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On trees and the multiplicative sum Zagreb index

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Abstract: For a graph G with edge set $E(G)$, the multiplicative sum Zagreb index of G is defined as $\Pi^*(G) = \Pi_{uv\in E(G)}[d_G(u)+d_G(v)]$, where $d_G(v)$ is the degree of vertex v in G. In this paper, we first introduce some graph transformations that decrease this index. In application, we identify the fourteen class of trees, with the first through fourteenth smallest multiplicative sum Zagreb indices among all trees of order $n \geq 13$.

Keywords: Multiplicative sum Zagreb index, Graph transformation, Branching point, Trees.

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1. Introduction

Throughout this paper, we consider connected graphs without loops and multiple edges. Let G be such a graph and $V(G)$ and $E(G)$ its vertex set and edge set, respectively. For a vertex v in G, the degree of v, $d_G(v)$, is the number of edges incident to v; $N[v,G]$ is the set of vertices adjacent to v. If $u \in V(G)$ and $u \in N[v,G]$, we then write $uv \in E(G)$. A pendent vertex is a vertex with degree one. We use $\Delta = \Delta(G)$ to denote the maximum degree of G. The

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number of edges of G connecting a vertex of degree i with a vertex of degree j will be denoted by $m_{i,j}(G)$.
For a subset W of $E(G)$, we denote by $G - W$ the subgraph of G obtained by deleting the edges of W. Similarly, for a subset U of $V(G)$, let $G - U$ be the subgraph of G obtained by deleting the vertices of U and the edges incident to them. For any two nonadjacent vertices u and v of graph G, $G + uv$ denotes the graph obtained from G by adding an edge uv.
A tree is a connected acyclic graph. Any tree with at least two vertices has at least two pendant vertices. The set of all n-vertex trees will be denoted by $\tau(n)$. We denote the path graph and the star graph (both with n vertices) with P_n and S_n, respectively.
We denote the distance between two arbitrary vertices x and y of a graph G by $d_G(x, y)$. This distance is defined as the number of edges in the minimal path connecting the vertices x and y. Given an edge $e = uv \in E(G)$ of G, let us denote the number of vertices lying closer to the vertex u than to the vertex v of e by $n_u(e|G)$ and the number of vertices lying closer to the vertex v than to the vertex u by $n_v(e|G)$. Thus,

$$n_u(e|G) := |\{a \in V(G)|d_G(u, a) < d_G(v, a)\}|.$$

A graph invariant (topological index) is a real number related to a graph, which is invariant under graph isomorphism. For a graph G, the graph invariant

$$M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)],$$

is called the first Zagreb index. It is easy to see that $M_1(G) = \sum_{v \in V(G)} d_G(v)^2$.
This index was introduced more than 40 years ago [10], and has many applications in chemistry [2, 11, 13]. Also, the first Zagreb index was subject to a large number of mathematical studies [3–5, 8, 9]. Todeschini et al. [12, 14] have recently proposed to consider multiplicative variants of additive graph invariants. Eliasi et al. [6] applied this idea to the first Zagreb index and defined the first multiplicative Zagreb index as:

$$\Pi^*(G) = \Pi_{uv \in E(G)}[d_G(u) + d_G(v)].$$

This graph invariant is called the multiplicative sum Zagreb index by Xu and Das [17]. Eliasi et al. [6] proved that among all connected graphs with a given number of vertices, the path has minimal Π^*, and they determined the trees with the second minimal Π^*. Also, Xu and Das [17] characterized the trees, unicylcic, and bicyclic graphs extremal (maximal and minimal) with respect to the multiplicative sum Zagreb index. Moreover, they used a method different but shorter than that in [6] for determining the minimal multiplicative sum Zagreb index of trees. Other results for this index can be found in [1, 7, 15]. In this paper, we first introduce some graph transformations, which decrease Π^*. By using these operations, we identify the fourteen class of trees, with the first through fourteenth smallest multiplicative sum Zagreb indices among all trees of order $n \geq 13$.

