Communications in Combinatorics and Optimization
Vol. 1 No. 2, 2016 pp.83-102
DOI: 10.22049/CCO.2016.13554

CCO
Commun. Comb. Optim.

A full Nesterov-Todd step interior-point method for circular cone optimization

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Received: 3 January 2016.; Accepted: 16 September 2016; Available Online: 17 September 2016.

Communicated by ***Tibor Illés***

Abstract: In this paper, we present a full Newton step feasible interior-point method for circular cone optimization by using Euclidean Jordan algebra. The search direction is based on the Nesterov-Todd scaling scheme, and only full-Newton step is used at each iteration. Furthermore, we derive the iteration bound that coincides with the currently best known iteration bound for small-update methods.

Keywords: Circular cone optimization, Full-Newton step, Interior-point methods, Euclidean Jordan algebra

2010 Mathematics Subject Classification: 90C51

1. Introduction

Circular cone optimization (CCO) problems are convex optimization problems in which a linear function is minimized over the intersection of an affine linear manifold with the Cartesian product of circular cones. The circular cone in R^n is given by

$$\mathcal{Q}_\theta^n := \big\{(x_0;\bar{x}) : x_0 \geq \cot(\theta)\|\bar{x}\|\big\},$$

where $\bar{x} = (x_1;\ldots;x_{n-1}) \in R^{n-1}$ and $\theta \in (0, \frac{\pi}{2})$ is a given angle. The circular cone $\mathcal{Q}_\theta^n$ with $\theta \neq \frac{\pi}{4}$ naturally arise in many real-life engineering problems [15]. We consider the following standard primal CCO problem:

$$\min\big\{\langle c, x\rangle_\theta :\ (Ax)_\theta = b,\ x \in \mathcal{Q}_\theta^n\big\}, \qquad (P)$$

where $\langle c, x\rangle_\theta := c^T I_{\theta,n}^2 x$ and $(Ax)_\theta := A I_{\theta,n}^2 x$ denote respectively the circular inner product and the circular matrix-vector product, and the matrix

$$I_{\theta,n} := \begin{bmatrix} 1 & 0^T \\ 0 & \cot(\theta) I_{n-1} \end{bmatrix} \in R^{n\times n},$$

is called the circular identity matrix as a generalization of the identity matrix $I_n \in R^{n\times n}$ and $\mathcal{Q}_\theta^n$ is the Cartesian product of several circular cones, i.e.,

$$\mathcal{Q}_\theta^n = \mathcal{Q}_{\theta_1}^{n_1} \times \mathcal{Q}_{\theta_2}^{n_2} \times \cdots \times \mathcal{Q}_{\theta_N}^{n_N},$$

with $n = n_1 + n_2 + \cdots + n_N$. When $\theta = \frac{\pi}{4}$, the circular cone reduces to the well-known second-order cone given by

$$\mathcal{L}^n := \left\{(x_0; \bar{x}) : x_0 \geq \|\bar{x}\|\right\} \quad \text{and} \quad I_n = I_{\frac{\pi}{4},n}.$$

In this case, $\langle c, x\rangle_{\frac{\pi}{4}} = c^T x$ and $(Ax)_{\frac{\pi}{4}} = Ax$, and therefore, the CCO reduces to second-order cone optimization (SOCO) problems [2]. So, CCO includes SOCO as a special case. Without loss of generality, we assume that A has full row rank, i.e., $\operatorname{rank}(A) = m$. Due to the fact that under the circular inner product $\mathcal{Q}_\theta^n$ is self-dual [4, Lemma 2], the dual problem of (P) is given by

$$\max\left\{b^T y : \ A^T y + s = c, y \in R^m, s \in \mathcal{Q}_\theta^n\right\}. \qquad (D)$$

In 1984, Karmarkar [10] developed a method for linear optimization (LO) which runs in provably polynomial time and, is also very efficient in practice. Karmarkar's worthy paper revitalized the study of interior point methods (IPMs), showing that it was possible to create an algorithm for LO in polynomial complexity and, moreover, that was competitive with the simplex method. The study of primal-dual IPMs for symmetric cone optimization (SCO) problems was started by Nesterov and Todd [19]. In [7], Faybusovich introduced a new concept and unifying frame to analyze IPMs for LO, SOCO and SDO problems. Schmieta and Alizadeh [22] extended the analysis of Monteiro and Zhang in [17] to symmetric cone by using Euclidean Jordan algebras. Illés and Nagy [9] investigated a version of Mizuno-Todd-Ye predictor-corrector algorithm for the $P_*(\kappa)$-LCP. The primal-dual full Newton-step feasible IPM for LO was first analyzed by Roos et al. in [21] and the authors obtained the currently best known iteration bound for small-update methods, namely, $O(\sqrt{n}\log\frac{n}{\epsilon})$. De Klerk [14], Wang et al. [25, 26] and Wang and Lesaja [24] generalized the results for LO obtained by Roos et al. in [21] to semidefinite optimization

