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The sum-annihilating essential ideal graph of a commutative ring

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Abstract: Let R be a commutative ring with identity. An ideal I of a ring R is called an annihilating ideal if there exists $r \in R \setminus \{0\}$ such that $Ir = (0)$ and an ideal I of R is called an essential ideal if I has non-zero intersection with every other non-zero ideal of R. The sum-annihilating essential ideal graph of R, denoted by $\mathcal{AE}_R$, is a graph whose vertex set is the set of all non-zero annihilating ideals and two vertices I and J are adjacent whenever $\mathrm{Ann}(I) + \mathrm{Ann}(J)$ is an essential ideal. In this paper we initiate the study of the sum-annihilating essential ideal graph. We first characterize all rings whose sum-annihilating essential ideal graphs are stars or complete graphs and then we establish sharp bounds on the domination number of this graph. Furthermore, we determine all isomorphism classes of Artinian rings whose sum-annihilating essential ideal graphs have genus zero or one.

Keywords: Commutative rings, annihilating ideal, essential ideal, genus of a graph

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1. Introduction

The history of studying a graph associated to a commutative ring has began by the paper [6, 9, 21], and then it followed over commutative and noncommutative rings (see for example [4, 5, 18, 22, 23]). Since then a huge number of works have

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been added to the literature about graphs associated to algebraic structures. In a recent study [11], the annihilating ideal graph, $\mathbb{AG}(R)$, is defined as follows: The vertex set of this graph is the set of all non-zero annihilating ideals of R, and two distinct vertices I and J are adjacent if and only if $IJ = (0)$. The interplay between the ring theoretic properties of a ring R and the graph theoretic properties of its annihilating ideal graph has been investigated in [1–3, 8, 10, 24]. In this paper, we continue the study of associating a graph to a commutative ring.

Throughout this paper, all rings are assumed to be commutative rings with identity that are not integral domain. If X is either an element or a subset of R, then the *annihilator* of X is defined as $\text{Ann}(X) = \{r \in R \mid rX = 0\}$. An ideal I of a ring R is called an annihilating ideal if $\text{Ann}(I) \neq (0)$. We denote the set of all maximal ideals, the set of all minimal prime ideals, and the set of all associated prime ideals of a ring R by $\text{Max}(R)$, $\text{Min}(R)$ and $\text{Ass}(R)$, respectively. The ring R is said to be *reduced* if it has no non-zero nilpotent element. An ideal I of R is called an essential ideal if I has non-zero intersection with every other non-zero ideal of R.

Let G be a simple graph with the vertex set $V(G)$ and edge set $E(G)$. The *degree* of a vertex $v \in V(G)$ is defined as $d_G(v) = |\{u \in V(G) \mid uv \in E(G)\}|$. A graph G is *regular* or *r-regular* if $d_G(v) = r$ for each vertex v of G. The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of the shortest uv-path in G. The greatest distance between any pair of vertices u and v in G is the *diameter* of G and is denoted by $\text{diam}(G)$. A *universal* vertex is a vertex that is adjacent to all other vertices of G. If a graph G contains a universal vertex with no extra edge, then G is called a *star graph*. A *clique* of G is a maximal complete subgraph of G and the number of vertices in the largest clique of G, denoted by $\omega(G)$, is called the *clique number* of G. A graph is planar if it has a drawing without crossings. The genus of a graph G, $\lambda(G)$, is the minimum integer k such that the graph can be drawn without crossing itself on a sphere with k handles (i.e. an oriented surface of genus k). Thus, a planar graph has genus 0, because it can be drawn on a sphere without self-crossing. Let $\chi(G)$ denote the *chromatic number* of G, that is, the minimal number of colors needed to color the vertices of G so that no two adjacent vertices have the same color. Obviously, $\chi(G) \geq \omega(G)$. We write K_n (resp. K_∞) for the complete graph of order n (resp. infinite complete graph), P_n for the path of length $n - 1$, and $K_{m,n}$ for the complete bipartite graph with partite sets of size m and n. For terminology and notation not defined here, the reader is referred to [25].

A set $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) - D$ has a neighbor in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. The concept of domination in graphs, with its many variations, is now well studied in graph theory and the literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [12, 13].

Motivated by the work done on annihilating ideal graph of a ring, in this paper we introduce the sum-annihilating essential ideal graph of a commutative ring which is not an integral domain and is defined as follows: The vertex set of this graph is the set of all non-zero annihilating ideals and two vertices I and J are adjacent whenever $\mathrm{Ann}(I) + \mathrm{Ann}(J)$ is an essential ideal. For convenience we denote this graph by $\mathcal{AE}_R$.

The aim of this article is to study some properties of $\mathcal{AE}_R$. We first characterize all rings whose sum-annihilating essential ideal graphs are stars or complete graphs and then we determine all isomorphism classes of Artinian rings whose sum-annihilating essential ideal graph has genus zero or one.

We make use of the following observations and results in this paper.

Observation 1. ([11]) Let R be ring. Then every descending chain (resp. ascending chain) of non-zero annihilating ideals terminates if and only if R is Artinian (resp. Noetherian).

Observation 2. If I, J are ideals of R such that I is essential and $I \subseteq J$, then J is essential.

Observation 3. For any ideal I of R, $I + \mathrm{Ann}(I)$ is an essential ideal.

Proof. Let I be an ideal of R such that $I + \mathrm{Ann}(I)$ is not essential. Then $J \cap (I + \mathrm{Ann}(I)) = (0)$ for some non-zero proper ideal J of R. It follows that $IJ = (0)$ and so $J \subseteq \mathrm{Ann}(I)$ which is a contradiction. Thus $I + \mathrm{Ann}(I)$ is essential. □

Observation 4. Let R be a ring and let I, J be two arbitrary non-zero proper ideals of R. Then

(1) If $\mathrm{Ann}(I)$ is essential, then I is a universal vertex in $\mathcal{AE}_R$.

(2) If $IJ = (0)$, then I and J are adjacent in $\mathcal{AE}_R$,

(3) If $I + J = R$, $\mathrm{Ann}(I) \neq (0)$ and $\mathrm{Ann}(J) \neq (0)$, then $\mathrm{Ann}(I)$ and $\mathrm{Ann}(J)$ are adjacent in $\mathcal{AE}_R$.

Proof. **(1)** Let J be an arbitrary vertex of $\mathcal{AE}_R$. By Observation 2, $\mathrm{Ann}(I) + \mathrm{Ann}(J)$ is an essential ideal of R and so I and J are adjacent in $\mathcal{AE}_R$. Hence, I is a universal vertex.

(2) Let $IJ = (0)$. Then $I \subseteq \mathrm{Ann}(J)$ and so $I + \mathrm{Ann}(I) \subseteq \mathrm{Ann}(I) + \mathrm{Ann}(J)$. It follows from Observations 2 and 3 that I and J are adjacent in $\mathcal{AE}_R$.

(3) Let $I + J = R$. Then $\mathrm{Ann}(I) \cap \mathrm{Ann}(J) = (0)$ and we deduce from (2) that $\mathrm{Ann}(I)$ and $\mathrm{Ann}(J)$ are adjacent in $\mathcal{AE}_R$.

□

By Observation 4 (part (2)), $\mathbb{AG}(R)$ is a spanning subgraph of $\mathcal{AE}_R$. Hence, we can use some of results obtained on the annihilating-ideal graph to obtain the same results for the sum-annihilating essential ideal graph of a ring. The following examples show that the graphs $\mathbb{AG}(R)$ and $\mathcal{AE}_R$ are not the same.

Example 1. If $R = \mathbb{Z}_{12}$, then $\mathcal{AE}_R$ is $K_4 - e$ and $\mathbb{AG}(R)$ is P_4.

