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Twin minus domination numbers in directed graphs

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Abstract: Let $D = (V, A)$ be a finite simple directed graph. A function $f : V \longrightarrow \{-1, 0, 1\}$ is called a twin minus dominating function if $f(N^-[v]) \geq 1$ and $f(N^+[v]) \geq 1$ for each vertex $v \in V$. The twin minus domination number of D is $\gamma_-^*(D) = \min\{w(f) \mid f$ is a twin minus dominating function of $D\}$. In this paper, we initiate the study of twin minus domination numbers in digraphs and present some lower bounds for $\gamma_-^*(D)$ in terms of the order, size and maximum and minimum in-degrees and out-degrees.

Keywords: Twin domination in digraphs, minus domination in graphs, twin minus domination in digraphs.

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1. Introduction

In this paper, D is a finite simple directed graph (digraph) with vertex set V and arc set A. The integers $n = n(D) = |V|$ and $m = m(D) = |A|$ are the order and size of D. A digraph without directed cycles of length 2 is an *oriented digraph*. We write $d_D^+(v) = d^+(v)$ for the out-degree of a vertex v

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and $d_D^-(v) = d^-(v)$ for its in-degree. The *minimum* and *maximum in-degrees* and *minimum* and *maximum out-degrees* of D are denoted by $\delta^-(D) = \delta^-$, $\Delta^-(D) = \Delta^-$, $\delta^+(D) = \delta^+$ and $\Delta^+(D) = \Delta^+$, respectively. If (u, v) is an arc of D, then we also write $u \to v$, and we say that v is an *out-neighbor* of u and u is an *in-neighbor* of v, and we also say that u dominates v or v is dominated by u. The sets $N^-(v) = N_D^-(v) = \{x \mid (x, v) \in A(D)\}$ and $N^+(v) = N_D^+(v) = \{x \mid (v, x) \in A(D)\}$ are called the in-neighborhood and out-neighborhood of the vertex v. Likewise, $N_D^-[v] = N^-[v] = N^-(v) \cup \{v\}$ and $N_D^+[v] = N^+[v] = N^+(v) \cup \{v\}$. If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by X. If $X \subseteq V(D)$ and $v \in V(D)$, then $A(X, v)$ is the set of arcs from X to v. We denote by $A(X, Y)$ the set of arcs from a subset X to a subset Y. The notation D^{-1} is used for the digraph obtained from D by reversing the arcs of D. The underlying graph of a digraph D is the graph obtained from D by removing the direction of each arc. The complete digraph of order n, K_n^*, is a digraph D such that $(u, v), (v, u) \in A(D)$ for any two distinct vertices $u, v \in V(D)$. For a real-valued function $f : V(D) \longrightarrow R$ the weight of f is $w(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$, we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V)$. Consult [15] for the notation and terminology which are not defined here.

Let $D = (V, A)$ be a finite simple digraph. A subset S of V is called a *dominating set* of D if every vertex in $V \setminus S$ has an in-neighbor in S. The minimum cardinality of a dominating set in D, denoted by $\gamma(D)$, is called the domination number of D. A dominating set of size $\gamma(D)$ is called a $\gamma(D)$-set. A dominating set of D is called a *twin dominating set* if it also is a dominating set of D^{-1}. The minimum size of a twin dominating set of D is denoted by $\gamma^*(D)$. This concept of domination was introduced by Chartrand et al. [8], and has been studied by several authors [2]. Similarly, the concept of twin Roman domination was studied in [1].

A *Signed Dominating Function* (SDF) of D is a function $f : V \to \{-1, 1\}$ such that $f(N^-[v])] \geq 1$ for every $v \in V$. The signed domination number of a digraph D is

$$\gamma_s(D) = \min\{w(f) \mid f \text{ is a SDF of } D\}.$$

A $\gamma_s(D)$-function is a SDF of D of weight $\gamma_s(D)$. The signed domination number of a digraph was introduced by Zelinka in [17], and has been studied by several authors [4, 11].

Recently, Atapour et al. [6] studied the twin signed domination numbers in digraphs. A signed dominating function of a digraph D is called a *Twin Signed Dominating Function* (TSDF) if it also is a signed dominating function of D^{-1}, i.e., $f(N^+[v]) \geq 1$ for every $v \in V$. The twin signed domination number of a digraph D is $\gamma_s^*(D) = \min\{w(f) \mid f \text{ is an TSDF of } D\}$.

A *Minus Dominating Function* (MDF) of D is a function $f : V \to \{-1, 0, 1\}$ such that $f(N^-[v])] \geq 1$ for every $v \in V$. The minus domination number for a digraph D is

$$\gamma_-(D) = \min\{w(f) \mid f \text{ is a MDF of } D\}.$$

A $\gamma_-(D)$-function is an MDF of D of weight $\gamma_-(D)$. The minus domination number of a digraph was introduced by Li and Xing in [12]. We define a Twin Minus Dominating Function (TMDF) of D as a minus dominating function of both D and D^{-1}, i.e., $f(N^-[v]) \geq 1$ and $f(N^+[v]) \geq 1$ for every $v \in V$. The twin minus domination number for a digraph D is $\gamma_-^*(D) = \min\{w(f) \mid f$ is an TMDF of $D\}$. an TMDF of a digraph D, of weight $\gamma_-^*(D)$ is called a $\gamma_-^*(D)$- function.

Let G be a graph with vertex V and edge set E. For every vertex $v \in V$, the open neighborhood of v, $N(v)$ is the set $\{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. A *minus dominating* function of G, introduced by Dunbar et al. [10], is a function $f : V \to \{-1, 0, 1\}$ such that $f(N[v]) \geq 1$ for every $v \in V$. The *minus domination number* of G, denoted by $\gamma_-(G)$, is the minimum weight of a minus dominating function on G. The minus domination in graphs was studied by several authors for example [5, 13, 14, 16].

