Bounds on the restrained Roman domination number of a graph

H. Abdollahzadeh Ahangar* and S.R. Mirmehdipour

Department of Basic Science, Babol Noshirvani University of Technology, Babol, I.R. Iran
ha.ahangar@nit.ac.ir, r.m.mehdipor@gmail.com

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Abstract: A Roman dominating function on a graph $G$ is a function $f : V(G) \rightarrow \{0, 1, 2\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. A restrained Roman dominating function $f$ is a Roman dominating function if the vertices with label 0 induce a subgraph with no isolated vertex. The weight of a restrained Roman dominating function is the value $\omega(f) = \sum_{u \in V(G)} f(u)$. The minimum weight of a restrained Roman dominating function of $G$ is called the restrained Roman domination number of $G$ and denoted by $\gamma_{rR}(G)$. In this paper we establish some sharp bounds for this parameter.

Keywords: Roman dominating function, Roman domination number, restrained Roman dominating function, restrained Roman domination number

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1. Introduction

Throughout this paper, we only consider finite connected graph $G$ with vertex set $V(G)$ and edge set $E(G)$ (briefly $V$ and $E$). A graph is simple if it has no loops and no two of its links join the same pair of vertices. For every vertex

* Corresponding Author
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Let $v \in V$, the open neighborhood of $v$ is the set $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood of $v$ is the set $N[v] = N(v) \cup \{v\}$. The open neighborhood of a set $S \subseteq V$ is the set $N(S) = \cup_{v \in S} N(v)$, and the closed neighborhood of $S$ is the set $N[S] = N(S) \cup S$. The minimum and maximum degree of $G$ are respectively denoted by $\delta(G)$ and $\Delta(G)$. A leaf of a tree $T$ is a vertex of degree 1, a support vertex is a vertex adjacent to a leaf and a strong support vertex is a vertex adjacent to at least two leaves. The number of leaves (support vertices, respectively) of a tree $T$ will be denoted by $\ell(T)$ ($s(T)$, respectively).

For $r, s \geq 1$, a double star $S(r, s)$ is a tree with exactly two vertices that are not leaves, with one adjacent to $r$ leaves and the other to $s$ leaves. For a real-valued function $f : V \to \mathbb{R}$ the weight of $f$ is $\omega(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$ we define $f(S) = \sum_{v \in S} f(v)$, so $\omega(f) = f(V)$. For a vertex $v$ in a rooted tree $T$, let $D(v)$ denote the set of descendants of $v$ and $D[v] = D(v) \cup \{v\}$. The maximal subtree at $v$ is the subtree of $T$ induced by $D[v]$, and is denoted by $T_v$. A subdivision of an edge $uv$ is obtained by removing the edge $uv$, adding a new vertex $w$, and adding edges $uw$ and $wv$. The subdivision graph $S(G)$ is the graph obtained from $G$ by subdividing each edge of $G$. The subdivision star $S(K_{1,t})$ for $t \geq 2$, is called a healthy spider $S_t$. A wounded spider $S_t$ is the graph formed by subdividing at most $t-1$ of the edges of the star $K_{1,t}$ for $t \geq 2$. Note that stars are wounded spiders. A spider is a healthy or wounded spider. We use [10] for terminology and notation which are not defined here.

A subset $S$ of vertices of $G$ is a restrained dominating set if $N[S] = V$ and the subgraph induced by $V - S$ has no isolated vertex. The restrained domination number $\gamma_r(G)$ is the minimum cardinality of a restrained dominating set of $G$. The restrained domination number was introduced by Domke et al. [1] and has been studied by several authors [2–5].