Example 2. If p is a prime number and $R = \mathbb{Z}_{p^4}$, then $\mathcal{AE}_R$ is K_3 and $\mathbb{AG}(R)$ is the path P_3.

Theorem A. *([11]) If R is an Artinian ring, then every non-zero proper ideal I of R is a vertex of $\mathbb{AG}(R)$.*

Corollary 1. If R is an Artinian ring, then every non-zero proper ideal I of R is a vertex of $\mathcal{AE}_R$.

Theorem B. *([11]) Let R be a ring. Then the following statements are equivalent.*

(1) *$\mathbb{AG}(R)$ is a finite graph.*

(2) *R has only finitely many ideals.*

(3) *Every vertex of $\mathbb{AG}(R)$ has finite degree.*

Moreover, $\mathbb{AG}(R)$ has $n \geq 1$ vertices if and only if R has n non-zero proper ideals.

Theorem C. *([11]) For every ring R, the annihilating ideal graph $\mathbb{AG}(R)$ is connected with $\mathrm{diam}(\mathbb{AG}(R)) \leq 3$.*

Next result is an immediate consequence of Theorems B and C and the fact that $\mathbb{AG}(R)$ is a spanning subgraph of $\mathcal{AE}_R$.

Corollary 2. Let R be a ring. Then

(1) $\mathcal{AE}_R$ is a connected graph and $\mathrm{diam}\mathcal{AE}_R \leq 3$.

(2) The degree of each vertex in $\mathcal{AE}_R$ is finite if and only if the number of non-zero proper ideals of R is finite.

If $R = F_1 \times F_2 \times F_3$ where F_i is a field for $i = 1, 2, 3$, then $\mathcal{AE}_R$ is a connected graph with diam$\mathcal{AE}_R = 3$ (see Figure 1), and hence the bound of Corollary 2 is sharp.

2. Properties of sum-annihilating essential ideal graphs

In this section, we investigate the basic properties of the sum-annihilating essential ideal graphs. We recall that a ring R is decomposable if there exist non-zero rings R_1 and R_2 such that $R = R_1 \times R_2$, otherwise it is indecomposable.
First we classify all commutative rings R whose sum-annihilating essential ideal graphs $\mathcal{AE}_R$ are stars. We start with the following lemma.

Lemma 1. Let R be a decomposable ring. Then $\mathcal{AE}_R$ is a star if and only if $R = F \times D$ where F is a field and D is an integral domain.

Proof. Let $R = R_1 \times R_2$ and let $\mathcal{AE}_R$ be a star. If R_i is not a field for $i = 1, 2$ and I_i is a non-zero proper ideal of R_i, then $(R_1 \times (0), (0) \times R_2, I_1 \times (0), (0) \times I_2)$ is C_4 which is a contradiction. Henceforth, we may assume that R_1 is a field. If R_2 is an integral domain, then we are done. Suppose R_2 is not an integral domain and J is a non-zero annihilating ideal of R_2. Then clearly $(0) \times R_2$ and $(0) \times J$ are adjacent to $R_1 \times (0)$. Since $\mathcal{AE}_R$ is a star, $R_1 \times (0)$ is the center of $\mathcal{AE}_R$ and so $R_1 \times (0)$ is adjacent to $R_1 \times J$. It follows that $\mathrm{Ann}_R(R_1 \times (0)) + \mathrm{Ann}_R(R_1 \times J) = (0) \times R_2 + (0) \times \mathrm{Ann}_{R_2}(J) = (0) \times R_2$ is essential, a contradiction with $((0) \times R_2) \cap (R_1 \times (0)) = (0)$. Thus R_2 is an integral domain. Conversely, let $R = F \times D$ where F is a field and D is an integral domain. Then $V(\mathcal{AE}_R) = \{F \times (0), (0) \times I \mid (0) \neq I \trianglelefteq D\}$. It is easy to see that $\mathcal{AE}_R$ is a star with center $F \times (0)$. □

Corollary 3. Let R be an Artinian ring with at least two non-zero annihilating ideals. Then $\mathcal{AE}_R$ is a star if and only if $\mathcal{AE}_R \simeq K_2$.

Proof. Since K_2 is a star graph, if part of the Corollary is clear.
Conversely, assume that $\mathcal{AE}_R$ is a star. Either R is local or R is not local. If R is local, then it follows from Lemma 12 that $\mathcal{AE}_R$ is isomorphic to K_2. If R is not local, then R is decomposable. We deduce from Lemma 1 that $R = F \times D$ where F is a field and D is an integral domain. Since R is Artinian, we conclude that D is Artinian and so D is a field. Then clearly $\mathcal{AE}_R \simeq K_2$. □

Theorem 1. Let R be a ring with at least two non-zero proper ideals. Then $\mathcal{AE}_R$ is a star if and only if one of the following holds:

(a) R has exactly two non-zero proper ideals.

(b) $R = F \times D$ where F is a field and D is an integral domain which is not a field.

(c) R has a minimal ideal I such that $I^2 = (0)$, I is not essential and I is the annihilator of every non-zero proper ideal different from I.

Proof. If R has exactly two non-zero proper ideals, then R is Artinian and so $|V(\mathcal{AE}_R)| = 2$, by Corollary 1. Since $\mathcal{AE}_R$ is connected, we have $\mathcal{AE}_R \simeq K_2$ as desired. If $R = F \times D$ where F is a field and D is an integral domain that is not a field, then it follows from Lemma 1 that $\mathcal{AE}_R$ is a star. Now, let R have a minimal ideal I such that $I^2 = (0)$, I is not essential and $\mathrm{Ann}(J) = I$ for each non-zero proper ideal different from I. Since $I \subseteq \mathrm{Ann}_R(I)$, $\mathrm{Ann}_R(I)$ is essential by Observation 3 and so I is a universal vertex. Let J, K be two arbitrary distinct vertices of $\mathcal{AE}_R$ different from I. If J is adjacent to K then $\mathrm{Ann}_R(J) + \mathrm{Ann}_R(K) = I$ is essential, a contradiction. Hence J and K are not adjacent and this implies that $\mathcal{AE}_R$ is a star.
Conversely, let $\mathcal{AE}_R$ be a star. If $|V(\mathcal{AE}_R)| < \infty$, then by Observation 1, R is Artinian and it follows from Corollary 3 that R is isomorphic to $F_1 \times F_2$. Hence, R satisfies (a).
Now, let $|V(\mathcal{AE}_R)| = \infty$. Let I be the universal vertex of $\mathcal{AE}_R$. We claim that I is a minimal ideal of R. Assume, to the contrary, that J is a non-zero ideal of R such that $J \subsetneqq I$. Then $\mathrm{Ann}_R(I) + \mathrm{Ann}_R(J) = \mathrm{Ann}_R(J)$. Since I and J are adjacent in $\mathcal{AE}_R$, $\mathrm{Ann}_R(J)$ is an essential ideal of R and so J is a universal vertex of $\mathcal{AE}_R$ by Observation 4 and this leads to a contradiction. Hence, I is a minimal ideal of R. Consider two cases.

Case 1. $I^2 \neq (0)$.
Then $I^2 = I$. By Brauer's Lemma ([15], p. 172, Lemma 10.22), R is decomposable. Since $|V(\mathcal{AE}_R)| \geq 3$, we have $R = F \times D$, where F is a field and D is an integral domain that is not a field. Hence R satisfies (b).

Case 2. $I^2 = (0)$.
Let $J \neq I$ be an arbitrary vertex of $\mathcal{AE}_R$. Then $\mathrm{Ann}(J) \neq J$, for otherwise , $\mathrm{Ann}(J)$ is essential by Observation 3 and this implies by Observation 4 (part (1)) that $J = I$, a contradiction. Since $J.\mathrm{Ann}(J) = (0)$, we deduce from Observation 4 (part (2)) that J and $\mathrm{Ann}(J)$ are adjacent in $\mathcal{AE}_R$. Since $J \neq I$, we have $\mathrm{Ann}_R(J) = I$ and I is not essential. Thus, R satisfies (c). □

Next, we characterize all Artinian rings R whose sum-annihilating essential ideal graphs are complete.