For any function $f : V(D) \to \{-1, 0, 1\}$, on a digraph D, we define $P = P_f = \{v \in V \mid f(v) = 1\}$, $Z = Z_f = \{v \in V \mid f(v) = 0\}$ and $M = M_f = \{v \in V \mid f(v) = -1\}$. Since every TMDF of D is a MDF on both D and D^{-1} and since the constant function 1 is an TMDF of D, we have

$$\max\{\gamma_-(D), \gamma_-(D^{-1})\} \leq \gamma_-^*(D) \leq n. \tag{1}$$

Let S be a twin dominating set in a digraph D. Then the function $f : V(D) \to \{-1, 0, 1\}$ that assigns $+1$ to every vertex in S and 0 to the others, is an TMDF of D. On the other hand, every TSDF of D is an TMDF and so we have

$$\gamma_-^*(D) \leq \min\{\gamma^*(D), \gamma_s^*(D)\}. \tag{2}$$

In this paper, we initiate the study of the twin minus domination number in digraphs and we present some lower bounds on this parameter.

We make use of the following results and remarks in this paper.

Theorem 1. *[2]* For the directed path $\overrightarrow{P_n}$, $n \geq 2$, $\gamma^*(\overrightarrow{P_n}) = \lceil \frac{n+1}{2} \rceil$ and for a directed cycle $\overrightarrow{C}_n$, with $n \geq 3$ vertices, $\gamma^*(\overrightarrow{C}_n) = \lceil \frac{n}{2} \rceil$.

Theorem 2. *[12] For any directed cycle $\overrightarrow{C_n}$, $n \geq 3$, $\gamma_-(\overrightarrow{C_n}) = \lceil \frac{n}{2} \rceil$.*

Next result is an immediate consequence of Propositions 1 and 2, and inequalities (1) and (2).

Corollary 1. *If $\overrightarrow{P_n}$ and $\overrightarrow{C_n}$ are directed path and cycle on n vertices, then $\gamma^*_-(\overrightarrow{P_n}) = \lceil \frac{n+1}{2} \rceil$ and $\gamma^*_-(\overrightarrow{C_n}) = \lceil \frac{n}{2} \rceil$.*

2. Basic properties

In this section, we present basic properties of the twin minus domination number.

Proposition 1. *Let D be a digraph of order n. Then $\gamma^*_-(D) = n$ if and only if $d^+(v) = 0$ or $d^-(v) = 0$ for every vertex $v \in V(D)$.*

Proof. The sufficiency is clear. Thus, we only verify the necessity of the condition. Assume, to the contrary, $d^-(v) \geq 1$ and $d^+(v) \geq 1$ for some $v \in V(D)$. Define $f : V(D) \to \{-1, 0, 1\}$ by $f(v) = 0$ and $f(x) = 1$ for $x \in V(D) \setminus \{v\}$. Obviously, f is an TMDF of D of weight less than n, a contradiction. This completes the proof. □

As we observed in (1), $\gamma^*_-(D) \geq \max\{\gamma_-(D), \gamma_-(D^{-1})\}$. Now we show that the difference $\gamma^*_-(D) - \max\{\gamma_-(D), \gamma_-(D^{-1})\}$ can be arbitrarily large.

Theorem 3. *For every positive integer k, there exists a digraph D such that*

$$\gamma^*_-(D) - \max\{\gamma_-(D), \gamma_-(D^{-1})\} \geq 4k - 4.$$

Proof. Let $k \geq 1$ be an integer and D be a digraph with vertex set $V(D) = \{x, y, u_i, v_i \mid 1 \leq i \leq 2k\}$ and edge set

$$A(D) = \{(x, u_i), (u_{k+i}, x), (y, v_{k+i}), (v_i, y), (v_i, u_i), (u_{k+i}, v_{k+i}) \mid 1 \leq i \leq k\}.$$

Clearly, $D \cong D^{-1}$ and so, $\gamma_-(D) = \gamma_-(D^{-1})$. It is easy to verify that the function $f : V(D) \to \{-1, 0, 1\}$ defined by $f(u_i) = f(v_{k+i}) = -1$ for $1 \leq i \leq k$ and $f(u) = 1$ otherwise, is an MDF of D, and so $\gamma_-(D) \leq 2$. Now let g be a $\gamma^*_-(D)$-function. Since $N^-[w] = \{w\}$ for each $w \in \{u_{k+i}, v_i \mid 1 \leq i \leq k\}$ and $N^+[w] = \{w\}$ for each $w \in \{u_i, v_{k+i} \mid 1 \leq i \leq k\}$, we must have $g(w) = 1$ for each $w \in V(D) - \{x, y\}$. It follows that $\gamma^*_-(D) \geq 4k - 2$. Thus $\gamma^*_-(D) - \max\{\gamma_-(D), \gamma_-(D^{-1})\} \geq 4k - 4$ and the proof is complete. □

As we observed in (2), $\gamma^*(D) \geq \gamma^*_-(D)$ and $\gamma^*_s(D) \geq \gamma^*_-(D)$. Next we show that $\gamma^*(D) - \gamma^*_-(D)$ and $\gamma^*_s(D) - \gamma^*_-(D)$ can be arbitrary large.

