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Bounds on the restrained Roman domination number of a graph

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Abstract: A *Roman dominating function* on a graph G is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. A *restrained Roman dominating* function f is a Roman dominating function if the vertices with label 0 induce a subgraph with no isolated vertex. The weight of a restrained Roman dominating function is the value $\omega(f) = \sum_{u\in V(G)} f(u)$. The minimum weight of a restrained Roman dominating function of G is called the *restrained Roman domination number* of G and denoted by $\gamma_{rR}(G)$. In this paper we establish some sharp bounds for this parameter.

Keywords: Roman dominating function, Roman domination number, restrained Roman dominating function, restrained Roman domination number

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1. Introduction

Throughout this paper, we only consider finite connected graph G with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E). A graph is simple if it has no loops and no two of its links join the same pair of vertices. For every vertex

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$v \in V$, the open neighborhood of v is the set $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v \in S} N(v)$, and the closed neighborhood of S is the set $N[S] = N(S) \cup S$. The *minimum* and *maximum* degree of G are respectively denoted by $\delta(G)$ and $\Delta(G)$. A *leaf* of a tree T is a vertex of degree 1, a *support vertex* is a vertex adjacent to a leaf and a *strong support vertex* is a vertex adjacent to at least two leaves. The number of leaves (support vertices, respectively) of a tree T will be denoted by $\ell(T)$ ($s(T)$, respectively). For $r, s \geq 1$, a double star $S(r, s)$ is a tree with exactly two vertices that are not leaves, with one adjacent to r leaves and the other to s leaves. For a real-valued function $f : V \to \mathbb{R}$ the *weight* of f is $\omega(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$ we define $f(S) = \sum_{v \in S} f(v)$, so $\omega(f) = f(V)$. For a vertex v in a rooted tree T, let $D(v)$ denotes the set of descendants of v and $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. A *subdivision* of an edge uv is obtained by removing the edge uv, adding a new vertex w, and adding edges uw and wv. The *subdivision graph* $S(G)$ is the graph obtained from G by subdividing each edge of G. The subdivision star $S(K_{1,t})$ for $t \geq 2$, is called a *healthy spider* S_t. A *wounded spider* S_t is the graph formed by subdividing at most $t-1$ of the edges of the star $K_{1,t}$ for $t \geq 2$. Note that stars are wounded spiders. A *spider* is a healthy or wounded spider. We use [10] for terminology and notation which are not defined here.

A subset S of vertices of G is a *restrained dominating set* if $N[S] = V$ and the subgraph induced by $V - S$ has no isolated vertex. The *restrained domination number* $\gamma_r(G)$ is the minimum cardinality of a restrained dominating set of G. The restrained domination number was introduced by Domke et al. [1] and has been studied by several authors [2–5].

A *Roman dominating function* (RDF) on a graph $G = (V, E)$ is defined in [8, 9] as a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex v for which $f(v) = 0$ is adjacent to at least one vertex u for which $f(u) = 2$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, equals the minimum weight of an RDF on G. A $\gamma_R(G)$-*function* is a Roman dominating function of G with weight $\gamma_R(G)$. A Roman dominating function $f : V \longrightarrow \{0, 1, 2\}$ can be represented by the ordered partition (V_0, V_1, V_2) (or (V_0^f, V_1^f, V_2^f) to refer f) of V, where $V_i = \{v \in V \mid f(v) = i\}$. In this representation, its weight is $\omega(f) = |V_1| + 2|V_2|$.

A Roman dominating function $f = (V_0, V_1, V_2)$ is called *restrained Roman dominating function* (RRDF) if the induced subgraph $G[V_0]$ has no isolated vertex. The restrained Roman domination number of G, denoted by $\gamma_{rR}(G)$, is the minimum weight of an RRDF on G. A $\gamma_{rR}(G)$-function is an RRDF of G with weight $\omega(f) = \gamma_{rR}(G)$. The restrained Roman domination number was introduced by P.R. Leely Pushpam and S. Padmapriea [7] and has been also studied in [6]. Pushpam and Padmapriea observed that

$$\max\{\gamma_R(G), \gamma_r(G)\} \leq \gamma_{rR}(G) \leq 2\gamma_r(G). \tag{1}$$

Our purpose in this paper is to establish two sharp bounds on the restrained Roman domination numbers in graphs. Some of our results improve some previous results.