A Roman dominating function (RDF) on a graph $G = (V,E)$ is defined in [8, 9] as a function $f : V \to \{0,1,2\}$ satisfying the condition that every vertex $v$ for which $f(v) = 0$ is adjacent to at least one vertex $u$ for which $f(u) = 2$. The Roman domination number of a graph $G$, denoted by $\gamma_R(G)$, equals the minimum weight of an RDF on $G$. A $\gamma_R(G)$-function is a Roman dominating function of $G$ with weight $\gamma_R(G)$. A Roman dominating function $f : V \to \{0,1,2\}$ can be represented by the ordered partition $(V_0, V_1, V_2)$ (or $(V_0^f, V_1^f, V_2^f)$ to refer $f$) of $V$, where $V_i = \{v \in V \mid f(v) = i\}$. In this representation, its weight is $\omega(f) = |V_1| + 2|V_2|$. A Roman dominating function $f = (V_0, V_1, V_2)$ is called restrained Roman dominating function (RRDF) if the induced subgraph $G[V_0]$ has no isolated vertex. The restrained Roman domination number of $G$, denoted by $\gamma_{rR}(G)$, is the minimum weight of an RRDF on $G$. A $\gamma_{rR}(G)$-function is an RRDF of $G$ with weight $\omega(f) = \gamma_{rR}(G)$. The restrained Roman domination number was introduced by P.R. Leely Pushpam and S. Padmapriya [7] and has been also
studied in [6]. Pushpam and Padmapriya observed that

\[ \max \{ \gamma_R(G), \gamma_r(G) \} \leq \gamma_{rR}(G) \leq 2\gamma_r(G). \]  (1)

Our purpose in this paper is to establish two sharp bounds on the restrained Roman domination numbers in graphs. Some of our results improve some previous results.

We make use of the following results.

Let \( C = (x_1 x_2 x_3 x_4 x_5) \) be a cycle of length 5. Assume \( B_p \) is the graph obtained from \( C \) by adding \( p \geq 1 \) pendant edges at some \( x_i \) and \( B_{p,q} \) is the graph obtained from \( C \) by adding \( p \geq 1 \) pendant edges at some \( x_i \) and \( q \geq 1 \) pendant edges at some \( x_j \) where \( d(x_i, x_j) = 2 \).

**Theorem 1.** [7] Let \( G \) be a connected graph of order \( n \geq 2 \). Then \( \gamma_{rR}(G) = n \) if and only if \( G \cong C_4, C_5, B_p, B_{p,q} \) or \( G \) is a tree with \( \text{diam}(G) \leq 5 \).

**Observation 1.** If \( H \) is a subgraph of \( G \), then \( \gamma_{rR}(G) \leq \gamma_{rR}(H) + |V(G)| - |V(H)| \).

### 2. Bounds on the restrained Roman domination number

In this section we establish two sharp bounds on the restrained Roman domination number, one of which improves a previous result.

The Dutch-windmill graph, \( K_3^{(m)} \) with \( m \geq 2 \), is a graph which consists of \( m \) copies of \( K_3 \) with a vertex in common. Clearly \( \gamma_{rR}(K_3^{(m)}) = 2 \). Jafari Rad and Krzywkowski in [6] proved the following lower bound for the restrained Roman domination number of general graphs and characterized all extreme graphs.

**Theorem 2.** For every connected graph \( G \) of order \( n \geq 3 \) with \( m \) edges we have \( \gamma_{rR}(G) \geq n + 1 \frac{2m}{3} \), with equality if and only if \( G \) is a Dutch-windmill graph of order at least 7.

**Observation 2.** If a graph \( G \) has \( f = (\emptyset, V(G), \emptyset) \) as a unique \( \gamma_{rR}(G) \)-function, then \( G \cong K_1 \) or \( K_{1,s} \) with \( s \geq 1 \).

Let \( \mathcal{I} \) denote the set of all mutually non-isomorphic multigraphs without isolated vertices and let \( \mathcal{H} = \{ H \mid H \text{ is obtained from some } F \in \mathcal{I}, \text{ by subdividing each edge of } F \text{ twice} \} \).

**Theorem 3.** Let \( G \) be a connected graph of order \( n \geq 4 \) and size \( m \). Then

\[ \gamma_{rR}(G) \geq 2n - \frac{4m}{3} \]
with equality if and only if either $G \cong K_{1,3}$ or $G \in \mathcal{H}$.

**Proof.** Let $f = (V_0, V_1, V_2)$ be a $\gamma_{rR}(G)$-function, so that $|V_2|$ is maximum. If $V_2 = \emptyset$, then $V_1 = V(G)$ and so $f = (\emptyset, V(G), \emptyset)$ is the unique $\gamma_{rR}$-function of $G$. Now Observation 2 implies that $G = K_{1,n-1}$, and we have $\gamma_{rR}(G) = n \geq (2n + 4)/3 = 2n - 4m/3$, with equality if and only if $G = K_{1,3}$.