Theorem 4. *For every positive integer $k \geq 1$, there exists a digraph D such that*

$$\gamma^*(D) - \gamma^*_-(D) \geq k.$$

Proof. Let $k \geq 1$ be an integer and D be a digraph obtained from the directed path $\overrightarrow{P}_{4k} : v_1v_2 \ldots v_{4k}$ by adding $4k$ new vertices u_i, x_i, y_i, z_i $(1 \leq i \leq k)$ and adding new arcs $(v_{4(i-1)+1}, u_i)$,$(v_{4(i-1)+2}, u_i)$, $(u_i, v_{4(i-1)+3})$, (u_i, v_{4i}), $(x_i, v_{4(i-1)+1})$, $(v_{4(i-1)+2}, x_i)$, $(y_i, v_{4(i-1)+2})$, $(v_{4(i-1)+3}, y_i)$, $(z_i, v_{4(i-1)+3})$, (v_{4i}, z_i), $(1 \leq i \leq k)$. It is easy to verify that the function $f : V(D) \to \{-1, 0, 1\}$ defined by $f(u_i) = -1$, $f(x_i) = f(y_i) = f(z_i) = 0$ for $1 \leq i \leq k$ and $f(v) = +1$ otherwise, is an TMDF of D and so $\gamma^*_-(D) \leq 4k - k = 3k$. Now let S be a $\gamma^*(D)$-set. Since $N^-(w) \cap S \neq \varnothing$ and $N^+(w) \cap S \neq \varnothing$ for each $w \in \{u_i, x_i, y_i, z_i \mid 1 \leq i \leq k\}$, we must have $|S| \geq 4k$. It follows that $\gamma^*(D) \geq 4k$. Thus $\gamma^*(D) - \gamma^*_-(D) \geq k$ and the proof is complete. □

Theorem 5. *For every positive integer k, there exists a digraph D such that*

$$\gamma^*_s(D) - \gamma^*_-(D) \geq k.$$

Proof. Let $k \geq 1$ be an integer and D be a digraph with vertex set $V(D) = \{u, v, x_1, \ldots, x_k\}$ and arc set $A(D) = \{(u, x_i), (x_i, v) \mid 1 \leq i \leq k\}$. It is easy to see that the function $f : V(D) \to \{-1, 0, 1\}$ defined by $f(u) = f(v) = +1$ and $f(x) = 0$ otherwise, is an TMDF of D and so $\gamma^*_-(D) \leq 2$. Now let g be a $\gamma^*_s(D)$-function. Since $N^+[x_i] = \{v, x_i\}$ and $N^-[x_i] = \{u, x_i\}$ for $1 \leq i \leq k$, $N^-[u] = \{u\}$ and $N^+[v] = \{v\}$, we must have $g(x) = +1$ for each $x \in V(D)$. It follows that $\gamma^*_s(D) = k + 2$. Thus $\gamma^*_s(D) - \gamma^*_-(D) \geq k$ and the proof is complete. □

Now we show that the twin minus domination number and the twin signed domination number of digraphs can be arbitrary small.

Theorem 6. *For any positive integer k, there exists a digraph D such that*

$$\gamma^*_-(D) \leq 6k - 8k^2.$$

Proof. Let $k \geq 1$ be an integer and let D be a digraph obtained from a complete digraph of order $4k$ with vertex set $V(K_{4k}^*) = \{u_{i_1} \dots u_{i_4} \mid 1 \leq i \leq k\}$ by adding the set $\{v_{i_j}, w_{i_j} \mid 1 \leq i \leq k \text{ and } 1 \leq j \leq 4k-1\}$ of new vertices and the set

$$\{(v_{i_j}, u_{i_1}), (v_{i_j}, u_{i_2}), (u_{i_3}, v_{i_j}), (u_{i_4}, v_{i_j}), (u_{i_1}, w_{i_j}), (u_{i_2}, w_{i_j}), (w_{i_j}, u_{i_3}), \\ (w_{i_j}, u_{i_4}) \mid 1 \leq i \leq k,\ 1 \leq j \leq 4k-1\},$$

of new arcs. It is easy to see that the function $f : V(D) \to \{-1, 0, 1\}$ defined by $f(v_{i_j}) = f(w_{i_j}) = -1$ and $f(x) = +1$ otherwise, is an TMDF of D and so $\gamma_-^*(D) \leq 4k - 2(k(4k-1)) = 6k - 8k^2$. □

The function defined in the proof of Theorem 6 is also an TSDF of D and so $\gamma_s^*(D) \leq 6k - 8k^2$. Hence, the twin signed domination number of digraphs can also be arbitrary small.
A tournament is a digraph in which for every pair of distinct vertices u and v, either $(u, v) \in A(D)$ or $(v, u) \in A(D)$, but not both. Next we determine the exact value of the twin minus domination number for two particular types of tournaments. The acyclic tournament $AT(n)$ with n vertices has the vertex set $V(\mathrm{AT}(n)) = \{u_1, u_2, \dots, u_n\}$ and arc goes from u_i into u_j if and only if $i < j$. Let $n = 2r + 1$ for some positive integer r. We define the circulant tournament $\mathrm{CT}(n)$ with n vertices as follows. The vertex set of $\mathrm{CT}(n)$ is $V(\mathrm{CT}(n)) = \{u_0, u_1, \dots, u_{n-1}\}$ and for each i, the arcs go from u_i to the vertices $u_{i+1}, \dots, u_{i+r}$, where r is a positive integer and the sum is modulo n.

Proposition 2. *For $n \geq 3$, $\gamma_-^*(\mathrm{AT}(n)) = 2$.*

Proof. Let f be an TMDF on $\mathrm{AT}(n)$. Since $f(N^-[u_i]) \geq 1$ and $f(N^+[u_i]) \geq 1$, we have

$$f(N^-[u_i]) + f(N^+[u_i]) = f(u_1) + \cdots + f(u_{i-1}) + 2f(u_i) + f(u_{i+1}) + \cdots + f(u_n) \geq 2,$$

for each $1 \leq i \leq n$. Summing these inequalities, we obtain $(n+1)w(f) \geq 2n$ and so $w(f) \geq \frac{2n}{n+1}$. Since $\gamma_-^*(\mathrm{AT}(n))$ is an integer, we conclude that $\gamma_-^*(\mathrm{AT}(n)) = w(f) \geq 2$. Now define $f : V(\mathrm{AT}(n)) \to \{-1, 0, +1\}$ by $f(u_1) = f(u_n) = +1$ and $f(x) = 0$ otherwise. It is easy to see that f is an TMDF on $\mathrm{AT}(n)$ of weight 2, which implies that $\gamma_-^*(\mathrm{AT}(n)) = 2$. □

Theorem 7. *[12] For $n \geq 3$, $\gamma_-(\mathrm{CT}(n)) = 2$.*

The next Proposition shows that $\gamma_-^*(\mathrm{CT}(n)) = \gamma_-(\mathrm{CT}(n))$ for all odd $n \geq 3$.