We make use of the following results.

Let $C = (x_1x_2x_3x_4x_5)$ be a cycle of length 5. Assume B_p is the graph obtained from C by adding $p \geq 1$ pendant edges at some x_i and $B_{p,q}$ is the graph obtained from C by adding $p \geq 1$ pendant edges at some x_i and $q \geq 1$ pendant edges at some x_j where $d(x_i, x_j) = 2$.

Theorem 1. [7] Let G be a connected graph of order $n \geq 2$. Then $\gamma_{rR}(G) = n$ if and only if $G \simeq C_4, C_5, B_p, B_{p,q}$ or G is a tree with $\operatorname{diam}(G) \leq 5$.

Observation 1. If H is a subgraph of G, then $\gamma_{rR}(G) \leq \gamma_{rR}(H) + |V(G)| - |V(H)|$.

2. Bounds on the restrained Roman domination number

In this section we establish two sharp bounds on the restrained Roman domination number, one of which improves a previous result.

The Dutch-windmill graph, $K_3^{(m)}$ with $m \geq 2$, is a graph which consists of m copies of K_3 with a vertex in common. Clearly $\gamma_{rR}(K_3^{(m)}) = 2$. Jafari Rad and Krzywkowski in [6] proved the following lower bound for the restrained Roman domination number of general graphs and characterized all extreme graphs.

Theorem 2. For every connected graph G of order $n \geq 3$ with m edges we have $\gamma_{rR}(G) \geq n + 1 - \frac{2m}{3}$, with equality if and only if G is a Dutch-windmill graph of order at least 7.

Observation 2. If a graph G has $f = (\emptyset, V(G), \emptyset)$ as a unique $\gamma_{rR}(G)$-function, then $G \simeq K_1$ or $K_{1,s}$ with $s \geq 1$.

Let $\mathcal{I}$ denote the set of all mutually non-isomorphic multigraphs without isolated vertices and let $\mathcal{H} = \{H \mid H \text{ is obtained from some } F \in \mathcal{I}, \text{ by subdividing each edge of } F \text{ twice}\}$.

Theorem 3. Let G be a connected graph of order $n \geq 4$ and size m. Then

$$\gamma_{rR}(G) \geq 2n - \frac{4m}{3}$$

with equality if and only if either $G \simeq K_{1,3}$ or $G \in \mathcal{H}$.

Proof. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{rR}(G)$-function, so that $|V_2|$ is maximum. If $V_2 = \emptyset$, then $V_1 = V(G)$ and so $f = (\emptyset, V(G), \emptyset)$ is the unique γ_{rR}-function of G. Now Observation 2 implies that $G = K_{1,n-1}$, and we have $\gamma_{rR}(G) = n \geq (2n+4)/3 = 2n - 4m/3$, with equality if and only if $G = K_{1,3}$.
Now, consider $V_2 \neq \emptyset$. Let m_i be the size of the induced subgraph $G[V_i]$, for $i = 1, 2, 3$, $m_{2,0}$ the number of edges between V_2 and V_0, and m_3 the number of edges between V_1 and $V_0 \cup V_2$. Then $m_0 = \frac{1}{2}\sum_{v \in V_0} \deg_{G[V_0]}(v) \geq \frac{1}{2}|V_0| = \frac{1}{2}(n - |V_1| - |V_2|)$ because $G[V_0]$ has no isolated vertex. Since V_2 dominates V_0, every vertex in V_0 is adjacent to at least one vertex in V_2 and hence $m_{2,0} \geq |V_0| = n - |V_1| - |V_2|$. On the other hand, since G is connected, we must have $m_1 + m_3 \geq |V_1|$. Thus