Now, consider $V_2 \neq \emptyset$. Let $m_i$ be the size of the induced subgraph $G[V_i]$, for $i = 1, 2, 3$, $m_{2,0}$ the number of edges between $V_2$ and $V_0$, and $m_3$ the number of edges between $V_1$ and $V_0 \cup V_2$. Then $m_0 = \frac{1}{2} \sum_{v \in V_0} \deg_{G[V_0]}(v) \geq \frac{1}{2} |V_0| = \frac{1}{2} (n - |V_1| - |V_2|)$ because $G[V_0]$ has no isolated vertex. Since $V_2$ dominates $V_0$, every vertex in $V_0$ is adjacent to at least one vertex in $V_2$ and hence $m_{2,0} \geq |V_0| = n - |V_1| - |V_2|$. On the other hand, since $G$ is connected, we must have $m_1 + m_3 \geq |V_1|$. Thus

$$m \geq m_0 + m_1 + m_2 + m_3 + m_{2,0} \geq \frac{3n}{2} - \frac{3|V_2|}{2} - \frac{|V_1|}{2} + m_2 \geq \frac{3n}{2} - \frac{6|V_2|}{4} - \frac{3|V_1|}{4} + \frac{|V_1|}{4} + m_2 \geq \frac{3n}{2} - \frac{3}{4} \gamma_{rR}(G) + \frac{|V_1|}{4} + m_2.$$

This implies that $\gamma_{rR}(G) \geq 2n - \frac{4m}{3}$.

Suppose $\gamma_{rR}(G) = 2n - \frac{4m}{3}$. Then $|V_1| = m_2 = 0$ and all inequalities occurring in the proof become equalities. Hence

(a) $V_1 = \emptyset$,
(b) $V_2$ is an independent dominating set of $G$;
(c) $G[V_0]$ is a 1-regular graph;
(d) every vertex in $V_0$ is adjacent to exactly one vertex in $V_2$.

Clearly (a)–(d) lead to $G \in \mathcal{H}$.

Conversely, let $G \in \mathcal{H}$. Hence $G$ is obtained from some $F \in \mathcal{I}$ by subdividing each edge of $F$ twice. If $F$ has order $n_1$ and size $m_1$, then $n = n_1 + 2m_1$, $m = 3m_1$ and $\gamma_{rR}(G) \geq 2n - \frac{4m}{3} = 2n_1$. Clearly, $V(F)$ is a restrained dominating set of $G$ and hence $\gamma_r(G) \leq n_1$. It follows from (1) that $\gamma_{rR}(G) \leq 2\gamma_r(G) \leq 2n_1$. Thus $\gamma_{rR}(G) = 2n_1$ and the proof is complete.

Since $\gamma_{rR}(G) \geq 2$ for every connected graph $G$ of order $n \geq 3$, the aforementioned bound is useless unless $n - 2m/3 \geq 1$. Thus, our first result improves the bound of Theorem 2.

Jafari Rad and Krzywkowski in [6] proved the following lower bound for the restrained Roman domination number of trees and characterized all extreme trees.

**Theorem 4.** For every tree $T$ of diameter at least three, order $n$, with $\ell(T)$ leaves and $s(T)$ support vertices, we have $\gamma_{rR}(T) \geq (2n + \ell(T) - s(T) + 4)/3$. 

In the sequel, we present a similar sharp upper bound for the restrained Roman domination number in trees.

**Theorem 5.** Let $T$ be a tree of order $n \geq 3$. Then

$$\gamma_{rR}(T) \leq \left\lceil \frac{2n + 5s(T) + \ell(T) - 4}{3} \right\rceil.$$ 

This bound is sharp for stars.

**Proof.** The proof is by induction on $n$. The statement holds for all trees of order $n = 3, 4$. For the inductive hypothesis, let $n \geq 5$ and suppose that for every nontrivial tree $T$ of order less than $n$ the result is true. Assume that $T$ is a tree of order $n$. If $\text{diam}(T) = 2$, then $T$ is the star $K_{1,n-1}$ for which $s(T) = 1$, $\ell(T) = n - 1$ and $\gamma_{rR}(T) = n$, and so $\gamma_{rR}(T) = n = \left\lceil \frac{2n + 5s(T) + \ell(T) - 4}{3} \right\rceil$. If $\text{diam}(T) = 3$, then $T$ is a double star $S(r, s)$ for some integers $r, s \geq 1$. In this case, $s(T) = 2$, $\ell(T) = r + s$ and $\gamma_{rR}(T) = n$. Hence $\gamma_{rR}(T) = n = \left\lceil \frac{3n+4}{3} \right\rceil$. Hence we may assume that $\text{diam}(T) \geq 4$.