2. Some Graph Transformations

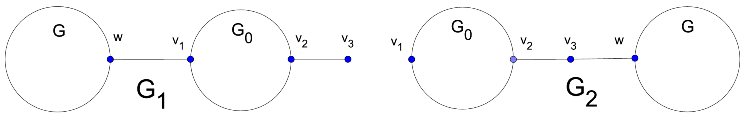
In this section, we introduce some graph transformations which decrease the multiplicative sum Zagreb index. We start with some definitions and notations which are taken from [16]. A vertex v of a tree T is called a branching point of T if $d_T(v) \geq 3$. Let $T_n(n_1, n_2, \ldots, n_m)$ be a starlike tree of order n obtained from the star S_{m+1} by replacing its m edges with m paths $P_{n_1}, P_{n_2}, \ldots, P_{n_m}$, with $\sum_{i=1}^{m} n_i = n - 1$. Any starlike tree has exactly one branching point. For a tree T of order n with two branching points v_1 and v_2, $d_T(v_1) = r$ and $d_T(v_2) = t$, and if the orders of $r-1$ components, which are paths of $T - \{v_1\}$, are $p_1, p_2, \ldots, p_{r-1}$, and the orders of $t-1$ components, which are paths of $T - \{v_2\}$, are $q_1, q_2, \ldots, q_{t-1}$, then we write the tree as $T = T_n(p_1, p_2, \ldots, p_{r-1}; q_1, q_2, \ldots, q_{t-1})$. In addition, if $v_1v_2 \in E(T)$ then we write the tree as $T = T_n^{\sim}(p_1, p_2, \ldots, p_{r-1}; q_1, q_2, \ldots, q_{t-1})$, and if $v_1v_2 \notin E(T)$, then we write $T = T_n^{\not\sim}(p_1, p_2, \ldots, p_{r-1}; q_1, q_2, \ldots, q_{t-1})$.

Lemma 1. *Suppose that G_0 is a tree with given vertices v_1, v_2, and v_3, such that $d_{G_0}(v_1) \geq 3$, $d_{G_0}(v_2) \geq 2$, $d_{G_0}(v_3) = 1$, and $v_2v_3 \in E(G_0)$. In addition, suppose that G is another tree, and w is a vertex in G such that $d_{G_0}(v_1) \geq d_G(w)$. let G_1 be the graph obtained from G_0 and G by attaching vertices w, v_1, and $G_2 = G_1 - wv_1 + wv_3$. Then $\Pi^*(G_2) < \Pi^*(G_1)$ (see Figure 1).*

Proof. Suppose that $d_{G_0}(v_1) = x$, $N[v_1, G_0] = \{l_1, \ldots l_x\}$, $d_{G_0}(l_i) = d_i$, for $i = 1, \ldots, x$. Let $d_{G_0}(v_2) = m$ and $d_G(w) = k$. If $v_1 \neq v_2$, then we have

$$\begin{aligned} \frac{\Pi^*(G_1)}{\Pi^*(G_2)} &= \frac{(k+x+2)(m+1)\prod_{i=1}^{x}(d_i+x+1)}{(m+2)(k+3)\prod_{i=1}^{x}(d_i+x)} \\ &> \frac{(k+x+2)(m+1)}{(m+2)(k+3)}. \end{aligned} \quad (1)$$

Figure 1. **The trees G_0, G, G_1 and G_2 in Lemma 1 .**



But $x \geq 3$ and $m \geq 2$, so $m(x-1) \geq 4$ and $xm \geq m+4$. According to the hypothesis, $x \geq k$. Hence,

$$\begin{aligned}(k+x+2)(m+1)-(m+2)(k+3) &= xm+x-(m+k+4) \\ &= [xm-(m+4)]+(x-k) \geq 0,\end{aligned}$$

and by (1) we have $\frac{\Pi^*(G_1)}{\Pi^*(G_2)} > 1$.
Now, suppose that $v_1 = v_2$. Without loss of generality, we may assume that $l_1 = v_3$. So,