$$\begin{aligned} m &\geq m_0 + m_1 + m_2 + m_3 + m_{2,0} \\ &\geq \tfrac{3n}{2} - \tfrac{3|V_2|}{2} - \tfrac{|V_1|}{2} + m_2 \\ &= \tfrac{3n}{2} - \tfrac{6|V_2|}{4} - \tfrac{3|V_1|}{4} + \tfrac{|V_1|}{4} + m_2 \\ &= \tfrac{3n}{2} - \tfrac{3}{4}\gamma_{rR}(G) + \tfrac{|V_1|}{4} + m_2. \end{aligned}$$

This implies that $\gamma_{rR}(G) \geq 2n - \frac{4m}{3}$.
Suppose $\gamma_{rR}(G) = 2n - \frac{4m}{3}$. Then $|V_1| = m_2 = 0$ and all inequalities occurring in the proof become equalities. Hence
(a) $V_1 = \emptyset$,
(b) V_2 is an independent dominating set of G;
(c) $G[V_0]$ is a 1-regular graph;
(d) every vertex in V_0 is adjacent to exactly one vertex in V_2.
Clearly (a)–(d) lead to $G \in \mathcal{H}$.
Conversely, let $G \in \mathcal{H}$. Hence G is obtained from some $F \in \mathcal{I}$ by subdividing each edge of F twice. If F has order n_1 and size m_1, then $n = n_1 + 2m_1$, $m = 3m_1$ and $\gamma_{rR}(G) \geq 2n - \frac{4m}{3} = 2n_1$. Clearly, $V(F)$ is a restrained dominating set of G and hence $\gamma_r(G) \leq n_1$. It follows from (1) that $\gamma_{rR}(G) \leq 2\gamma_r(G) \leq 2n_1$. Thus $\gamma_{rR}(G) = 2n_1$ and the proof is complete. □

Since $\gamma_{rR}(G) \geq 2$ for every connected graph G of order $n \geq 3$, the aforementioned bound is useless unless $n - 2m/3 \geq 1$. Thus, our first result improves the bound of Theorem 2
Jafari Rad and Krzywkowski in [6] proved the following lower bound for the restrained Roman domination number of trees and characterized all extreme trees.

Theorem 4. For every tree T of diameter at least three, order n, with $\ell(T)$ leaves and $s(T)$ support vertices, we have $\gamma_{rR}(T) \geq (2n + \ell(T) - s(T) + 4)/3$.

In the sequel, we present a similar sharp upper bound for the restrained Roman domination number in trees.

Theorem 5. Let T be a tree of order $n \geq 3$. Then

$$\gamma_{rR}(T) \leq \left\lceil \frac{2n + 5s(T) + \ell(T) - 4}{3} \right\rceil.$$

This bound is sharp for stars.

Proof. The proof is by induction on n. The statement holds for all trees of order $n = 3, 4$. For the inductive hypothesis, let $n \geq 5$ and suppose that for every nontrivial tree T of order less than n the result is true. Assume that T is a tree of order n. If $\operatorname{diam}(T) = 2$, then T is the star $K_{1,n-1}$ for which $s(T) = 1, \ell(T) = n - 1$ and $\gamma_{rR}(T) = n$, and so $\gamma_{rR}(T) = n = \left\lceil \frac{2n+5s(T)+\ell(T)-4}{3} \right\rceil$. If $\operatorname{diam}(T) = 3$, then T is a double star $S(r, s)$ for some integers $r, s \geq 1$. In this case, $s(T) = 2, \ell(T) = r + s$ and $\gamma_{rR}(T) = n$. Hence $\gamma_{rR}(T) = n < \left\lceil \frac{3n+4}{3} \right\rceil = \left\lceil \frac{2n+5s(T)+\ell(T)-4}{3} \right\rceil$. Hence we may assume that $\operatorname{diam}(T) \geq 4$.
Now, consider T has a strong support vertex. Assume that u is a strong support vertex and v, w are two leaves adjacent to u. Let $T' = T - v$ and $f = (V_0, V_1, V_2)$ be a $\gamma_{rR}(T')$-function. Then $|V(T')| = n - 1$, $s(T') = s(T)$ and $\ell(T') = \ell(T) - 1$. It is easy to see that $g = (V_0, V_1 \cup \{v\}, V_2)$ is an RRDF of T and hence $\gamma_{rR}(T) \leq \gamma_{rR}(T') + 1$. It follows from the inductive hypothesis that