Now, consider $T$ has a strong support vertex. Assume that $u$ is a strong support vertex and $v, w$ are two leaves adjacent to $u$. Let $T' = T - v$ and $f = (V_0, V_1, V_2)$ be a $\gamma_{rR}(T')$-function. Then $|V(T')| = n - 1$, $s(T') = s(T)$ and $\ell(T') = \ell(T) - 1$. It is easy to see that $g = (V_0, V_1 \cup \{v\}, V_2)$ is an RRDF of $T$ and hence $\gamma_{rR}(T) \leq \gamma_{rR}(T') + 1$. It follows from the inductive hypothesis that

$$\gamma_{rR}(T) \leq \gamma_{rR}(T') + 1 \leq \left\lceil \frac{2(n-1) + 5s(T') + \ell(T') - 4}{3} \right\rceil + 1 = \left\lceil \frac{2n + 5s(T) + \ell(T) - 12}{3} \right\rceil + 1 < \left\lceil \frac{2n + 5s(T) + \ell(T) - 4}{3} \right\rceil,$$

as desired.

Now, consider $T$ has no a strong support vertex. If $\text{diam}(T) = 4$, then $T$ is a spider and we have $s(T) \geq 2$ and $\ell(T) + s(T) \geq n - 1$. It follows from Theorem 1 that $\gamma_{rR}(T) = n \leq \left\lceil \frac{2n + 5s(T) + \ell(T) - 4}{3} \right\rceil$. Suppose $\text{diam}(T) \geq 5$.

Let $v_1v_2 \ldots v_D$ be a diametral path in $T$ and root $T$ at $v_D$. Since $T$ has no strong support vertex, we have $\text{deg}(v_2) = \text{deg}(v_{D-1}) = 2$ and $v_3$ is only adjacent to a leaf or to a support vertex of degree 2. We consider two cases:

**Case 1:** $\text{deg}(v_3) \geq 3$.

Let $T' = T - \{v_1, v_2\}$. Then $|V(T')| = n - 2$, $s(T') = s(T) - 1$ and $\ell(T') =$
\( \ell(T) - 1 \). It is easy to see that \( \gamma_{rR}(T) \leq \gamma_{rR}(T') + 2 \). By the inductive hypothesis, we obtain

\[
\gamma_{rR}(T) \leq \gamma_{rR}(T') + 2 \\
\leq \left\lceil \frac{2(n-2)+5s(T') + \ell(T') - 4}{3} \right\rceil + 2 \\
= \left\lceil \frac{2n+5s(T) + \ell(T) - 14}{3} \right\rceil + 2 \\
\leq \left\lceil \frac{2n+5s(T) + \ell(T) - 14}{3} \right\rceil.
\]

**Case 2:** \( \text{deg}(v_3) = 2 \).
We distinguish the following subcases.

**Subcase 2.1:** \( \text{deg}(v_4) \geq 3 \).
Let \( T' = T - T_{v_3} \) and \( f \) be a \( \gamma_{rR}(T') \)-function. Then \( |V(T')| = n - 3 \), \( s(T') = s(T) - 1 \) and \( \ell(T') = \ell(T) - 1 \). Clearly, \( f \) can be extended to an RRDF of \( T \) by assigning 1 to \( v_1, v_2, v_3 \). Thus \( \gamma_{rR}(T) \leq \gamma_{rR}(T') + 3 \). By the inductive hypothesis, we have

\[
\gamma_{rR}(T) \leq \gamma_{rR}(T') + 3 \\
\leq \left\lceil \frac{2(n-3)+5s(T') + \ell(T') - 4}{3} \right\rceil + 3 \\
= \left\lceil \frac{2n+5s(T) + \ell(T) - 16}{3} \right\rceil + 3 \\
\leq \left\lceil \frac{2n+5s(T) + \ell(T) - 14}{3} \right\rceil.
\]