Lemma 2. If $(R, \mathfrak{m})$ is an Artinian local ring, then $\mathcal{AE}_R$ is a complete graph.

Proof. Let I be an arbitrary vertex of $\mathcal{AE}_R$. Then $\mathrm{Ann}_R(\mathfrak{m}) \subseteq \mathrm{Ann}_R(I)$. Since R is an Artinian local ring, $\mathfrak{m}$ is nilpotent ([7], p.89, Proposition 8.4). Hence, $\mathfrak{m}^n = (0)$ and $\mathfrak{m}^{n-1} \neq (0)$ for some positive integer n and this implies that $\mathfrak{m}^{n-1} \subseteq \mathrm{Ann}(I)$. Let t be the smallest positive integer such that $\mathfrak{m}^t I = (0)$ and $\mathfrak{m}^{t-1} I \neq (0)$. It follows that $\mathfrak{m}^{t-1} I \subseteq \mathrm{Ann}(\mathfrak{m}) \cap I$ and so $\mathrm{Ann}(\mathfrak{m})$ is essential. Since $\mathrm{Ann}(\mathfrak{m}) \subseteq \mathrm{Ann}_R(I)$, we have that $\mathrm{Ann}_R(I)$ is essential. It follows from Observation 4 that I is a universal vertex. Since I is an arbitrary vertex, we conclude that $\mathcal{AE}_R$ is a complete graph and the proof is complete. □

Theorem 2. Let R be an Artinian ring. Then $\mathcal{AE}_R$ is a complete graph if and only if one of the following holds:

1. $R = F_1 \times F_2$ where F_1 and F_2 are fields.
2. R is a local ring.

Proof. One side is clear. Let $\mathcal{AE}_R$ be a complete graph. Since R is Artinian, $R = R_1 \times R_2 \times \cdots \times R_n$ where R_i is an Artinian local ring for each $1 \leq i \leq n$. If $n \geq 3$, then the vertices $R_1 \times (0) \times \cdots \times (0)$ and $R_1 \times R_2 \times (0) \times \cdots \times (0)$ are not adjacent in $\mathcal{AE}_R$ which is a contradiction. If $n = 1$, then R is an Artinian local ring and we are done. Henceforth, we have $n = 2$. We claim that R_1 and R_2 are fields. Assume, to the contrary, that R_1 is not a field. Let I_1 be a non-zero proper ideal of R_1. By Theorem A, $\mathrm{Ann}_{R_1}(I_1) \neq (0)$ and so $\mathrm{Ann}_R((0) \times R_2) + \mathrm{Ann}_R(I_1 \times R_2) = R_1 \times (0)$. Since $R_1 \times (0)$ is not essential, the vertices $(0) \times R_2$ and $I_1 \times R_2$ are not adjacent in $\mathcal{AE}_R$, a contradiction. Hence R_1 and R_2 are fields. This completes the proof. □

Next result classifies all rings with at least one minimal ideal whose sum-annihilating essential ideal graphs are complete bipartite graph.

Theorem 3. Let R be a ring with at least one minimal ideal. Then $\mathcal{AE}_R$ is a complete bipartite graph with at least two vertices in each partition if and only if $R = D_1 \times D_2$ where D_1 and D_2 are integral domains which are not fields.

Proof. Let $R = D_1 \times D_2$ where D_1 and D_2 are integral domains which are not fields. Let $X = \{I \times (0) | (0) \neq I \trianglelefteq D_1\}$ and $Y = \{(0) \times J | (0) \neq J \trianglelefteq D_2\}$. Since D_1 and D_2 are integral domains, $V(\mathcal{AE}_R) = X \cup Y$. Clearly, $|X|, |Y| \geq 2$, X and Y are independent sets and every vertex of X is adjacent to every vertex of Y. Thus, $\mathcal{AE}_R$ is a complete bipartite graph with desired property.
Conversely, let $\mathcal{AE}_R \simeq K_{m,n}$ where $m, n \geq 2$. Suppose that I is a minimal ideal of R. If $I^2 = (0)$, then $\mathrm{Ann}_R(I)$ is essential by Observation 3 and so I is a universal vertex, a contradiction. Thus $I^2 \neq (0)$. Since I is minimal, $I^2 = I$.

By Brauer's Lemma, $R = R_1 \times R_2$. We claim that R_1 and R_2 are integral domains. Assume that R_2 is not an integral domain and I_2 is a non-zero proper ideal of R_2 such that $\mathrm{Ann}_{R_2}(I_2) \neq (0)$. As above, we have $((0) \times I_2)^2 \neq (0)$ and this implies that $I_2 \neq \mathrm{Ann}_{R_2}(I_2)$. By Observation 3, $(R_1, 0)$, $(0, I_2)$ and $(0, \mathrm{Ann}_{R_2}(I_2))$ induced a triangle in $\mathcal{AE}_R$ which is a contradiction. Hence, R_2 is an integral domain. Similarly, R_1 is an integral domain. Since $m, n \geq 2$, it follows from Theorem 1 that R_1 and R_2 are not fields. This completes the proof. □

Theorem 4. If R is an Artinian ring, then $\mathcal{AE}_R \simeq H \vee K_m$ where H is a multipartite graph and $m \in \mathbb{N} \cup \{\infty\}$.

Proof. Since R is Artinian, then $R = R_1 \times \cdots \times R_n$ where $n = |\mathrm{Max(R)}|$. Let

$$\begin{aligned} X &= \{I_1 \times \cdots \times I_n | I_i \lhd R_i \text{ for } 1 \leq i \leq n\} - \{(0) \times \cdots \times (0)\} \\ X_1 &= \{(R_1 \times I_2 \times \cdots \times I_n) | I_i \unlhd R_i,\ 2 \leq i \leq n\} - \{R_1 \times \cdots \times R_n\} \ \ \textit{and} \\ X_i &= \{I_1 \times \cdots \times I_{i-1} \times R_i \times I_{i+1} \times \cdots \times I_n | I_j \lhd R_j \text{ for } 1 \leq j \leq i-1 \\ &\quad \text{and } I_j \unlhd R_j \text{ for } i+1 \leq j \leq n\} \end{aligned}$$

for each $2 \leq i \leq n$.
It is easy to verify that $V(\mathcal{AE}_R) = X \cup X_1 \cup \ldots \cup X_n$ and that the sets $X_1, \ldots, X_n$ are independent sets. Let H denote the induced subgraph $\mathcal{AE}_R[X_1 \cup \ldots \cup X_n]$. Then H is a n-partite graph. Assume now that $I_1 \times \cdots \times I_n \in X$. Since R_i is an Artinian local ring, $\mathrm{Ann}_{R_i}(I_i)$ is an essential ideal of R_i for each i. It follows that $\mathrm{Ann}_R(I_1 \times \cdots \times I_n)$ is an essential ideal of R which implies that $I_1 \times \cdots \times I_n$ is adjacent to all vertices of $\mathcal{AE}_R$. In particular, the subgraph induced by X is a clique. Thus $\mathcal{AE}_R \simeq H \vee K_{|X|}$ and the proof is complete. □

Corollary 4. Let $R = R_1 \times \cdots \times R_n$ $(n \geq 2)$ be the product of Artinian local rings $R_1, \ldots, R_n$. If R_i has only finitely many ideals for each i, then $\chi(\mathcal{AE}_R) = \omega(\mathcal{AE}_R) = n - 1 + \prod_{i=1}^{n} n_i$, where n_i is the number of proper ideals of R_i.