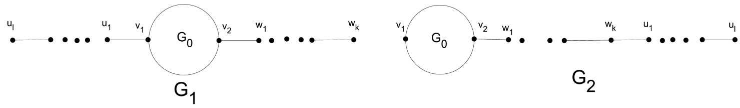
$$\begin{aligned}\frac{\Pi^*(G_1)}{\Pi^*(G_2)} &= \frac{(k+x+2)(x+2)\prod_{i=2}^{x}(d_i+x+1)}{(x+2)(k+3)\prod_{i=2}^{x}(d_i+x)} \\ &> \frac{(k+x+2)}{(k+3)} > 1, \qquad (2)\end{aligned}$$

because $x \geq 3$. □

Lemma 2. *Suppose that G_0 is a tree with given vertices v_1 and v_2 such that $d_{G_0}(v_1) \geq 3$, and $P_k := w_1w_2\dots w_k$ and $Q_l := u_1u_2\dots u_l$ are two paths, with k and l vertices, respectively. Let G_1 be the graph obtained from G_0, P_k, and Q_l by adding the edges v_1u_1 and v_2w_1. Also, let $G_2 = G_1 - v_1u_1 + w_ku_1$. Then $\Pi^*(G_2) < \Pi^*(G_1)$ (see Figure 2). This inequality holds, when $d_{G_0}(v_1) = 2$ and at least one of the neighborhoods of v_1 has degree less than 13 in G_0.*

Proof. We first suppose that $d_{G_0}(v_1) \geq 3$ and $k \geq 2$. Let H_0 be the tree obtained by joining G_0 and P_k by the edge v_2w_1. Then, H_0 is a tree with given vertices v_1, w_{k-1}, and w_k, such that $d_{H_0}(v_1) \geq 3$, $d_{H_0}(w_k) = 1$, and $w_{k-1}w_k \in E(H_0)$. Since u_1 is a vertex in Q_l and $d_{H_0}(v_1) \geq d_{Q_l}(u_1)$, Lemma 1 implies $\Pi^*(G_2) < \Pi^*(G_1)$.
If $k = 1$, then H_0 is a tree with given vertices v_1, v_2, and w_k, such that

Figure 2. **The trees G_0, P_k, Q_l, G_1 and G_2 in Lemma 2.**



$d_{H_0}(v_1) \geq 3$, $d_{H_0}(w_k) = 1$, and $v_2w_k \in E(H_0)$. Since u_1 is a vertex in Q_l and $d_{H_0}(v_1) \geq d_{Q_l}(u_1)$, Lemma 1 implies that $\Pi^*(G_2) < \Pi^*(G_1)$ (see Figure 2).
Now, suppose that $d_{G_0}(v_1) = 2$ and $N[v_1, G_0] = \{l_1, l_2\}$ and $d_{G_0}(l_i) = d_i$, for $i = 1, 2$ and $d_1 \leq 13$. We distinguish the following eight cases:

(1) $v_1 \neq v_2$ and $l, k \geq 2$, (2) $v_1 \neq v_2$ and $l \geq 2, k = 1$,
(3) $v_1 \neq v_2$ and $l = 1, k \geq 2$, (4) $v_1 \neq v_2$ and $l = 1, k = 1$,
(5) $v_1 = v_2$ and $l, k \geq 2$, (6) $v_1 = v_2$ and $l \geq 2, k = 1$,
(7) $v_1 = v_2$ and $l = 1, k \geq 2$, (8) $8v_1 = v_2$ and $l = 1, k = 1$.

Here, we only give the proof of (1).
Suppose that $v_1 \neq v_2$ and $l, k \geq 2$. Then we have

$$\frac{\Pi^*(G_1)}{\Pi^*(G_2)} = \frac{5 \times 3 \times (d_1 + 3) \times (d_2 + 3)}{4 \times 4 \times (d_1 + 2) \times (d_2 + 2)}. \tag{3}$$

But

$$\begin{aligned} 15(d_1 + 3)(d_2 + 3) - 16(d_1 + 2)(d_2 + 2) &= -d_1d_2 + 13d_1 + 13d_2 + 71 \\ &= d_2(13 - d_1) + 13d_1 + 71 > 0, \end{aligned}$$