Proposition 3. *Let $n \geq 3$ and $n = 2r + 1$, where r is a positive integer. Then $\gamma^*_-(\mathrm{CT}(n)) = \gamma_-(\mathrm{CT}(n))$.*

Proof. By (1) and Proposition 7, we have $\gamma^*_-(\mathrm{CT}(n)) \geq 2$. On the other hand, the function $f : V(\mathrm{CT}(n)) \to \{-1, 0, +1\}$ defined by $f(u_0) = f(u_{r+1}) = +1$ and $f(x) = 0$ otherwise, is an TMDF of $\mathrm{CT}(n)$ of weight 2. This completes the proof. □

3. Lower Bounds on $\gamma^*_-(D)$

In this section we present some lower bounds for $\gamma^*_-(D)$ in terms of the order, size, the maximum and minimum in-degrees and out-degrees of D. We begin with some results on the minus domination number of a digraph.

Remark 1. Let f be any $\gamma_-(D)$- function of a digraph D of order n. Then
(i) $|M_f| + |P_f| + |Z_f| = n$.
(ii) $w(f) = |P_f| - |M_f|$.

Theorem 8. *Let f be an MDF on a digraph D of order n. If $\Delta^+ = \Delta^+(D)$, $\delta^+ = \delta^+(D)$, $P = P_f$, $M = M_f$ and $Z = Z_f$, then*

(a) $\Delta^+|P| \geq (\delta^+ + 2)|M| + |Z|$.

(b) $(\Delta^+ + \delta^+ + 2)|P| + (\delta^+ + 1)|Z| \geq (\delta^+ + 2)n$.

(c) $(\delta^+ + 1)w(f) \geq (\delta^+ - \Delta^+)|P| + n$.

(d) $w(f) \geq (\frac{2\delta^+ - \Delta^+ + 2}{\Delta^+ - \delta^+})n + |P|$.

Proof. (a) It follows from Remark 1 (i), that

$$\begin{aligned}
|P| + |M| + |Z| &= n \leq \sum_{v\in V} \sum_{x\in N^-[v]} f(x) = \sum_{v\in V}(d^+(v) + 1)f(v) \\
&= \sum_{v\in P}(d^+(v) + 1) - \sum_{v\in M}(d^+(v) + 1) \\
&\leq (\Delta^+ + 1)|P| - (\delta^+ + 1)|M|.
\end{aligned}$$

This inequality chain yields to the desired bound in (a).
(b) Remark 1 (i) implies that $|Z| = n - |P| - |M|$. Using this identity and Part (a), we arrive at (b).
(c) According to Remark 1 and Part (b), we obtain Part (c) as follows:

$$w(f) = 2|P| - n + |Z|,$$

and

$$\begin{aligned}(\delta^+ + 1)w(f) &= (\delta^+ + 1)(2|P| - n + |Z|)\\ &= (\Delta^+ + \delta^+ + 2)|P| + (\delta^+ - \Delta^+)|P| - (\delta^+ + 1)n + (\delta^+ + 1)|Z|\\ &\geq (\delta^+ - \Delta^+)|P| - (\delta^+ + 1)n + (\delta^+ + 2)n\\ &= (\delta^+ - \Delta^+)|P| + n.\end{aligned}$$

(d) The inequality chain in the proof of Part (a) and Remark 1 (i) show that

$$\begin{aligned}n &\leq (\Delta^+ + 1)|P \cup Z| - (\delta^+ + 1)(n - |P \cup Z|)\\ &= (\Delta^+ - \delta^+)|P \cup Z| - (\delta^+ + 1)n,\end{aligned}$$

and thus

$$|P \cup Z| \geq (\frac{\delta^+ + 2}{\Delta^+ - \delta^+})n.$$

Using this inequality and Remark 1, we obtain

$$\begin{aligned}w(f) &= |P| - n + |P \cup Z| \geq (\tfrac{\delta^+ + 2}{\Delta^+ - \delta^+})n - n + |P|\\ &= (\tfrac{2\delta^+ - \Delta^+ + 2}{\Delta^+ - \delta^+})n + |P|,\end{aligned}$$

as required. □

Corollary 2. *Let D be a digraph of order n, minimum out-degree δ^+ and maximum out-degree Δ^+. If $\delta^+ < \Delta^+$, then*

$$\gamma_-(D) \geq (\frac{2\delta^+ - \Delta^+ + 3}{\Delta^+ + 1})n.$$

Proof. Multiplying both sides of the inequality in Theorem 8 (Part (d)) by $(\Delta^+ - \delta^+)$ and adding the resulting inequality to the inequality in Theorem 8 (Part (c)), we obtain the desired lower bound. □

Corollary 3. *Let D be a digraph of order n, minimum in-degree δ^- and maximum in-degree Δ^-. If $\delta^- < \Delta^-$, then*

$$\gamma_-(D^{-1}) \geq (\frac{2\delta^- - \Delta^- + 3}{\Delta^- + 1})n.$$

The next Corollary is a consequence of (1) and Corollaries 2 and 3.