Proof. Using notation of Theorem 4, we have $|X| = \prod_{i=1}^{n} n_i - 1$ and $\mathcal{AE}_R = H \vee K_{|X|}$. Since X_i is an independent set for each i, we conclude that $\omega(\mathcal{AE}_R) \leq \chi(\mathcal{AE}_R) \leq n - 1 + \prod_{i=1}^{n} n_i$.
To prove the inverse inequality, we observe that the subgraph induced by the set

$$X \cup \{R_1 \times (0) \times \cdots \times (0), (0) \times R_2 \times \cdots \times (0), \ldots, (0) \times \cdots \times (0) \times R_n\}$$

is a clique which implies that $\chi(\mathcal{AE}_R) \geq \omega(\mathcal{AE}_R) \geq n + |X| = n - 1 + \prod_{i=1}^{n} n_i$, as desired. This completes the proof. □

Corollary 5. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$ $(m \geq 2)$ is the decomposition of the positive integer n into primes, then $\chi(\mathcal{AE}_{\mathbb{Z}_n}) = \omega(\mathcal{AE}_{\mathbb{Z}_n}) = m + \alpha_1\alpha_2 \dots \alpha_m - 1$.

Corollary 6. If $R = F_1 \times \cdots \times F_n$ is the product of fields $F_1, \dots, F_n$, then $\chi(\mathcal{AE}_R) = \omega(\mathcal{AE}_R) = n$.

Theorem 5. *Let R be a ring. Then $\mathcal{AE}_R$ is a k- regular graph if and only if $R = F_1 \times F_2$ where F_1 and F_2 are fields and $k = 1$ or R is an Artinian local ring with exactly $k + 1$ non-zero proper ideals.*

Proof. One side is clear. Let $\mathcal{AE}_R$ be a k-regular graph. By Corollary 2 (part 2), R is Artinian and so $R = R_1 \times \cdots \times R_n$ where R_i is an Artinian local ring for each $1 \leq i \leq n$. If R_i is a field for each i, then $n = 2$. For otherwise, $\deg((0) \times R_2 \times \cdots \times R_n) \neq \deg(R_1 \times (0) \times \cdots \times (0))$ contradicting regularity of $\mathcal{AE}_R$.
Thus, we may assume that R_1 is not a field. Suppose that $\mathfrak{m}_1$ is the maximal ideal of R_1. Then $\mathfrak{m}_1 \neq (0)$ and so $\mathfrak{m}_1 \times (0) \times \cdots \times (0) \in V(\mathcal{AE}_R)$. Since $\mathrm{Ann}(\mathfrak{m}_1)$ is an essential ideal in R_1 (see the proof of Lemma 2), we deduce that $\mathfrak{m}_1 \times (0) \times \cdots \times (0)$ is adjacent to every vertex of $\mathcal{AE}_R$. Since $\mathcal{AE}_R$ is k-regular, $\mathcal{AE}_R$ is a complete graph of order $k + 1$. It follows from Theorem 2 that R is an Artinian local and the proof is complete. □

3. Domination number of $\mathcal{AE}_R$

In this section, we investigate the domination number of the sum-annihilating essential ideal graph of a ring. The first observation shows that $\gamma(\mathcal{AE}_R)$ can be arbitrary large.

Observation 5. If $R = F_1 \times F_2 \times \cdots \times F_n$ $(n \geq 3)$, where F_i is a field for each $i \in \{1, 2, \dots, n\}$, then $\gamma(\mathcal{AE}_R) = n$.

Proof. Let $D = \{F_1, F_2, \dots, F_n\}$. We show that D is a dominating set of $\mathcal{AE}_R$. Assume $I_1 \times \cdots \times I_n$ is a vertex of $\mathcal{AE}_R$. Since $I_1 \times \cdots \times I_n \neq R$, we have $I_i = (0)$ for some i. Then $\mathrm{Ann}(I_1 \times \cdots \times I_n) + \mathrm{Ann}(F_i) = R$ and so $I_1 \times \cdots \times I_n$ and F_i are adjacent in $\mathcal{AE}_R$. Therefore D is a dominating set and hence $\gamma(\mathcal{AE}_R) \leq n$.
To show that $\gamma(\mathcal{AE}_R) \geq n$, let D be any $\gamma(\mathcal{AE}_R)$-set and $\hat{F}_i = F_1 \times \cdots \times F_{i-1} \times (0) \times F_{i+1} \times \cdots \times F_n$ for each i. Obviously $\deg(\hat{F}_i) = 1$ and $\hat{F}_i$ is only adjacent

to F_i for each i. Hence $|D \cap \{F_i, \hat{F_i}\}| \geq 1$ for each i which implies that $|D| \geq n$ and so $\gamma(\mathcal{AE}_R) = n$. $\square$

Next we provide some sufficient conditions for a ring R to have $\gamma(\mathcal{AE}_R) = 1$.

Proposition 1. Let R be a ring. Then $\gamma(\mathcal{AE}_R) = 1$ if and only if R has a non-zero proper ideal I such that $\mathrm{Ann}(I)$ is essential.

Proof. If R has a non-zero proper ideal I such that $\mathrm{Ann}(I)$ is essential, then it follows from Observation 4 (1) that $\gamma(\mathcal{AE}_R) = 1$.
Conversely, let $\gamma(\mathcal{AE}_R) = 1$ and let $\{I\}$ be a dominating set of $\mathcal{AE}_R$. If I is not maximal, then there is a maximal ideal $\mathfrak{m}$ such that $I \subsetneqq \mathfrak{m}$. Since I and $\mathfrak{m}$ are adjacent, we deduce that $\mathrm{Ann}(I) + \mathrm{Ann}(\mathfrak{m}) = \mathrm{Ann}(I)$ is essential and we are done. Henceforth, we assume that I is a maximal ideal of R. First let R be local. If R has exactly one non-zero proper ideal, then clearly $I = \mathrm{Ann}(I)$ is essential. Let J be a non-zero proper ideal of R different from I. Since J and I are adjacent and since $J \subsetneqq I$, we conclude that $\mathrm{Ann}(J) + \mathrm{Ann}(I) = \mathrm{Ann}(J)$ is essential and we are done again.
Now R is not local. We consider two cases.

Case 1. R has exactly two maximal ideals.
Let I and $\mathfrak{m}$ be two distinct maximal ideals of R and let $J(R) = I \cap \mathfrak{m}$. Clearly, $J(R) \subsetneqq I$. If $J(R) = (0)$, then by Chinese remainder theorem ([7], Proposition 1.10, pp 7) we have $R \simeq \frac{R}{I} \times \frac{R}{\mathfrak{m}}$ which contradicts Observation 5. Thus $J(R) \neq (0)$. Since I and $J(R)$ are adjacent, we deduce that $\mathrm{Ann}(I) + \mathrm{Ann}(J(R)) = \mathrm{Ann}(J(R))$ is essential and we are done.

Case 2. R has at least three maximal ideals.
Let $I, \mathfrak{m}_1$ and $\mathfrak{m}_2$ be three distinct maximal ideals of R. Then we have $I\mathfrak{m}_1 \neq (0)$, for otherwise we have $I\mathfrak{m}_1 \subseteq \mathfrak{m}_2$ and this implies that either $I \subseteq \mathfrak{m}_2$ or $\mathfrak{m}_1 \subseteq \mathfrak{m}_2$ which is a contradiction. Since $I\mathfrak{m}_1 \subsetneqq I$ and since I and $I\mathfrak{m}_1$ are adjacent, we conclude that $\mathrm{Ann}(I) + \mathrm{Ann}(I\mathfrak{m}_1) = \mathrm{Ann}(I\mathfrak{m}_1)$ is essential and the proof is complete. $\square$

Corollary 7. Let $R = R_1 \times \cdots \times R_n$ $(n \geq 2)$ where R_i is not an integral domain for some i. Then $\gamma(\mathcal{AE}_R) = 1$ if and only if $\gamma(\mathcal{AE}_{R_i}) = 1$ for some i.