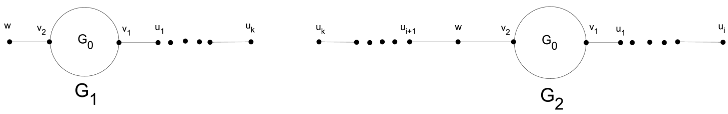
and by (3) we have $\frac{\Pi^*(G_1)}{\Pi^*(G_2)} > 1$.
Now, suppose that $v_1 \neq v_2$, $l \geq 2$, and $k = 1$ (Case 2). Let $d_{G_0}(v_2) = z$. Then

$$\frac{\Pi^*(G_1)}{\Pi^*(G_2)} = \frac{5 \times 3 \times (d_1 + 3) \times (d_2 + 3) \times (z + 2)}{4 \times 3 \times (d_1 + 2) \times (d_2 + 2) \times (z + 3)}. \tag{4}$$

But

$$\begin{aligned} &\Pi^*(G_1) - \Pi^*(G_2) \\ &= 15(d_1 + 3)(d_2 + 3)(z + 2) - [12(d_1 + 2)(d_2 + 2)(z + 3)] \\ &= 3\,d_1d_2z - 6\,d_1d_2 + 21\,d_1z + 18\,d_1 + 21\,d_2z + 18\,d_2 + 87\,z + 126 \\ &= 3\,d_1d_2(z - 2) + 21\,d_1z + 18\,d_1 + 21\,d_2z + 18\,d_2 + 87\,z + 126. \end{aligned} \tag{5}$$

Figure 3. **The trees G_0, P_k, G_1 and G_2 in Lemma 3.**



In (5), take $z = 1$. Since $d_1 \leq 13$, we obtain

$$\Pi^*(G_1) - \Pi^*(G_2) \;=\; 3[d_2(13 - d_1) + 13d_1 + 71] > 0.$$

In (5), if $z \geq 2$, then it is clear that $\Pi^*(G_1) - \Pi^*(G_2) > 0$, which is our claim. The proofs of the remaining cases are similar, and we omit them. □

Lemma 3. *Suppose that G_0 is a tree with given vertices v_1, v_2, and $w \in V(G_0)$, such that $d_{G_0}(v_1) \geq 2$, $d_{G_0}(v_2) \geq 3$ and w is a pendent vertex in $N[v_2, G_0]$. In addition, suppose that $P_k := u_1u_2 \dots u_k$ is a path, with $k \geq 3$ vertices. Let G_1 be the tree obtained from G_0 and P_k by joining v_1 and u_1. Let $2 \leq i \leq k-1$. If $G_2 = G_1 - u_iu_{i+1} + wu_{i+1}$, then $\Pi^*(G_2) < \Pi^*(G_1)$ (see Figure 3).*

Proof. Suppose that $d_{G_0}(v_2) = x$. We consider the following cases:
(a) $v_1 \neq v_2$. In this case, it is easy to see that

$$\frac{\Pi^*(G_1)}{\Pi^*(G_2)} \;=\; \frac{(x+1)\times 4}{(x+2)\times 3} = \frac{4x+4}{3x+6} > 1,$$

because, $x \geq 3 \Rightarrow 4x + 4 - (3x + 6) = x - 2 \geq 1 \Rightarrow 4x + 4 > 3x + 6$.
(b) $v_1 = v_2$. In this case, we have

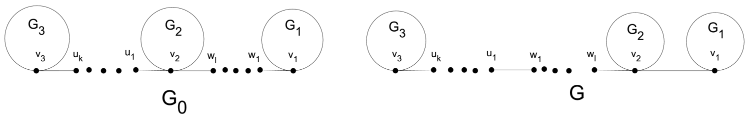
$$\frac{\Pi^*(G_1)}{\Pi^*(G_2)} \;=\; \frac{(x+2)\times 4}{(x+3)\times 3} = \frac{4x+8}{3x+9}, \tag{6}$$

but

$$4x + 8 - (3x + 9) = x - 1 > 0, \tag{7}$$

which completes the proof. □

Figure 4. **The trees G_i (i=1,2,3), P_k, and Q_l in Lemma 4.**



Remark 1. By considering (6) and (7), one can see that if $v_1 = v_2$ and $d_{G_0}(v_2) = 2$ in Lemma 3, then the inequality $\Pi^*(G_2) < \Pi^*(G_1)$ holds.