(SDO), SCO, convex quadratic symmetric cone optimization (CQSCO) and the Cartesian $P_*(\kappa)$-SCLCP, respectively. Achache and Goutali [1] proposed a primal-dual short-step feasible interior-point algorithm for linearly constrained convex optimization and proved that the complexity of their algorithm coincides with the best iteration bound for this class of algorithms. Darvay [6] proposed a new primal-dual path-following interior point algorithm for LO with full Newton step. His algorithm is based on an equivalent transformation on the centering equations of the central path, and the search directions are obtained by applying Newton's method to the resulting system. Kheirfam [12] presented a predictor-corrector interior-point algorithm for $P_*(\kappa)$-horizontal linear complementarity problems based on Darvay's technique. Based on a modified Nesterov-Todd (NT)-direction, Kheirfam and Mahdavi-Amiri [13] and Kheirfam [11] presented a feasible IPM for linear complementarity problem over symmetric cone (SCLCP) and the Cartesian $P_*(\kappa)$-SCLCP.

Recently, Alzalg [3] showed that the circular cone $\mathcal{Q}_\theta^n$ is symmetric under a certain inner product, which is called the circular inner product. The author proposed Euclidean Jordan algebra associated with the circular cone $\mathcal{Q}_\theta^n$ by introducing a new spectral decomposition associated with this circular cone. Furthermore, the author showed that the cone of square of this Euclidean Jordan algebra is indeed the circular cone itself, i.e., the cone is symmetric with respect to the circular inner product introduced in [3]. An equivalent form of this inner product was proposed by Ma et al. [16], but they analyzed the circular cone as a non-symmetric one.

Motivated by the aforementioned works, we present a full-NT step feasible IPM for the circular cone optimization. We provide some new tools which are needed in the analysis of algorithm. The algorithm uses only full-NT steps. It is proved that this algorithm stops after at most $O\big(\sqrt{N}\log\frac{N\mu_\theta^0}{\epsilon}\big)$ iterations. To our best knowledge, this is the first full-NT step feasible IPM for CCO.

The remainder of this paper is organized as follows: In Section 2, after reviewing the Jordan algebra associated with the circular cone, some new tools and properties are provided which are widely used in the subsequent sections. In Section 3, we first describe the notions of the central path and search directions. Then the algorithm is presented in Section 3.4. Section 4 is devoted to the analysis of the algorithm. The iteration bound of the algorithm is given in Section 5.

Notations used throughout the paper are as follows. A partial ordering " $\succeq_{\mathcal{Q}_\theta^n}$ " of R^n related to a circular cone $\mathcal{Q}_\theta^n$ is defined by $x \succeq_{\mathcal{Q}_\theta^n} s$ if $x - s \in \mathcal{Q}_\theta^n$. The interior of $\mathcal{Q}_\theta^n$ is denoted as $\mathrm{int}\mathcal{Q}_\theta^n$. We write $x \succ_{\mathcal{Q}_\theta^n} s$ if $x - s \in \mathrm{int}\mathcal{Q}_\theta^n$.

2. Algebraic properties of circular cone and associated Jordan algebra

In this section, we investigate some algebraic properties of the circular cone and its associated Euclidean Jordan algebra. To ease discussion, we first assume the circular cone $\mathcal{Q}_\theta^n$ is defined when $N = 1$. Finally, we generalize some definitions to the case where $N > 1$.
Here, we first review associated Jordan algebra with the circular cone. Then, we recall the quadratic representation and the circular spectral decomposition, closely following [3].
For $x, s \in R^n$, the bilinear operator $\circ$ with respect to θ is defined as:

$$(x \circ s)_\theta := (x^T I_{\theta,n}^2 s;\ x_0\bar{s} + s_0\bar{x}) = (x_0 s_0 + \cot^2(\theta)\bar{x}^T\bar{s};\ x_0\bar{s} + s_0\bar{x}), \tag{1}$$

where $\bar{x} = (x_1; \ldots; x_{n-1})$ and $\bar{s} = (s_1; \ldots; s_{n-1})$. It is easily seen that $(R^n, \theta, \circ)$ is a Euclidean Jordan algebra under the circular inner product $\langle \cdot, \cdot \rangle_\theta$, with the vector $e = (1; 0) \in R^n$ as identity element; i.e., for all $x, s, y \in R^n$:

(i) $(x \circ s)_\theta = (s \circ x)_\theta$.