Proof. Let $\gamma(\mathcal{AE}_{R_i}) = 1$ for some i, say $i = 1$. By Proposition 1, R_1 has a non-zero proper ideal I_1 such that $\mathrm{Ann}(I_1)$ is essential. Then clearly

$$\mathrm{Ann}_R(I_1 \times (0) \times \cdots \times (0)) = \mathrm{Ann}_{R_1}(I_1) \times R_2 \times \cdots \times R_n$$

is an essential ideal of R and hence $\gamma(\mathcal{AE}_R) = 1$ by Proposition 1.

Now let $\gamma(\mathcal{AE}_R) = 1$. By Proposition 1, R has a non-zero proper ideal $I_1 \times \cdots \times I_n$ whose annihilator is essential. Thus $\mathrm{Ann}_{R_1}(I_1) \times \cdots \times \mathrm{Ann}_{R_n}(I_n)$ is essential. Assume without loss of generality that $\mathrm{Ann}_{R_1}(I_1) \neq (0)$. Clearly $\mathrm{Ann}_{R_1}(I_1)$ is an essential ideal of R_1 and hence $\gamma(\mathcal{AE}_{R_1}) = 1$ by Proposition 1. □

Corollary 8. If R is a non-reduced ring, then $\gamma(\mathcal{AE}_R) = 1$.

Proof. Since R is non-reduced, R has a non-zero ideal I such that $I^2 = (0)$. It follows from Observation 3 that $\mathrm{Ann}(I) = I + \mathrm{Ann}(I)$ is an essential ideal of R and hence $\gamma(\mathcal{AE}_R) = 1$ by Proposition 1. □

Next, we determine the domination number of the sum-annihilating essential ideal graph of a Noetherian reduced ring.

Observation 6. Let S be a multiplicatively closed subset of a commutative ring R such that $S \cap Z(R) = \emptyset$ and I be a finitely generated ideal of R. Then $\mathrm{Ann}(I) \neq (0)$ if and only if $\mathrm{Ann}_{S^{-1}R}(S^{-1}I) \neq (0)$.

Proof. Since I is finitely generated ideal, we have $\mathrm{Ann}_{S^{-1}R}(S^{-1}I) \cong S^{-1}(\mathrm{Ann}_R(I))$. Now the result follows from the fact that $I = (0)$ if and only if $S^{-1}I = (0)$. □

Observation 7. Let R be a ring, I a non-zero ideal of R and S a multiplicatively closed subset of R with $S \cap Z(R) = \emptyset$ such that $I \cap S = \emptyset$. Then I is an essential ideal of R if and only if $S^{-1}I$ is an essential ideal of $S^{-1}R$.

Proof. Let I be an essential ideal of R. Assume $\mathcal{J}$ is a non-zero ideal of $S^{-1}R$. Then there is a non-zero ideal K of R such that $\mathcal{J} = S^{-1}K$. Since I is essential, $I \cap K \neq (0)$ which implies that $S^{-1}I \cap \mathcal{J} = S^{-1}I \cap S^{-1}K = S^{-1}(I \cap K) \neq (0)$. Thus $S^{-1}(I)$ is an essential ideal of $S^{-1}R$. The proof of other side is similar. □

Observation 8. Let R be a Noetherian ring and S be a multiplicatively closed subset of R with $S \cap Z(R) = \emptyset$. Then $\gamma(\mathcal{AE}_{S^{-1}R}) \leq \gamma(\mathcal{AE}_R)$.

Proof. Assume that E is a $\gamma(\mathcal{AE}_R)$-set and let $D = \{S^{-1}I \mid I \in E\}$. Suppose $\mathcal{J}$ is an arbitrary vertex in $V(\mathcal{AE}_{S^{-1}R}) \setminus D$. Then there is an ideal I of R such that $\mathcal{J} = S^{-1}I$. Clearly $I \notin E$. Since $\mathrm{Ann}_{S^{-1}R}(\mathcal{J}) \neq (0)$, we have $\mathrm{Ann}_R(I) \neq 0$ and so $I \in V(\mathcal{AE}_R) \setminus E$. Since E is a $\gamma(\mathcal{AE}_R)$-set, I is adjacent to a vertex $K \in E$. Therefore $\mathrm{Ann}(I)+\mathrm{Ann}(K)$ is an essential ideal of R. It follows from Observation 7 that $S^{-1}(\mathrm{Ann}(I) + \mathrm{Ann}(K)) = \mathrm{Ann}(S^{-1}I) + \mathrm{Ann}(S^{-1}K)$

is an essential ideal of $S^{-1}R$. This implies that $S^{-1}K$ and $\mathcal{J}$ are adjacent in $\mathcal{AE}_{S^{-1}R}$ where $S^{-1}K \in D$. So D is a dominating set of $\mathcal{AE}_{S^{-1}R}$ and the proof is complete. □

Proposition 2. Let R be a Noetherian reduced ring which is not an integral domain. Then $\gamma(\mathcal{AE}_R) = |\mathrm{Min}(R)|$.

Proof. Since R is Noetherian and $\mathrm{Min}(R) \subseteq \mathrm{Ass}(R)$, we have $|\mathrm{Min}(R)| \leq |\mathrm{Ass}(R)| < \infty$. Let $\mathrm{Min}(R) = \{P_1, \ldots, P_k\}$ and let $P_i = \mathrm{Ann}(x_i)$ for some $x_i \in R - \{0\}$ $(1 \leq i \leq k)$. Let I be a vertex of $\mathcal{AE}_R$. Then $I \subseteq P_i$ for some $1 \leq i \leq k$ ([16], Proposition 1.2(part 2)). As above, we conclude that $\{Rx_1, Rx_2, \ldots, Rx_k\}$ is a dominating set of $\mathcal{AE}_R$ implying that $\gamma(\mathcal{AE}_R) \leq |\mathrm{Min}(R)| \leq k$.
Now we show that $\gamma(\mathcal{AE}_R) \geq |\mathrm{Min}(R)|$. Let $S = R - \cup_{i=1}^{k} P_i$. Then $S^{-1}R = R_{P_1} \times \cdots \times R_{P_k}$ where $R_{P_1}, \ldots, R_{P_i}$ are fields ([16], Propositions 1.1 and 1.5). By Observations 5 and 8, we have $\gamma(\mathcal{AE}_R) \geq \gamma(\mathcal{AE}_{S^{-1}R}) = k$. Thus $\gamma(\mathcal{AE}_R) = |\mathrm{Min}(R)|$ and the proof is complete. □

Nikandish and Maimani [17] proved the if R a Noetherian ring, then $\gamma(\mathbb{AG}(R)) \leq |\mathrm{Ass}(R)| < \infty$. Since $\mathbb{AG}(R)$ is a spanning subgraph of $\mathcal{AE}_R$, we have the next result.

Corollary 9. For any Noetherian ring R, $\gamma(\mathcal{AE}_R) \leq |\mathrm{Ass}(R)| < \infty$.

4. Classification of Artinian rings with genus zero or one

In this section we characterize all Artinian rings with genus zero or one. The proof of the following result can be found in [14].

Theorem D. A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.

Proposition 3. Let R be an Artinian ring. Then $\mathcal{AE}_R$ is a planar graph if and only if one of the following holds:

(a) R has at most four non-zero proper ideals.

(b) $R = F_1 \times F_2 \times F_3$, where F_1, F_2 and F_3 are fields.