Corollary 4. *Let D be a digraph of order n, minimum in-degree δ^-, maximum in-degree Δ^-, minimum out-degree δ^+ and maximum out-degree Δ^+. If $\delta^- < \Delta^-$ and $\delta^+ < \Delta^+$, then*

$$\gamma^*_-(D) \geq \max\{\frac{(2\delta^+ - \Delta^+ + 3}{\Delta^+ + 1})n, (\frac{2\delta^- - \Delta^- + 3}{\Delta^- + 1})n\}.$$

Let $G = (V, A)$ be a digraph. A *subdivision digraph* of G is a digraph $SD(G)$ obtained from G by adding new vertex x_{uv} for each arc $(u, v) \in A(G)$ and replacing the arc (u, v) with a directed path $ux_{uv}v$. (The new vertices are all of in-degree and out-degree 1.)

Theorem 9. *Let D be a digraph of order n and size m. Then $\gamma^*_-(D) \geq n - \frac{m}{2}$ with equality if and only if D is a subdivision digraph.*

Proof. Let f be a $\gamma^*_-(D)-$ function. For any $v \in M$, we have $|A(v, P)| \geq 2$ and $|A(P, v)| \geq 2$, which implies that $|A(M, P)| \geq 2|M|$ and $|A(P, M)| \geq 2|M|$. Also for any $v \in Z$, we have $|A(v, P)| \geq 1$ and $|A(P, v)| \geq 1$, which implies that $|A(Z, P)| \geq |Z|$ and $|A(P, Z)| \geq |Z|$. On the other hand, if $w \in P$, then it follows from $f(N^+[w]) \geq 1$ that $|A(w, P)| \geq |A(w, M)|$, which implies $|A(P, P)| \geq |A(P, M)| \geq 2|M|$. Therefore,

$$\begin{aligned} m &\geq |A(M, P)| + |A(P, M)| + |A(Z, P)| + |A(P, Z)| + |A(P, P)| \\ &\geq 4|M| + 2|Z| + |A(P, P)|/2 + |M|. \end{aligned} \tag{3}$$

Hence, we have

$$\gamma^*_-(D) = w(f) = |P| - |M| = n - 2|M| - |Z| \geq n - \frac{m}{2}.$$

If D is a subdivision digraph of a graph G, then $n = |V(D)| = |V(G)| + |A(G)|$ and $m = |A(D)| = 2|A(G)|$. Define the function $f : V(D) \to \{-1, 0, 1\}$ by $f(w) = +1$ for $w \in V(G)$ and $f(w) = 0$ otherwise. Then f is an TMDF of D, which implies that

$$\gamma^*_-(D) \leq f(V(D)) = |V(G)| = n - \frac{m}{2}.$$

Thus $\gamma^*_-(D) = n - \frac{m}{2}$.
Now let $\gamma^*_-(D) = n - \frac{m}{2}$. It follows from the chain inequality (3) that $|M| = 0, |A(P, P)| = 0$ and $|A(Z, P)| = |A(P, Z)| = |Z|$. This implies that $|A(P, w)| = |A(w, P)| = +1$ and so $d^+(w) = d^-(w) = 1$ for each $w \in Z$. Now let G be the digraph obtained from D by replacing every directed path of length 2 with central vertex in Z by an arc in the same direction as the path. Obviously, D is the subdivision digraph of G and the proof is complete. $\square$

A set $S \subseteq V(G)$ is a 2-packing if for each pair of distinct vertices $x, y \in S$, $N[x] \cap N[y] = \varnothing$. The 2-packing number $\rho(G)$ is the cardinality of a maximum 2-packing.

Proposition 4. *Let G be a graph of order n with minimum degree δ and let D be an orientation of G. Then*

$$\gamma_{-}^{*}(D) \geq \rho(G)(\delta + 2) - n.$$

Proof. Let S be a maximum 2-packing of G and f be a $\gamma_{-}^{*}(D)$-function. Since $f(N^{+}[v]) \geq 1$ and $f(N^{-}[v]) \geq 1$, we have $f(N_G[v]) = f(N^{+}[v]) + f(N^{-}[v]) - f(v) \geq 1$ for each $v \in S$. This implies that

$$\begin{aligned} \gamma_{-}^{*}(D) &= \textstyle\sum_{v \in S} f(N_G[v]) + \sum_{v \in V(G) - N_G[S]} f(v) \\ &\geq |S| + \textstyle\sum_{v \in V(G) - N_G[S]} (-1) \\ &\geq |S| - (n - |S|(\delta + 1)) \\ &= \rho(G)(\delta + 2) - n, \end{aligned}$$

and the proof is complete. □

The next proposition presents a lower bound on twin signed domination numbers in oriented graphs.

Proposition 5. *Let D be an oriented graph of order n. Then*

$$\gamma_{s}^{*}(D) \geq \sqrt{9 + 16n} - (n + 3).$$

Proof. Let f be a $\gamma_{s}^{*}(D)$- function. If $M_f = \varnothing$, then $\gamma_{s}^{*}(D) = w(f) = n \geq \sqrt{9 + 16n} - (n + 3)$. Let $M_f \neq \varnothing$. Every vertex in M_f has at least 2 out-neighbors and at least 2 in-neighbors in P_f. By the Pigeonhole principle, at least one vertex in P_f say y satisfies $|((N^{+}(y) \cup N^{-}(y)) \cap M_f| \geq \frac{4|M_f|}{|P_f|}$. So we have

$$\begin{aligned} 2 &\leq f(N^{+}[y]) + f(N^{-}[y]) = 2 + |(N^{+}(y) \cap P_f| + |(N^{-}(y) \cap P_f| \\ &\quad -|(N^{+}(y) \cap M_f| - |(N^{-}(y) \cap M_f| \\ &= 2 + |((N^{+}(y) \cup N^{-}(y)) \cap P_f| - |(N^{+}(y) \cup N^{-}(y)) \cap M_f| \\ &\leq 2 + |P_f| - 1 - \tfrac{4|M_f|}{|P_f|} = 2 + p - 1 - \tfrac{4(n-p)}{p}, \end{aligned}$$

where $p = |P_f|$. This implies that $p \geq \frac{-3+\sqrt{9+16n}}{2}$ and so $\gamma_{s}^{*}(D) = w(f) \geq 2p - n \geq \sqrt{9 + 16n} - (n + 3)$. □

The next proposition presents a lower bound on twin minus domination numbers in an oriented graph in terms of its order.