Lemma 4. *Suppose that for $i = 1, 2, 3$, G_i are trees with $v_i \in V(G_i)$, $d_{G_1}(v_1) \geq 2$ and $d_{G_i}(v_i) \geq 1 (i = 2, 3)$. In addition, suppose that $P_l := w_1w_2 \ldots w_l$ and $Q_k := u_1u_2 \ldots u_k$ are two paths, with l and k vertices, respectively. Let G_0 be the graph obtained from G_i $(i = 1, 2, 3)$, P_l, and Q_k by adding the edges v_1w_1, w_lv_2, v_2u_1 and u_kv_3. Also, let $G = G_0 - \{v_1w_1, v_2u_1\} + \{v_1v_2, w_1u_1\}$. Then $\Pi^*(G) < \Pi^*(G_0)$ (see Figure 4).*

Proof. Let $d_{G_1}(v_1) = x$ and $d_{G_2}(v_2) = h$. Then

$$\frac{\Pi^*(G)}{\Pi^*(G_0)} = \frac{4(x+h+3)}{(h+4)(x+3)} = \frac{4x+4h+12}{hx+3h+4x+12} < 1,$$

since $x \geq 2$. □

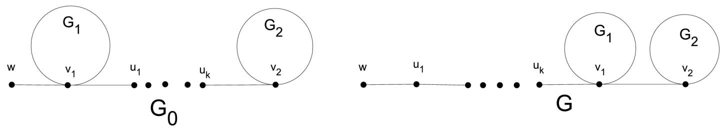
Lemma 5. *Suppose that for $i = 1, 2$, G_i are trees such that $\{v_1, w\} \subseteq V(G_1)$, $v_2 \in V(G_2)$, $d_{G_i}(v_i) \geq 2(i = 1, 2)$ and $d_{G_1}(w) = 1$. In addition, suppose that $P_k := u_1u_2 \ldots u_k$ is a path, with k vertices. Let G_0 be the graph obtained from G_i $(i = 1, 2)$ and P_k by adding the edges v_1u_1 and u_kv_2. Also, let $G = G_0 - \{v_1u_1, u_kv_2, v_1w\} + \{v_1v_2, wu_1, v_1u_k\}$. Then $\Pi^*(G) < \Pi^*(G_0)$ (see Figure 5).*

Proof. Let $d_{G_1}(v_1) = h$ and $d_{G_2}(v_2) = x$. Then

$$\begin{aligned} \frac{\Pi^*(G)}{\Pi^*(G_0)} &= \frac{3(x+h+3)}{(h+3)(x+3)} \\ &= \frac{3x+3h+9}{hx+3h+3x+9} < 1, \end{aligned}$$

since $x, h \geq 2$. □

Figure 5. The trees G_i (i=1,2), P_k, G_0, and G in Lemma 5.



3. Main Theorems

For positive integers $x_1, \ldots, x_m$, and $y_1, \ldots, y_m$, let $T(x_1^{(y_1)}, \ldots, x_m^{(y_m)})$ be the class of trees with y_i vertices of the degree x_i, $i = 1, \ldots, m$.

Theorem 1. *Let $\acute{T}$ be a tree in $\tau(n)$, where $n \geq 13$. If $\Delta(\acute{T}) \geq 4$ and $\acute{T} \notin T_n(n_1, n_2, n_3, n_4)$, then for each $T \in T_n(n_1, n_2, n_3, n_4)$, we have $\Pi^*(T) < \Pi^*(\acute{T})$. ($n_i \geq 2$, for $i = 1, 2, 3, 4$.)*