Proof. If R satisfies (a), then the result is immediate. Let $R = F_1 \times F_2 \times F_3$ where F_1, F_2 and F_3 are fields. Then $\mathcal{AE}_R$ is the graph illustrated in Figure 1 and so $\mathcal{AE}_R$ is planar.

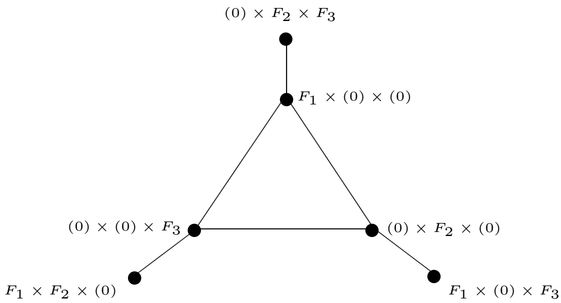

Figure 1. **The graph** $\mathcal{AE}_R(F_1 \times F_2 \times F_3)$

Conversely, let $\mathcal{AE}_R$ be a planar graph. Since R is Artinian, $V(\mathcal{AE}_R)$ contains all non-zero proper ideals of R. If $|V(\mathcal{AE}_R)| \leq 4$, then R satisfies (a). Suppose that $|V(\mathcal{AE}_R)| \geq 5$. Since R is Artininan, $R = R_1 \times R_2 \times \cdots \times R_n$ where R_i is an Artinian local ring for each $1 \leq i \leq n$. If $n \geq 4$, then the subgraph induced by the set
$\{R_1 \times (0) \times \cdots \times (0), (0) \times \cdots (0) \times R_n, R_1 \times (0) \times \cdots (0) \times R_n, (0) \times R_2 \times (0) \times \cdots \times (0), (0) \times R_2 \times R_3 \times (0) \times \cdots \times (0), (0) \times (0) \times R_3 \times (0) \times \cdots \times (0)\}$ contains the complete bipartite graph $K_{3,3}$ with partite sets $X = \{R_1 \times (0) \times \cdots \times (0), (0) \times \cdots (0) \times R_n, R_1 \times (0) \times \cdots (0) \times R_n\}$ and $Y = \{(0) \times R_2 \times (0) \times \cdots \times (0), (0) \times R_2 \times R_3 \times (0) \times \cdots \times (0), (0) \times (0) \times R_3 \times (0) \times \cdots \times (0)\}$, a contradiction. Thus $n \leq 3$. If $n = 1$, then R is an Artinian local ring and we get a contradiction by Lemma 2 and Theorem D. Hence $n = 2$ or 3. Let $\mathfrak{m}_i$ be the maximal ideal of R_i for $i = 1, 2, 3$.
First let $n = 3$. We claim that R_1, R_2 and R_3 are fields. Assume, to the contrary, that R_i is not a field for some i, say $i = 3$. Then the induced subgraph by the set

$$\{R_1 \times 0 \times (0), R_1 \times (0) \times \mathfrak{m}_3, (0) \times (0) \times \mathfrak{m}_3, (0) \times R_2 \times (0), (0) \times R_2 \times \mathfrak{m}_3, (0) \times R_2 \times R_3\}$$

contains $K_{3,3}$ with partite sets $X = \{R_1 \times (0) \times (0), R_1 \times (0) \times \mathfrak{m}_3, (0) \times (0) \times \mathfrak{m}_3\}$ and $Y = \{(0) \times R_2 \times (0), (0) \times R_2 \times \mathfrak{m}_3, (0) \times R_2 \times R_3\}$, a contradiction. Thus R satisfies (b).
Let now $n = 2$. Since $|V(\mathcal{AE}_R)| \geq 5$, R_i is not a field for some i, say $i = 2$. If R_1 is a field, then it follows from $|V(\mathcal{AE}_R)| \geq 5$ that R_2 has at least two distinct non-zero proper ideals I and J. It is easy to see that the subgraph induced by the set $\{R_1 \times (0), R_1 \times I, R_1 \times J, (0) \times R_2, (0) \times I, (0) \times J\}$ contains $K_{3,3}$ with

partite sets $X = \{R_1 \times (0), R_1 \times I, R_1 \times J\}$ and $Y = \{(0) \times R_2, (0) \times I, (0) \times J\}$ contradicting planarity of $\mathcal{AE}_R$. Assume that R_1 is not a field. Then the subgraph induced by the set $\{\mathfrak{m}_1 \times (0), (0) \times \mathfrak{m}_2, R_1 \times \mathfrak{m}_2, (0) \times R_2, \mathfrak{m}_1 \times (0), \mathfrak{m}_1 \times R_2\}$ contains $K_{3,3}$ with partite sets $X = \{\mathfrak{m}_1 \times (0), (0) \times \mathfrak{m}_2, R_1 \times \mathfrak{m}_2\}$ and $Y = \{(0) \times R_2, \mathfrak{m}_1 \times (0), \mathfrak{m}_1 \times R_2\}$ contradicting the planarity of $\mathcal{AE}_R$. This completes the proof. □

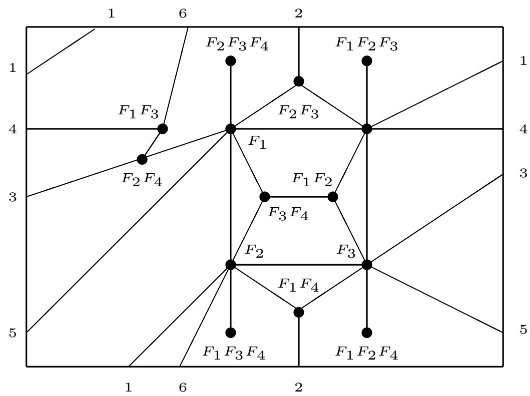

Figure 2. **toroidal embedding of** $\mathcal{AE}_R(F_1 \times F_2 \times F_3 \times F_4)$

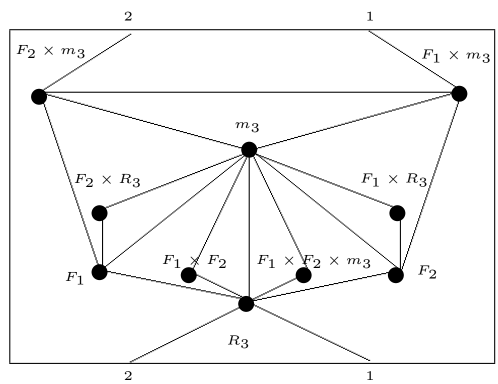

Figure 3. toroidal embedding of $\mathcal{AE}_R(F_1 \times F_2 \times R_3)$ where R_3 has exactly one non-zero proper ideal

Next, we characterize all Artinian rings whose the sum-annihilating essential ideal graphs have genus one. The proof of the following results can be found in Ringel and Youngs [20]; Ringel [19], respectively.

Theorem E. *For $n \geq 3$, $\lambda(K_n) = \lceil \frac{1}{12}(n-3)(n-4) \rceil$. In particular, $\lambda(K_n) = 1$ if $n = 5, 6, 7$.*

Theorem F. *For $m, n \geq 2$, $\lambda(K_{m,n}) = \lceil \frac{(m-2)(n-2)}{4} \rceil$.*

We begin with the following lemma.

Lemma 3. Let $R = F_1 \times \cdots \times F_n$ $(n \geq 4)$ where F_i is a field for each $1 \leq i \leq n$. Then $\lambda(\mathcal{AE}_R) = 1$ if and only if $n = 4$.