Proposition 6. *Let D be an oriented graph of order n. Then*

$$\gamma^*_-(D) \geq \lfloor\sqrt{9+16n}\rfloor - (n+3).$$

Proof. Let f be a $\gamma^*_-(D)$- function. If $Z_f = \varnothing$, then f is an TSDF on D and by Proposition 5, $\gamma^*_-(D) = w(f) \geq \gamma^*_s(D) \geq \sqrt{9+16n}-(n+3) \geq \lfloor\sqrt{9+16n}\rfloor - (n+3)$. Let $Z_f \neq \varnothing$. Let $n_1 = n - |Z_f|$ and D_1 be a subdigraph of D induced by the set $V(D) - Z_f$. Then $f|_{V(D_1)}$ is an TSDF on D_1 and by Proposition 5, $\gamma^*_-(D) = w(f) \geq \gamma^*_s(D_1) \geq \sqrt{9+16n_1} - (n_1+3) \geq \lfloor\sqrt{9+16n_1}\rfloor - (n_1+3)$. Now we can easily see that the function $g(x) = \lfloor\sqrt{9+16x}\rfloor - (x+3)$ is a non increasing function for any integer $x \geq 1$ and so $g(n_1) \geq g(n)$. This implies that $\gamma^*_-(D) \geq \lfloor\sqrt{9+16n_1}\rfloor - (n_1+3) \geq \lfloor\sqrt{9+16n}\rfloor - (n+3)$. $\square$

Next we prove that the bounds given in Propositions 5 and 6 are sharp.
For $1 \leq i \leq 4$ and $r \geq 1$, let D_i be the circulant Tournament $CT(2r+1)$ with vertex set $\{u^i_0, \ldots, u^i_{2r}\}$. Let H be a digraph obtained from the union of D_i's by adding the set

$$\{(u^1_j, u^2_t), (u^2_j, u^3_t), (u^3_j, u^4_t), (u^4_j, u^1_t)(u^1_j, u^3_t), (u^2_j, u^4_t) \mid 0 \leq j, t \leq 2r, \}$$

of new arcs. Let D be obtained from H by adding the set

$$\{v^t_1, \ldots, v^t_{3r+1}, w^t_1, \ldots, w^t_{3r+1}, z^t_1, \ldots, z^t_{2r+1} \mid 0 \leq t \leq 2r\},$$

of new vertices and the set

$$\{(u^1_t, v^t_k), (u^2_t, v^t_k), (v^t_k, u^3_t), (v^t_k, u^4_t), (w^t_k, u^1_t), (w^t_k, u^2_t), (u^3_t, w^t_k), (u^4_t, w^t_k), (u^1_t, z^t_s), \\ (u^2_t, z^t_s), (z^t_s, u^3_t), (z^t_s, u^4_t) \mid 0 \leq t \leq 2r,\ 1 \leq k \leq 3r+1,\ 1 \leq s \leq 2r+1\},$$

of new arcs. Then the order of D is $n = 16r^2 + 22r + 7$ and $\sqrt{9+16n} - (n+3) = -16r^2 - 6r + 1$. Now define $f : V(D) \to \{-1, +1\}$ by $f(x) = +1$ for $x \in \{u^i_0, \ldots, u^i_{2r} \mid 1 \leq i \leq 4\}$ and $f(x) = -1$ otherwise. It is easy to see that f is an TSDF and also an TMDF on D of weight $-16r^2 - 6r + 1$, which implies that $\gamma^*_s(D) \leq -16r^2 - 6r + 1$ and $\gamma^*_-(D) \leq -16r^2 - 6r + 1$. By Propositions 5 and 6, we have $\gamma^*_s(D) = \gamma^*_-(D) - 16r^2 - 6r + 1$.

4. Twin Minus Domination in Oriented Graphs

Let G be the complete bipartite graph $K_{4,5}$ with partite sets $U = \{u_1, u_2, u_3, u_4, u_5\}$ and $V = \{v_1, v_2, v_3, v_4\}$. Let D_1 be an orientation of G such that all arcs go from V_1 into V_2 and D_2 be an orientation of G such that $A(D_2) = \{(v_i, u_j), (u_j, v_r) \mid i = 1, 2, r = 3, 4 \text{ and } 1 \le j \le 5\}$. It is easy to see that $\gamma^*_-(D_1) = 9$ and $\gamma^*_-(D_2) = 4$. Thus two distinct orientations of a graph can have distinct twin minus domination numbers. Motivated by this observation, we define lower orientable twin minus domination number $\mathrm{dom}^*_-(G)$ and upper orientable twin minus domination number $\mathrm{Dom}^*_-(G)$ of a graph G as follows:

$$\mathrm{dom}^*_-(G) = \min\{\gamma^*_-(D) \mid \text{D is an orientation of } G\},$$

and

$$\mathrm{Dom}^*_-(G) = \max\{\gamma^*_-(D) \mid \text{D is an orientation of } G\}.$$

Corresponding concepts have been defined and studied for orientable domination (out-domination) [9], twin domination number [8], twin signed domination number [6], twin signed total domination number [3] and twin signed Roman domination number [7].

Since for any orientation D of a graph G, $\gamma^*_-(D) \le \gamma^*(D)$ and $\gamma^*_-(D) \le \gamma^*_s(D)$, we have

$$\mathrm{dom}^*_-(G) \le \mathrm{dom}^*(G) \tag{4}$$

$$\mathrm{dom}^*_-(G) \le \mathrm{dom}^*_s(G) \tag{5}$$

The proof of the following three theorems can be found in [2].