Proof. We consider the following cases:
Case 1. $\Delta(\acute{T}) = 4$. Since $\acute{T} \notin T_n(n_1, n_2, n_3, n_4)$, thus $\acute{T} \in T(4^{(1)}, 2^{(n-5)}, 1^{(4)})$ and there exists $i \in \{1, 2, 3, 4\}$ such that $n_i = 1$ or $\acute{T} \notin T(4^{(1)}, 2^{(n-5)}, 1^{(4)})$.
Subcase 1.1 Suppose that $\acute{T} \in T(4^{(1)}, 2^{(n-5)}, 1^{(4)})$ and there exists $i \in \{1, 2, 3, 4\}$ such that $n_i = 1$. Since $n \geq 13$, there exists $j \in \{1, 2, 3, 4\}$ such that $n_j \geq 3$. In Lemma 3, put $P_k = P_{n_j}$ and $w = P_{n_i} = P_1$. Using Lemma 3 gives us a tree, say $Q \in T_n(m_1, m_2, m_3, m_4)$ such that $|\{m_i | 1 \leq i \leq 4 \ and \ m_i = 1\}| < |\{n_i | 1 \leq i \leq 4 \ and \ n_i = 1\}|$. If $|\{m_i | 1 \leq i \leq 4 \ and \ m_i = 1\}| = 0$, then $Q \cong T$, and by Lemma 3, $\Pi^*(T) = \Pi^*(Q) < \Pi^*(\acute{T})$. Otherwise, we obtain the result by replacing Q with $\acute{T}$ and by repeating the above process.
Subcase 1.2 Suppose that $\acute{T} \notin T(4^{(1)}, 2^{(n-5)}, 1^{(4)})$.Then by repeated application of Lemmas 2 we obtain a tree, for example H such that $H \in T(4^{(1)}, 2^{(n-5)}, 1^{(4)})$. If for $i = 1, 2, 3, 4$, $n_i \geq 2$, Then $H \cong T$ and by Lemma 2, $\Pi^*(T) < \Pi^*(\acute{T})$. Otherwise, we obtain the result by replacing H with $\acute{T}$ in Subcase 1.1.
Case 2. If $\Delta(\acute{T}) > 4$, then by using Lemma 1, in finite stages, we can obtain a tree with maximal degree 4 such that the Π^* of this tree is less than $\Pi^*(\acute{T})$. Suppose that $u \in V(\acute{T})$, $d_{\acute{T}}(u) = \Delta(\acute{T})$, and $\bar{u} \in N[u, \acute{T}]$. In Lemma 1, we put $v_1 = u$ and $w = \bar{u}$. By cutting edge $u\bar{u}$ and attaching $\bar{u}$ to a pendent vertex in $n_u(u\bar{u}|\acute{T})$, we obtain the new tree T_1 and $d_{T_1}(u) = \Delta(\acute{T}) - 1$. We continue this process until all the vertices of $\acute{T}$ with degree $\Delta(\acute{T})$ are used. In this way, a tree

$\acute{T}_1$ of order n, and $\Delta(\acute{T}_1) = \Delta(\acute{T}) - 1$ is obtained. Now, by replacing $\acute{T}_1$ with $\acute{T}$, and by repeating above process we obtain a tree, say M such that $\Delta(M) = 4$. If $T \in T_n(n_1, n_2, n_3, n_4)$, and for $i = 1, 2, 3, 4$, $n_i \geq 2$. Then, $M \cong T$ and by Lemma 1, $\Pi^*(T) < \Pi^*(\acute{T})$. Otherwise, we obtain the result by replacing M with $\acute{T}$ in case 1. □

Notations: For a positive number $n \geq 13$, let:

$$\begin{aligned} A(n) \; = \; & \{T \in T(3^{(3)}, 2^{(n-8)}, 1^{(5)}) | \;\; m_{1,2}(T) = 5, \;\; m_{2,3}(T) = 5, \\ & m_{3,3}(T) = 2, \; \textit{and} \; m_{2,2}(T) = n - 13\}. \end{aligned}$$

It is easy to see that for each $T \in A(n)$, we have

$$\Pi^*(T) = 5^5 \times 6^2 \times 3^5 \times 4^{n-13} \; . \tag{8}$$

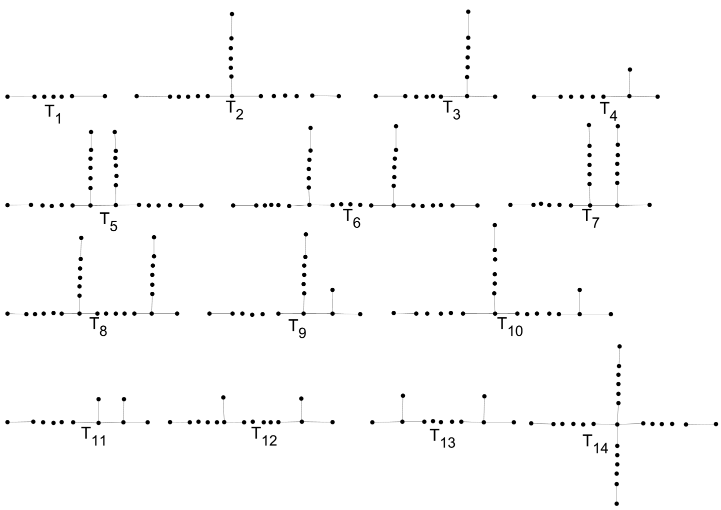
Theorem 2. *Let $\acute{T}$ be a tree with $\Delta(\acute{T}) = 3$ such that the number of its vertices of degree 3 is at least 3. Then, if $\acute{T} \notin A(n)$ for each $T \in A(n)$, we have $\Pi^*(T) < \Pi^*(\acute{T})$.*