Proof. If $n = 4$, then $\mathcal{AE}_R$ is the graph illustrated in Figure 2 and it follows from Proposition 3 that $\lambda(\mathcal{AE}_R) = 1$.
Conversely, let $\lambda(\mathcal{AE}_R) = 1$. If $n \geq 5$, then the subgraph induced by the set $\{F_1, F_2, F_3, F_4, F_5, F_1 \times F_2, F_3 \times F_4, F_3 \times F_4 \times F_5, F_3 \times F_5, F_4 \times F_5\}$ contains $K_{3,7}$ with partite sets $X = \{F_1, F_2, F_1 \times F_2\}$ and $Y = \{F_3, F_4, F_5, F_3 \times F_4, F_3 \times F_4 \times F_5, F_3 \times F_5, F_4 \times F_5\}$. By Theorem F, we have $\lambda(\mathcal{AE}_R) \geq \lambda(K_{3,7}) = 2$, a contradiction. Thus $n = 4$ and the proof is complete. □

Theorem 6. Let R be an Artinian ring. Then $\lambda(\mathcal{AE}_R) = 1$ if and only if one of the following holds:

(a) $R = F_1 \times F_2 \times F_3 \times F_4$ where F_1, F_2, F_3 and F_4 are fields.

(b) $R = F_1 \times F_2 \times R_3$ where F_1 and F_2 are fields and R_3 has exactly one non-zero proper ideal.

(c) $R = R_1 \times R_2$ where each of R_1 and R_2 has exactly one non-zero proper ideal.

(d) $R = F \times R_2$ where F is a field and R_2 is a local ring with exactly two non-zero proper ideals.

(e) $R = F \times R_2$ where F is a field and R_2 is a local ring with exactly three non-zero proper ideals.

(f) R is a local ring with r non-zero proper ideals where $5 \leq r \leq 7$.

Proof. If $R = F_1 \times F_2 \times F_3 \times F_4$, then the result follows by Lemma 3. If $R = F_1 \times F_2 \times R_3$ where F_1 and F_2 are fields and R_3 has exactly one non-zero proper ideal, then $\mathcal{AE}_R$ is the graph illustrated in Figure 3 and so $\lambda(\mathcal{AE}_R) = 1$. If $R = R_1 \times R_2$ where R_i has exactly one non-zero proper ideal $\mathfrak{m}_i$ for $i = 1, 2$, then $|V(\mathcal{AE}_R)| = 7$ and it is easy to verify that the subgraph induced by the set $\{R_1 \times (0), \mathfrak{m}_1 \times (0), R_1 \times \mathfrak{m}_2, (0) \times R_2, (0) \times \mathfrak{m}_2, \mathfrak{m}_1 \times R_2\}$ contains $K_{3,3}$ which implies that $K_{3,3} \leq \mathcal{AE}_R \leq K_7$. It follows from Theorems E and F that $1 = \lambda(\mathcal{AE}_R)$.

If $R = F \times R_2$ and R_2 is a local ring with exactly two non-zero proper ideals I and $\mathfrak{m}$ where $\mathfrak{m}$ is a maximal ideal of R_2, then $|V(\mathcal{AE}_R)| = 6$ and $\mathcal{AE}_R$ contains $K_{3,3}$ with partite sets $\{F \times (0), F \times I, F \times \mathfrak{m}\}$ and $\{(0) \times R_2, (0) \times I, (0) \times \mathfrak{m}\}$ implying that $\lambda(\mathcal{AE}_R) = 1$. If $R = F \times R_2$ and R_2 has exactly three non-zero proper ideals I, J and $\mathfrak{m}_2$ where $\mathfrak{m}_2$ is a maximal ideal of R_2, then clearly $\mathcal{AE}_R \simeq \overline{K_4} \vee K_4$ and $V(\mathcal{AE}_R) = \{F \times (0), F \times I, F \times J, F \times \mathfrak{m}_2, (0) \times R_2, (0) \times I, (0) \times J, (0) \times \mathfrak{m}_2\}$. As illustrated in Figure 4, we have $\lambda(\mathcal{AE}_R) = 1$.
Finally, If R satisfies (f), then $K_5 \leq \mathcal{AE}_R \leq K_7$ and so $\lambda(\mathcal{AE}_R) = 1$ by Theorem E.

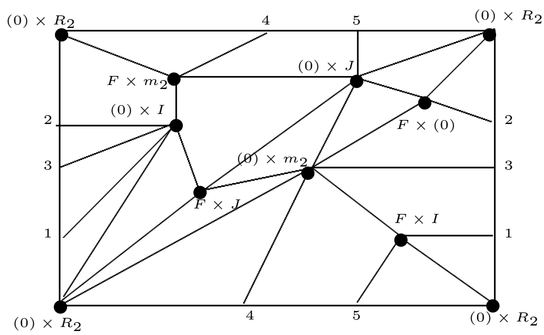

Figure 4. **toroidal embedding of** $\overline{K_4} \vee K_4$

Conversely, let $\lambda(\mathcal{AE}_R) = 1$. Since R is Artinian, we have $R = R_1 \times \cdots \times R_n$ where R_i is an Artinian local ring for each i. If $n \geq 5$, then using an argument similar to that described in the proof of Theorem 3, we deduce that $\mathcal{AE}_R$ contains $K_{3,7}$ which leads to a contradiction by Theorem F. Therefore $n \leq 4$. If R_1, R_2, R_3, R_4 are fields, then R satisfies (a). Assume without loss of generality that R_4 is not a field. Let $\mathfrak{m}_4$ be the maximal ideal of R_4. Then the subgraph induced by the set

$$\{R_1 \times (0) \times (0) \times (0),$$
$$R_1 \times (0) \times (0) \times \mathfrak{m}_4, (0) \times R_2 \times (0) \times (0), (0) \times R_2 \times (0) \times \mathfrak{m}_4,$$

$$R_1 \times R_2 \times (0) \times \mathfrak{m}_4,$$
$$(0) \times (0) \times R_3 \times (0), (0) \times (0) \times R_3 \times \mathfrak{m}_4, (0) \times (0) \times (0) \times R_4,$$

$$(0) \times (0) \times (0) \times \mathfrak{m}_4\}$$

contains $K_{4,5}$ with partite sets $X = \{R_1 \times (0) \times (0) \times (0), R_1 \times (0) \times (0) \times \mathfrak{m}_4, (0) \times R_2 \times (0) \times (0), (0) \times R_2 \times (0) \times \mathfrak{m}_4, R_1 \times R_2 \times (0) \times \mathfrak{m}_4\}$ and $Y =$

$\{(0)\times(0)\times R_3\times(0),(0)\times(0)\times R_3\times\mathfrak{m}_4,(0)\times(0)\times(0)\times R_4,(0)\times(0)\times(0)\times\mathfrak{m}_4\}$ contradicting $\lambda(\mathcal{AE}_R)=1$ by Theorem F. Therefore $n\leq 3$. We consider the following cases.

Case 1. $n=3$, $R=F_1\times F_2\times R_3$ where F_1 and F_2 are fields.
If R_3 has a non-zero proper ideal I different from $\mathfrak{m}_3$, then the subgraph induced by the set

$$\{F_1\times(0)\times(0),F_1\times(0)\times I,F_1\times(0)\times\mathfrak{m}_3,(0)\times F_2\times(0),(0)\times F_2\times I,$$
$$(0)\times F_2\times\mathfrak{m}_3,F_1\times F_2\times I,(0)\times(0)\times R_3,(0)\times(0)\times I,(0)\times(0)\times\mathfrak{m}_3\}$$

contains $K_{3,7}$ with partite sets $X=\{F_1\times(0)\times(0),F_1\times(0)\times I,F_1\times(0)\times\mathfrak{m}_3,(0)\times F_2\times(0),(0)\times F_2\times I,(0)\times F_2\times\mathfrak{m}_3,F_1\times F_2\times I\}$ and $Y=\{(0)\times(0)\times R_3,(0)\times(0)\times I,(0)\times(0)\times\mathfrak{m}_3\}$ contradicting $\lambda(\mathcal{AE}_R)=1$ by Theorem F. Thus R_3 has exactly one non-zero proper ideal and so R satisfies (b).