Theorem 10. *For the path P_n, we have $\mathrm{dom}^*(P_n) = \lceil \frac{n+1}{2} \rceil$.*

Theorem 11. *For $n \ge 3$, $\mathrm{dom}^*(K_n) = 2$.*

Theorem 12. *For any two positive integers m, n, $\mathrm{dom}^*(K_{m,n}) \le 4$.*

Proposition 7. *For any graph G of order n, $\gamma_-(G) \le \mathrm{dom}^*_-(G)$.*

Proof. Let D be an orientation of G and let f be a $\gamma^*_-(D)$-function. Then $f(N_G[v]) = f(N^+_D[v]) + f(N^-_D[v]) - f(v)$ for each $v \in V$. Since $f(N^+_D[v]) \ge 1$ and $f(N^-_D[v]) \ge 1$, we have $f(N_G[v]) \ge 1$ for each $v \in V$, and so f is an MDF of G. Therefore $\gamma_-(G) \le w(f) = \mathrm{dom}^*_-(G)$ as desired. □

The proof of the next two results are straightforward and therefore omitted.

Proposition 8. *Let G be a graph of order n and $v \in V(G)$. If $\deg(v) = 1$, then for any orientation D of G and any $\gamma_-^*(D)$-function f, we have $f(v) = +1$ and $f(u) \geq 0$ for $u \in N[v]$.*

Proposition 9. *Let G be a graph of order n and $v \in V(G)$. If $\deg(v) \leq 3$, then for any orientation D of G and any $\gamma_-^*(D)$-function f, we have $f(v) \geq 0$.*

Proposition 10. *Let G be a graph of order n. Then $\mathrm{dom}_-^*(G) = n$ if and only if $\deg(v) \leq 1$ for every $v \in V(G)$.*

Proof. One side is clear by Proposition 8. Let $\mathrm{dom}_-^*(G) = n$. Assume, to the contrary, there exists a vertex $v \in V(G)$ such that $\deg(v) \geq 2$. Let u and w be two vertices adjacent to v and D be an orientation of G such that $(u, v), (v, w) \in A(D)$. Then the function $f : V(D) \to \{-1, 0, 1\}$ that assigns 0 to v and $+1$ to the remaining vertices, is an TMDF of D of weight $n - 1$ and so $\mathrm{dom}_-^*(G) \leq n - 1$, a contradiction. This completes the proof. □

Proposition 11. *For the path P_n, we have $\mathrm{dom}_-^*(P_n) = \lceil \frac{n+1}{2} \rceil$.*

Proof. Let D be an orientation of P_n and let f be a $\gamma_-^*(D)$-function such that $\gamma_-^*(D) = \mathrm{dom}_-^*(P_n)$. By Proposition 9, $M_f = \varnothing$ and so P_f is a twin dominating set on D. Hence

$$\mathrm{dom}^*(P_n) \leq \gamma^*(D) \leq w(f) = \mathrm{dom}_-^*(P_n).$$

Now the result follows by (4) and Proposition 10. □

We now proceed to determine the lower orientable twin domination numbers of several classes of graphs including complete graphs and complete bipartite graphs.

Theorem 13. *For $n \geq 3$, $\mathrm{dom}_-^*(K_n) = 2$.*

Proof. We first show that $\mathrm{dom}_-^*(K_n) \geq 2$. Let D be an orientation of K_n and let f be a $\gamma_-^*(D)$-function. If $M_f = \varnothing$ and $Z_f = \varnothing$, then $w(f) = n$. Let $Z_f = \varnothing$ and $v \in M_f$. Since $f(N_D^+[v]) \geq 1$ and $f(N_D^-[v]) \geq 1$, we have

$$w(f) = f(N_D^+[v]) + f(N_D^-[v]) - f(v) \geq 3.$$

Let now $M_f = \varnothing$ and $v \in Z_f$. As above we have

$$w(f) = f(N_D^+[v]) + f(N_D^-[v]) - f(v) \geq 2.$$

This implies that $\mathrm{dom}^*_-(K_n) \geq 2$. Now the result follows by (4) and Proposition 11. $\square$

Proposition 12. *For $n \geq 2$, $\mathrm{dom}^*_-(K_{1,n}) = n$.*

Proof. Let $V(K_{1,n}) = \{v, u_1, \ldots, u_n\}$ where v is the central vertex of $K_{1,n}$. Let D be an orientation of $K_{1,n}$ and let f be a $\gamma^*_-(D)$-function. By Proposition 8, $f(u_i) = +1$ for $1 \leq i \leq n$, which implies that $w(f) \geq n$ and so $\mathrm{dom}^*_-(K_{1,n}) \geq n$. Let now D be an orientation of $K_{1,n}$ such that $(v, u_1), (u_2, v) \in A(D)$. Then the function $f : V(D) \to \{-1, 0, 1\}$ defined by $f(v) = 0$ and $f(x) = +1$ otherwise, is an TMDF of D and so $\gamma^*_-(D) \leq n$. This completes the proof. $\square$