Proof. We consider the following cases:
Case 1. The number of vertices of degree 3 in $\acute{T}$ is equal to 3. Since $\acute{T} \notin A(n)$, $m_{1,2}(\acute{T}) \neq 5 (m_{2,3}(\acute{T}) \neq 5)$, or $m_{3,3}(\acute{T}) \neq 2$ or both.
Subcase 1.1 Suppose that $m_{1,2}(\acute{T}) \neq 5 (m_{2,3}(\acute{T}) \neq 5)$ and $m_{3,3}(\acute{T}) \neq 2$. Then by repeated application of Lemmas 4, and 5, we obtain a tree, for example Q such that $m_{3,3}(Q) = 2$. If $m_{1,2}(Q) = 5 (m_{2,3}(Q) = 5)$, then $Q \in A(n)$, and $\Pi^*(Q) < \Pi^*(\acute{T})$, which completes the proof. If $m_{1,2}(Q) \neq 5 (m_{2,3}(Q) \neq 5)$, then since $n \geq 13$, by repeated application of Lemma 3 we obtain a tree in $A(n)$, with the first multiplicative Zagreb index less than Q, and therefore less than $\acute{T}$.
Subcase 1.2 Suppose that $m_{1,2}(\acute{T}) \neq 5 (m_{2,3}(\acute{T}) \neq 5)$ and $m_{3,3}(\acute{T}) = 2$. Since $n \geq 13$, by repeated application of Lemma 3 we obtain a tree in $A(n)$, with the first multiplicative Zagreb index less than $\acute{T}$.
Subcase 1.3 Suppose that $m_{1,2}(\acute{T}) = 5 (m_{2,3}(\acute{T}) = 5)$ and the $m_{3,3}(\acute{T}) \neq 2$. Then by repeated application of Lemmas 4, and 5, we obtain a tree, for example Q such that the vertices of $m_{3,3}(Q) = 2$, $Q \in A(n)$, and $\Pi^*(Q) < \Pi^*(\acute{T})$, which completes the proof.
Case 2. The number of vertices of degree 3 in $\acute{T}$ is greater than 3. In this case, by repeated application of Lemma 2, we obtain a tree, $T_m \in T(3^{(3)}, 2^{(n-8)}, 1^{(5)})$. If $m_{1,2}(T_m) = 5 (m_{2,3}(T_m) = 5)$ and $m_{3,3}(T_m) = 2$. Then $T_m \in A(n)$, and by Lemma 2, $\Pi^*(T_m) < \Pi^*(\acute{T})$, which completes the proof. Otherwise, we obtain the result by replacing $\acute{T}$ with T_m in case 1. □