**Subcase 2.2:** \( \text{deg}(v_4) = 2 \) and \( \text{deg}(v_5) \geq 3 \).
Let \( T' = T - T_{v_4} \) and \( f \) be a \( \gamma_{rR}(T') \)-function. Then \( |V(T')| = n - 4 \), \( s(T') = s(T) - 1 \) and \( \ell(T') = \ell(T) - 1 \). Obviously, \( f \) can be extended to an RRDF of \( T \) by assigning 1 to \( v_1, v_2, v_3, v_4 \). Thus \( \gamma_{rR}(T) \leq \gamma_{rR}(T') + 4 \). It follows from the inductive hypothesis that

\[
\gamma_{rR}(T) \leq \gamma_{rR}(T') + 4 \\
\leq \left\lceil \frac{2(n-4)+5s(T') + \ell(T') - 4}{3} \right\rceil + 4 \\
= \left\lceil \frac{2n+5s(T) + \ell(T) - 18}{3} \right\rceil + 4 \\
\leq \left\lceil \frac{2n+5s(T) + \ell(T) - 14}{3} \right\rceil.
\]

**Subcase 2.3:** \( \text{deg}(v_4) = \text{deg}(v_5) = 2 \) and \( \text{deg}(v_6) \geq 3 \).
If $v_5$ is a support vertex, then $T = P_5$ for which the result is true. So, suppose $v_5$ is not a support vertex. Let $T' = T - v_5$ and $f$ be a $\gamma_{rR}(T')$-function. Then $|V(T')| = n - 5$, $s(T') = s(T) - 1$ and $\ell(T') = \ell(T) - 1$. Then $f$ can be extended to an RRDF of $T$ by assigning 1 to $v_1, v_2, v_3, v_4, v_5$. Thus $\gamma_{rR}(T) \leq \gamma_{rR}(T') + 5$. By the inductive hypothesis, we have

$$\gamma_{rR}(T) \leq \gamma_{rR}(T') + 5 \leq \left\lceil \frac{2(n-5)+5s(T')-\ell(T')}{3} \right\rceil + 5 \leq \left\lceil \frac{2n+5s(T)+\ell(T)-20}{3} \right\rceil + 5 \leq \left\lceil \frac{2n+5s(T)+\ell(T)-4}{3} \right\rceil.$$

**Subcase 2.4:** $\deg(v_4) = \deg(v_5) = \deg(v_6) = 2.$

If $v_6$ is a support vertex, then $T = P_7$ for which the result is true. So, suppose $v_6$ is not a support vertex. If $\deg(v_7) \geq 3$, then assume that $T' = T - v_6$ and $f$ is a $\gamma_{rR}(T')$-function. Then $|V(T')| = n - 6$, $s(T') = s(T) - 1$ and $\ell(T') = \ell(T) - 1$. Clearly, $f$ can be extended to an RRDF of $T$ by assigning 1 to $v_1, v_2, v_3, v_4, v_5, v_6$. Thus $\gamma_{rR}(T) \leq \gamma_{rR}(T') + 6$. By the inductive hypothesis we have

$$\gamma_{rR}(T) \leq \gamma_{rR}(T') + 6 \leq \left\lceil \frac{2(n-6)+5s(T')-\ell(T')}{3} \right\rceil + 6 \leq \left\lceil \frac{2n+5s(T)+\ell(T)-22}{3} \right\rceil + 6 \leq \left\lceil \frac{2n+5s(T)+\ell(T)-4}{3} \right\rceil.$$

Now let $\deg(v_7) = 2$. If $v_7$ is a support vertex, then $T = P_8$ and the result is clearly true. Hence, we suppose that $v_7$ is not a support vertex. Let $T' = T - v_7$ and $f$ be a $\gamma_{rR}(T')$-function. Then $|V(T')| = n - 7$, $s(T') = s(T) - 1$ and $\ell(T') = \ell(T) - 1$. Then $f$ can be extended to an RRDF of $T$ by assigning 2 to $v_1, v_4, v_7$ and 0 to $v_2, v_3, v_5, v_6$. Thus $\gamma_{rR}(T) \leq \gamma_{rR}(T') + 6$. It follows from the inductive hypothesis that $\gamma_{rR}(T) < \left\lceil \frac{2n+5s(T)+\ell(T)-4}{3} \right\rceil$ and the proof is complete.

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