4} + 2 \leq i \leq r \text{ or } \frac{3r-1}{2} + \frac{k+!}{4} + 2 \leq i \leq 2r. \end{cases}$$

Obviously, g is an STRkDF on $CT(n)$ of weight $2k+2$ and thus $\gamma^k_{stR}(CT(n)) \leq 2k+2$ when $k \equiv 3$ (mode 4). Then $\gamma^k_{stR}(CT(n)) = 2k+2$ when r is odd and this completes the proof. □

The complement $\overline{D}$ of a digraph D is the digraph with vertex set $V(D)$ such that for any two distinct vertices u and v the arc (u, v) belongs to $\overline{D}$ if and only if (u, v) does not belong to D. Finally, we present a so called Nordhaus-Gaddum type inequality for the signed total Roman k-domination number of regular digraphs.

Theorem 6. *If D is an r-regular digraph of order n such that $r \geq k$ and $n-r-1 \geq k$, then*

$$\gamma^k_{stR}(D) + \gamma^k_{stR}(\overline{D}) \geq \frac{4kn}{n-1}.$$

If n is even, then $\gamma^k_{stR}(D) + \gamma^k_{stR}(\overline{D}) \geq \frac{4k(n-1)}{n-2}$.

Proof. Since D is r-regular, the complement $\overline{D}$ is $(n-r-1)$-regular. Therefore it follows from Corollary 3 that

$$\gamma^k_{stR}(D) + \gamma^k_{stR}(\overline{D}) \geq kn(\frac{1}{r} + \frac{1}{n-r-1}).$$

The conditions $r \geq k$ and $n-r-1 \geq k$ imply that $k \leq r \leq n-k-1$. As the function $f(x) = \frac{1}{x} + \frac{1}{n-x-1}$ has its minimum for $x = \frac{(n-1)}{2}$ when $k \leq x \leq n-k-1$, we obtain

$$\gamma^k_{stR}(D) + \gamma^k_{stR}(\overline{D}) \geq kn(\frac{1}{r} + \frac{1}{n-r-1}) \geq kn(\frac{2}{n-1} + \frac{2}{n-1}) = \frac{4kn}{n-1},$$

and this is the desired bound. If n is even, then the function f has its minimum for $r = x = \frac{n-2}{2}$ or $r = x = \frac{n}{2}$, since r is an integer. Hence this case leads to

$$\gamma^k_{stR}(D) + \gamma^k_{stR}(\overline{D}) \geq kn(\frac{1}{r} + \frac{1}{n-r-1}) \geq kn(\frac{2}{n} + \frac{2}{n-2}) = \frac{4k(n-1)}{n-2},$$

and the proof is complete. □

Let $k \geq 2$ be an even integer, and D and $\overline{D}$ be k-regular digraphs of order $n = 2k+1$. By Example 1, we have $\gamma^k_{stR}(D) = \gamma^k_{stR}(\overline{D}) = n$. Consequently,

$$\gamma^k_{stR}(D) + \gamma^k_{stR}(\overline{D}) = 2n = \frac{4kn}{n-1}.$$

Thus the Nordhaus-Gaddum bound of Theorem 6 is sharp for even k.

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