Case 2. $n=3$ and $R=F\times R_2\times R_3$, where F is a field and R_2,R_3 are not fields.
Let $\mathfrak{m}_2$ and $\mathfrak{m}_3$ be the maximal ideals of R_2 and R_3, respectively. Then the subgraph induced by the set

$$\{F\times(0)\times(0),F\times\mathfrak{m}_2\times\mathfrak{m}_3,(0)\times R_2\times(0),(0)\times R_2\times\mathfrak{m}_3,F\times(0)\times\mathfrak{m}_3,$$
$$(0)\times(0)\times R_3,(0)\times(0)\times\mathfrak{m}_3),(0)\times\mathfrak{m}_2\times(0),(0)\times\mathfrak{m}_2\times\mathfrak{m}_3\}$$

contains the graph $K_{4,5}$ with partitions $X=\{F\times(0)\times(0),F\times\mathfrak{m}_2\times\mathfrak{m}_3,(0)\times R_2\times(0),(0)\times R_2\times\mathfrak{m}_3,F\times(0)\times\mathfrak{m}_3\}$ and $Y=\{(0)\times(0)\times R_3,(0)\times(0)\times\mathfrak{m}_3),(0)\times\mathfrak{m}_2\times(0),(0)\times\mathfrak{m}_2\times\mathfrak{m}_3\}$ a contradiction with $\lambda(\mathcal{AE}_R)=1$.

Case 3. $n=2$ and $R=R_1\times R_2$, where R_1,R_2 are not fields.
We claim that each of R_1 and R_2 has exactly one non- zero proper ideal. Let $\mathfrak{m}_1$ be the maximal ideal of R_1 and I be a non-zero proper ideal of R_1 different from $\mathfrak{m}_1$. Then the subgraph induced by the set

$$\{R_1\times(0),\mathfrak{m}_1\times(0),R_1\times\mathfrak{m}_2,I\times(0),I\times\mathfrak{m}_2,(0)\times R_2,(0)\times\mathfrak{m}_2,\mathfrak{m}_1\times R_2,\mathfrak{m}_1\times\mathfrak{m}_2\}$$

contains the graph $K_{4,5}$ with partitions $X=\{R_1\times(0),\mathfrak{m}_1\times(0),R_1\times\mathfrak{m}_2,I\times(0),I\times\mathfrak{m}_2\}$ and $Y=\{(0)\times R_2,(0)\times\mathfrak{m}_2,\mathfrak{m}_1\times R_2,\mathfrak{m}_1\times\mathfrak{m}_2\}$ a contradiction with $\lambda(\mathcal{AE}_R)=1$. Thus R_1 has exactly one non-zero proper ideal. Similarly, R_2 has exactly one non-zero proper ideal and so R satisfies (c).

Case 4. $n=2$ and $R=F\times R_2$, where F is a field and R_2 is a local ring which is not a field.

If R_2 has exactly one non-zero proper ideal, then $|V(\mathcal{AE}_R)| = 4$ which leads to a contradiction. If R_2 has four non-zero proper ideals I, J, K and $\mathfrak{m}_2$, then the subgraph induced by the set

$$\{F \times (0), F \times I, F \times J, F \times K, F \times \mathfrak{m}_2, (0) \times R_2, (0) \times I, (0) \times J, (0) \times K\}$$

contains $K_{4,5}$ with partitions

$$\{F \times (0), F \times I, F \times J, F \times K, F \times \mathfrak{m}_2\} \cup \{(0) \times R_2, (0) \times I, (0) \times J, (0) \times K\}$$

contradicting with $\lambda(\mathcal{AE}_R) = 1$. Thus R_2 has exactly two or three non- zero proper ideals and so R satisfies either (d) or (e).

Case 5. $n = 1$.
Then R is an Artinian local ring. By Theorem 2, $\mathcal{AE}_R$ is a complete graph. Since $\lambda(\mathcal{AE}_R) = 1$, we deduce that R has r non-zero proper ideals where $5 \leq r \leq 7$. □

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References

[1] G. Aalipour, S. Akbari, R. Nikandish, M.J. Nikmehr, and F. Shaveisi, *On the coloring of the annihilating-ideal graph of a commutative ring*, Discrete Math. **312** (2012), no. 17, 2620–2626.

[2] S. Akbari, A. Alilou, J. Amjadi, and S.M. Sheikholeslami, *The co-annihilating ideal graphs of commutative rings*, Canadian Math. Bull. **to appear**.

[3] S. Akbari and F. Heydari, *The regular graph of a commutative ring*, Period. Math. Hungar. **67** (2013), no. 2, 211–220.

[4] S. Akbari and A. Mohammadian, *Zero-divisor graphs of non-commutative rings*, J. Algebra **296** (2006), no. 2, 462–479.

[5] D.F. Anderson and A. Badawi, *On the zero-divisor graph of a ring*, Comm. Algebra **36** (2008), 3073–3092.

[6] D.F. Anderson and P.S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra **217** (1999), no. 2, 434–447.

[7] M.F. Atiyah and I.G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Company, 1969.

[8] A. Badawi, *On the annihilator graph of a commutative ring*, Comm. Algebra **42** (2014), 108–121.

[9] I. Beck, *Coloring of a commutative ring*, J. Algebra **116** (1988), no. 1, 208–226.

[10] M. Behboodi and Z. Rakeei, *The annihilating-ideal graph of commutative rings ii*, J. Algebra Appl. **10** (2001), no. 4, 741–753.

[11] ______, *The annihilating-ideal graph of commutative rings i*, J. Algebra Appl. **10** (2011), no. 4, 727–739.

[12] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Domination in graphs: Advanced topics*, Marcel Dekker, Inc., New York, 1998.

[13] ______, *Fundamentals of domination in graphs*, Marcel Dekker, Inc., New York, 1998.

[14] K. Kuratowski, *Sur le* problèm *des courbes gauches en topologie*, Fund. Math. **15** (1930), 271–283.

[15] T.Y. Lam, *A first course in non-commutative rings*, Springer-Verlag (New York, Inc., 1991.

[16] M. Matlis, *The minimal prime spectrum of a reduced ring*, Illinois J. Math. **27** (1983), no. 3, 353–391.

[17] R. Nikandish and H.R. Maimani, *Dominating sets of the annihilating-ideal graphs*, Electron. Notes Discrete Math. **45** (2014), 17–22.

[18] S.P. Redmond, *The zero-divisor graph of a non-commutative ring*, Internat. J. Commutative rings **1** (2002), 203–211.

[19] G. Ringel, *Das gescblecht des vollstandingen paaren graphen*, Abh. Math. Sem. Univ. Hamburg **28** (1965), 139–150.

[20] G. Ringel and J.W.T. Youngs, *Solution of the heawood map-coloring problem*, Proc. Nat. Acad. Sci. U.S.A. **60** (1968), 438–445.

[21] P.K. Sharma and S.M. Bhatwadekar, *A note on graphical representation of rings*, J. Algebra **176** (1995), no. 1, 124–127.

[22] A. Tehranian and H.R. Maimani, *A study of the total graph*, Iranian J. Math. Sci. Info. **6** (2011), no. 2, 75–80.

[23] S. Visweswaran and H.D. Patel, *A graph associated with the set of all nonzero annihilating ideals of a commutative ring*, Discrete Math. Algorithms Appl. **6** (2014), no. 4, 1450047 (22 pages).

[24] ______, *When is the annihilating ideal graph of a zero-dimensional quasisemilocal commutative ring complemented?*, Arab. J. Math. Sci. **22** (2016), no. 1, 1–21.

[25] D.B. West, *Introduction to graph theory (second edition)*, Prentice Hall, USA, 2001.