Table 1. **Trees with smallest values of Π^* ($n_i, m_i, \geq 2$).**

Notation	Notation	Π^*
P_n		$3^2 \times 4^{n-3}$
$T_n(n_1, n_2, n_3)$		$5^3 \times 3^3 \times 4^{n-7}$
$T_n(n_1, n_2, 1)$		$5^2 \times 3^2 \times 4^{n-5}$
$T_n(n_1, 1, 1)$		$5 \times 3 \times 4^{n-3}$
$T_n^{\sim}(p_{n_1}, p_{n_2} : q_{m_1}, q_{m_2})$		$5^4 \times 6 \times 3^4 \times 4^{n-10}$
$T_n^{\not\sim}(p_{n_1}, p_{n_2} : q_{m_1}, q_{m_2})$		$5^6 \times 3^4 \times 4^{n-11}$
$T_n^{\sim}(p_{n_1}, p_{n_2} : q_{m_1}, 1)$		$5^3 \times 6 \times 3^3 \times 4^{n-8}$
$T_n^{\not\sim}(p_{n_1}, p_{n_2} : q_{m_1}, 1)$		$5^5 \times 3^3 \times 4^{n-9}$
$T_n^{\sim}(p_{n_1}, 1 : q_{m_1}, 1)$	$T_n^{\sim}(p_{n_1}, p_{n_2} : 1, 1)$	$5^2 \times 6 \times 3^2 \times 4^{n-6}$
$T_n^{\not\sim}(p_{n_1}, 1 : q_{m_1}, 1)$	$T_n^{\not\sim}(p_{n_1}, p_{n_2} : 1, 1)$	$5^4 \times 3^2 \times 4^{n-7}$
$T_n^{\sim}(p_{n_1}, 1 : 1, 1)$		$5 \times 6 \times 3 \times 4^{n-4}$
$T_n^{\not\sim}(p_{n_1}, 1 : 1, 1)$		$5^3 \times 3 \times 4^{n-5}$
$T_n^{\not\sim}(1, 1 : 1, 1)$		$5^2 \times 4^{n-3}$
$T_n(n_1, n_2, n_3, n_4)$		$6^4 \times 3^4 \times 4^{n-9}$

Figure 6. **The trees in Theorem 3.**



Theorem 3. *Let G be a tree with n vertices, except the trees given in Table 1. If $n \geq 13$ and $T_1 := P_n, T_2 \in T_n(n_1, n_2, n_3), T_3 \in T_n(n_1, n_2, 1), T_4 \in T_n(n_1, 1, 1), T_5 \in T_n^{\sim}(p_{n_1}, p_{n_2} : q_{m_1}, q_{m_2}), T_6 \in T_n^{\not\sim}(p_{n_1}, p_{n_2} : q_{m_1}, q_{m_2}), T_7 \in T_n^{\sim}(p_{n_1}, p_{n_2} : q_{m_1}, 1), T_8 \in T_n^{\not\sim}(p_{n_1}, p_{n_2} : q_{m_1}, 1), T_9 \in T_n^{\sim}(p_{n_1}, 1 : q_{m_1}, 1) \cup T_n^{\sim}(p_{n_1}, p_{n_2} : 1, 1), T_{10} \in T_n^{\not\sim}(p_{n_1}, 1 : q_{m_1}, 1) \cup T_n^{\not\sim}(p_{n_1}, p_{n_2} : 1, 1), T_{11} \in T_n^{\sim}(p_{n_1}, 1 : 1, 1), T_{12} \in T_n^{\not\sim}(p_{n_1}, 1 : 1, 1), T_{13} \in T_n^{\not\sim}(1, 1 : 1, 1)$, and $T_{14} \in T_n(n_1, n_2, n_3, n_4)$, then we have*

$$\Pi^*(T_1) < \Pi^*(T_2) < \Pi^*(T_3) < \Pi^*(T_4) < \Pi^*(T_5) < \Pi^*(T_6) < \Pi^*(T_7) < \Pi^*(T_8) <$$
$$\Pi^*(T_9) < \Pi^*(T_{10}) < \Pi^*(T_{11}) < \Pi^*(T_{12}) < \Pi^*(T_{13}) < \Pi^*(T_{14}) < \Pi^*(G).$$

Proof. Table 1 shows that:

$$\Pi^*(T_1) < \Pi^*(T_2) < \Pi^*(T_3) < \Pi^*(T_4) < \Pi^*(T_5) < \Pi^*(T_6) < \Pi^*(T_7) < \Pi^*(T_8) <$$
$$\Pi^*(T_9) < \Pi^*(T_{10}) < \Pi^*(T_{11}) < \Pi^*(T_{12}) < \Pi^*(T_{13}) < \Pi^*(T_{14}).$$

If $\Delta(G) \geq 4$, then Theorem 1 gives us the result. If $\Delta(G) = 3$ and the number of vertices of degree 3 is at least 3, then Theorem 2, and since for each $T \in A(n)$, $\Pi^*(T_{14}) < \Pi^*(T)$, completes the proof.
Otherwise, G is included in Table 1. □

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