$$\begin{aligned}
\gamma_{rR}(T) &\leq \gamma_{rR}(T') + 1 \\
&\leq \left\lceil \tfrac{2(n-1)+5s(T')+\ell(T')-4}{3} \right\rceil + 1 \\
&= \left\lceil \tfrac{2n+5s(T)+\ell(T)-12}{3} \right\rceil + 1 \\
&< \left\lceil \tfrac{2n+5s(T)+\ell(T)-4}{3} \right\rceil,
\end{aligned}$$

as desired.
Now, consider T has no a strong support vertex. If $\operatorname{diam}(T) = 4$, then T is a spider and we have $s(T) \geq 2$ and $\ell(T) + s(T) \geq n - 1$. It follows from Theorem 1 that $\gamma_{rR}(T) = n \leq \left\lceil \frac{2n+5s(T)+\ell(T)-4}{3} \right\rceil$. Suppose $\operatorname{diam}(T) \geq 5$. Let $v_1 v_2 \ldots v_D$ be a diametral path in T and root T at v_D. Since T has no strong support vertex, we have $\deg(v_2) = \deg(v_{D-1}) = 2$ and v_3 is only adjacent to a leaf or to a support vertex of degree 2. We consider two cases:

Case 1: $\deg(v_3) \geq 3$.
Let $T' = T - \{v_1, v_2\}$. Then $|V(T')| = n - 2$, $s(T') = s(T) - 1$ and $\ell(T') =$

$\ell(T)-1$. It is easy to see that $\gamma_{rR}(T) \leq \gamma_{rR}(T')+2$. By the inductive hypothesis, we obtain

$$\begin{aligned}\gamma_{rR}(T) &\leq \gamma_{rR}(T')+2\\ &\leq \left\lceil \tfrac{2(n-2)+5s(T')+\ell(T')-4}{3}\right\rceil+2\\ &= \left\lceil \tfrac{2n+5s(T)+\ell(T)-14}{3}\right\rceil+2\\ &\leq \left\lceil \tfrac{2n+5s(T)+\ell(T)-4}{3}\right\rceil.\end{aligned}$$

Case 2: $\deg(v_3)=2$.
We distinguish the following subcases.

Subcase 2.1: $\deg(v_4)\geq 3$.
Let $T'=T-T_{v_3}$ and f be a $\gamma_{rR}(T')$-function. Then $|V(T')|=n-3$, $s(T')=s(T)-1$ and $\ell(T')=\ell(T)-1$. Clearly, f can be extended to an RRDF of T by assigning 1 to v_1,v_2,v_3. Thus $\gamma_{rR}(T)\leq\gamma_{rR}(T')+3$. By the inductive hypothesis, we have

$$\begin{aligned}\gamma_{rR}(T) &\leq \gamma_{rR}(T')+3\\ &\leq \left\lceil \tfrac{2(n-3)+5s(T')+\ell(T')-4}{3}\right\rceil+3\\ &= \left\lceil \tfrac{2n+5s(T)+\ell(T)-16}{3}\right\rceil+3\\ &\leq \left\lceil \tfrac{2n+5s(T)+\ell(T)-4}{3}\right\rceil.\end{aligned}$$