Proposition 13. *For $n \geq 2$, $\mathrm{dom}^*_-(K_{2,n}) = 2$.*

Proof. Consider $K_{2,n}$ with partite sets $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, v_2\}$. Let D be an orientation of $K_{2,n}$ and let f be a $\gamma^*_-(D)$-function. By Proposition 9, $f(u_i) \geq 0$ for $1 \leq i \leq n$. First let $M_f \neq \varnothing$. Let $f(v_1) = -1$. Then since $f(N^-_D[v]) \geq 1$ and $f(N^+_D[v]) \geq 1$ for each $v \in V(D)$, it follows that $f(v_2) = f(u_i) = +1$ for all $1 \leq i \leq n$. This implies that $w(f) \geq n$ and so $\mathrm{dom}^*_-(K_{1,n}) = n \geq 2$. Let now $M_f = \varnothing$. If $Z_f = \varnothing$, then $w(f) = n + 1 \geq 2$. Let $v \in Z_f$. Since $f(N^-_D[v]) \geq 1$ and $f(N^+_D[v]) \geq 1$, we have $w(f) \geq f(N^-_D[v]) + f(N^+_D[v]) - f(v) \geq 2$.
Let now D be an orientation of $K_{2,n}$ such that $A(D) = \{(v_1, u_i), (u_i, v_2) | \ 1 \leq i \leq n\}$. Then the function $f : V(D) \to \{-1, 0, 1\}$ defined by $f(v_1) = f(v_2) = +1$ and $f(x) = 0$ otherwise, is an TMDF of D and so $\gamma^*_-(D) \leq 2$. This completes the proof. $\square$

Proposition 14. *For $n \geq 3$, $\mathrm{dom}^*_-(K_{3,n}) = 3$.*

Proof. Consider $K_{3,n}$ with partite sets $U = \{u_1, u_2, \ldots, u_n\}$ and $V = \{v_1, v_2, v_3\}$. Let D be an orientation of $K_{3,n}$ and let f be a $\gamma^*_-(D)$-function. By Proposition 9, $f(u_i) \geq 0$ for $1 \leq i \leq n$. First let $M_f \neq \varnothing$. Let $f(v_1) = -1$. Then obviously, $f(v_2) = +1$ or $f(v_3) = +1$ and $f(u_i) = +1$ for all $1 \leq i \leq n$ and so $w(f) \geq n \geq 3$.
Let now $M_f = \varnothing$. If $f(x) = +1$ for every $x \in U$ or $f(y) = +1$ for every $y \in V$, then obviously $w(f) \geq 3$. If $f(x) = 0$ and $f(y) = 0$ for some $x \in U$ and $y \in V$, then $f(N^-_D[x]) \geq 1$, $f(N^+_D[x]) \geq 1$, $f(N^-_D[y]) \geq 1$ and $f(N^+_D[y]) \geq 1$ implies that $f(X) \geq 2$ and $f(Y) \geq 2$ and so $w(f) \geq 4$.
Let now D be an orientation of $K_{3,n}$ such that $A(D) = \{(v_1, u_i), (u_i, v_2), (u_i, v_3) | \ 1 \leq i \leq n\}$. Then the function $f : V(D) \to \{-1, 0, 1\}$

defined by $f(v_1) = f(v_2) = f(v_3) = +1$ and $f(x) = 0$ otherwise, is an TMDF of D and so $\gamma^*_-(D) \leq 3$. This completes the proof. $\square$

Proposition 15. *For $n, m \geq 4$, $\mathrm{dom}^*_-(K_{m,n}) = 4$.*

Proof. Consider $K_{m,n}$ with partite sets X and Y. By (4) and Proposition 12 we have $\mathrm{dom}^*_-(K_{m,n}) \leq 4$. Let now D be an orientation of $K_{m,n}$ and let f be a $\gamma^*_-(D)$-function. First let $M_f \neq \varnothing$. Let $v \in X$ and $f(v) = -1$. Since $f(N^-_D[v]) \geq 1$ and $f(N^+_D[v]) \geq 1$, we have $f(Y) \geq 4$. On the other hand, $f(N^-_D(u)) \geq 0$ and $f(N^+_D(u)) \geq 0$ for any $u \in Y$ and so $f(X) \geq 0$. This implies that $w(f) \geq 4$.
Let now $M_f = \varnothing$. If $f(x) = +1$ for any $x \in X$ or $f(y) = +1$ for any $y \in Y$, then obviously $w(f) \geq 4$. Let $f(x) = 0$ and $f(y) = 0$ for some $x \in X$ and $y \in Y$. Since $f(N^-_D[x]) \geq 1$, $f(N^+_D[x]) \geq 1$, $f(N^-_D[y]) \geq 1$ and $f(N^+_D[y]) \geq 1$, we have $f(X) \geq 2$ and $f(Y) \geq 2$ and so $w(f) \geq 4$. This completes the proof. $\square$

The wheel, W_n, is a graph with vertex set $\{v_0, v_1, \ldots, v_n\}$ and edge set $\{v_0v_i \mid 1 \leq i \leq n\} \cup \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$. Next we determine the lower orientable twin minus domination number of wheels.

Theorem 14. *[2] For $n \geq 4$, $\mathrm{dom}^*(W_n) = \begin{cases} \lceil \frac{n-1}{3} \rceil & n = 5 \\ \lceil \frac{n-1}{3} \rceil + 1 & \text{otherwise.} \end{cases}$*

Proposition 16. *For $n \geq 5$, $\mathrm{dom}^*_-(W_n) = \mathrm{dom}^*(W_n)$.*

Proof. By (4), we have $\mathrm{dom}^*_-(W_n) \leq \mathrm{dom}^*(W_n)$. We show that $\mathrm{dom}^*_-(W_n) \geq \mathrm{dom}^*(W_n)$. Let $V(W_n) = \{v_0, v_1, \ldots, v_n\}$ and $E(W_n) = \{v_iv_{i+1}, v_0v_i \mid 1 \leq i \leq n-1\} \cup \{v_0v_n, v_nv_1\}$. Let D be an orientation of W_n and f be a $\gamma^*_s(D)$-function. It follows from Proposition 9 that $f(v_i) \geq 0$ for each $i \geq 1$. If $f(v_0) = -1$, then $f(v_i) = +1$ for $i \geq 1$ and so $w(f) = n - 1$, which leads to a contradiction. Therefore $f(v_0) \geq 0$ and so P_f is a twin dominating set of D. This implies that

$$\gamma^*_-(D) = w(f) = |P_f| \geq \gamma^*(D) \geq \mathrm{dom}^*(W_n).$$

$\square$

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