Subcase 2.2: $\deg(v_4)=2$ and $\deg(v_5)\geq 3$.
Let $T'=T-T_{v_4}$ and f be a $\gamma_{rR}(T')$-function. Then $|V(T')|=n-4$, $s(T')=s(T)-1$ and $\ell(T')=\ell(T)-1$. Obviously, f can be extended to an RRDF of T by assigning 1 to v_1,v_2,v_3,v_4. Thus $\gamma_{rR}(T)\leq\gamma_{rR}(T')+4$. It follows from the inductive hypothesis that

$$\begin{aligned}\gamma_{rR}(T) &\leq \gamma_{rR}(T')+4\\ &\leq \left\lceil \tfrac{2(n-4)+5s(T')+\ell(T')-4}{3}\right\rceil+4\\ &= \left\lceil \tfrac{2n+5s(T)+\ell(T)-18}{3}\right\rceil+4\\ &\leq \left\lceil \tfrac{2n+5s(T)+\ell(T)-4}{3}\right\rceil.\end{aligned}$$

Subcase 2.3: $\deg(v_4)=\deg(v_5)=2$ and $\deg(v_6)\geq 3$.

If v_5 is a support vertex, then $T = P_6$ for which the result is true. So, suppose v_5 is not a support vertex. Let $T' = T - T_{v_5}$ and f be a $\gamma_{rR}(T')$-function. Then $|V(T')| = n-5$, $s(T') = s(T)-1$ and $\ell(T') = \ell(T)-1$. Then f can be extended to an RRDF of T by assigning 1 to v_1, v_2, v_3, v_4, v_5. Thus $\gamma_{rR}(T) \leq \gamma_{rR}(T')+5$. By the inductive hypothesis, we have

$$\begin{aligned}
\gamma_{rR}(T) &\leq \gamma_{rR}(T') + 5 \\
&\leq \left\lceil \tfrac{2(n-5)+5s(T')+\ell(T')-4}{3} \right\rceil + 5 \\
&= \left\lceil \tfrac{2n+5s(T)+\ell(T)-20}{3} \right\rceil + 5 \\
&\leq \left\lceil \tfrac{2n+5s(T)+\ell(T)-4}{3} \right\rceil .
\end{aligned}$$

Subcase 2.4: $\deg(v_4) = \deg(v_5) = \deg(v_6) = 2$.
If v_6 is a support vertex, then $T = P_7$ for which the result is true. So, suppose v_6 is not a support vertex. If $deg(v_7) \geq 3$, then assume that $T' = T - T_{v_6}$ and f is a $\gamma_{rR}(T')$-function. Then $|V(T')| = n - 6$, $s(T') = s(T) - 1$ and $\ell(T') = \ell(T) - 1$. Clearly, f can be extended to an RRDF of T by assigning 1 to $v_1, v_2, v_3, v_4, v_5, v_6$. Thus $\gamma_{rR}(T) \leq \gamma_{rR}(T')+6$. By the inductive hypothesis we have

$$\begin{aligned}
\gamma_{rR}(T) &\leq \gamma_{rR}(T') + 6 \\
&\leq \left\lceil \tfrac{2(n-6)+5s(T')+\ell(T')-4}{3} \right\rceil + 6 \\
&\leq \left\lceil \tfrac{2n+5s(T)+\ell(T)-22}{3} \right\rceil + 6 \\
&= \left\lceil \tfrac{2n+5s(T)+\ell(T)-4}{3} \right\rceil .
\end{aligned}$$

Now let $\deg(v_7) = 2$. If v_7 is a support vertex, then $T = P_8$ and the result is clearly true. Hence, we suppose that v_7 is not a support vertex. Let $T' = T - T_{v_7}$ and f be a $\gamma_{rR}(T')$-function. Then $|V(T')| = n - 7$, $s(T') = s(T) - 1$ and $\ell(T') = \ell(T) - 1$. Then f can be extended to an RRDF of T by assigning 2 to v_1, v_4, v_7 and 0 to v_2, v_3, v_5, v_6. Thus $\gamma_{rR}(T) \leq \gamma_{rR}(T') + 6$. It follows from the inductive hypothesis that $\gamma_{rR}(T) < \left\lceil \frac{2n+5s(T)+\ell(T)-4}{3} \right\rceil$ and the proof is complete. $\square$

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