(ii) $(x \circ (x^2 \circ s)_\theta)_\theta = (x^2 \circ (x \circ s)_\theta)_\theta$, where $x^2 = (x \circ x)_\theta$.

(iii) $\langle (x \circ s)_\theta, y \rangle_\theta = \langle x, (s \circ y)_\theta \rangle_\theta$.

(iv) $(x \circ e)_\theta = (e \circ x)_\theta = x$.

One easily checks that each $x \in R^n$ satisfies the quadratic equation

$$x^2 - 2x_0 x + (x_0^2 - \cot^2(\theta)\|\bar{x}\|^2)e = 0.$$

This means that $\lambda_\theta^2 - 2x_0\lambda_\theta + (x_0^2 - \cot^2(\theta)\|\bar{x}\|^2) = 0$ is the characteristic polynomial of x. Hence the eigenvalues of x are

$$\lambda_{\theta,\min}(x) = x_0 - \cot(\theta)\|\bar{x}\|, \qquad \lambda_{\theta,\max}(x) = x_0 + \cot(\theta)\|\bar{x}\|.$$

Therefore, the trace and the determinant of x are

$$\begin{aligned} \mathrm{tr}(x) &= \lambda_{\theta,\min}(x) + \lambda_{\theta,\max}(x) = 2x_0,\ \det(x) \\ &= \lambda_{\theta,\min}(x)\lambda_{\theta,\max}(x) = x_0^2 - \cot^2(\theta)\|\bar{x}\|^2. \end{aligned}$$

The circular spectral decomposition of x with respect to the angle $\theta \in (0, \frac{\pi}{2})$ is given in [3]

$$x = \lambda_{\theta,\max}(x)c_{\theta,1} + \lambda_{\theta,\min}(x)c_{\theta,2}, \tag{2}$$

where

$$c_{\theta,1} = \frac{1}{2}\Big(1; \frac{\tan(\theta)\bar{x}}{\|\bar{x}\|}\Big), \quad c_{\theta,2} = \frac{1}{2}\Big(1; \frac{-\tan(\theta)\bar{x}}{\|\bar{x}\|}\Big).$$

One easily verifies that $c_{\theta,1} + c_{\theta,2} = e$ which is the identity element of R^n. It is quite easy to see that $\mathrm{tr}(e) = 2$ and $\det(e) = 1$. For any real valued continuous function f_θ, we define the image of x under f_θ with respect to θ as follows:

$$f_\theta(x) := f_\theta\big(\lambda_{\theta,\max}(x)\big)c_{\theta,1} + f_\theta\big(\lambda_{\theta,\min}(x)\big)c_{\theta,2}.$$

In particular, we can obtain

$$\begin{aligned}
x^{-1} &:= \frac{1}{\lambda_{\theta,\min}(x)}c_{\theta,2} + \frac{1}{\lambda_{\theta,\max}(x)}c_{\theta,1} \\
&= \frac{1}{2(x_0 - \cot(\theta)\|\bar{x}\|)}\Big(1; -\frac{\tan(\theta)\bar{x}}{\|\bar{x}\|}\Big) + \frac{1}{2(x_0 + \cot(\theta)\|\bar{x}\|)}\Big(1; \frac{\tan(\theta)\bar{x}}{\|\bar{x}\|}\Big) \\
&= \frac{\big(x_0 + \cot(\theta)\|\bar{x}\|\big)\Big(1; -\frac{\tan(\theta)\bar{x}}{\|\bar{x}\|}\Big) + \big(x_0 - \cot(\theta)\|\bar{x}\|\big)\Big(1; \frac{\tan(\theta)\bar{x}}{\|\bar{x}\|}\Big)}{2\det_\theta(x)} \\
&= \frac{\big(x_0 + \cot(\theta)\|\bar{x}\|\big)\Big(1; \frac{\tan(\theta)(-\bar{x})}{\|\bar{x}\|}\Big) + \big(x_0 - \cot(\theta)\|\bar{x}\|\big)\Big(1; -\frac{\tan(\theta)(-\bar{x})}{\|\bar{x}\|}\Big)}{2\det_\theta(x)} \\
&= \frac{1}{\det_\theta(x)}\big(x_0; -\bar{x}\big).
\end{aligned}$$

Obviously, we have $c_{\theta,1}^2 = (c_{\theta,1} \circ c_{\theta,1})_\theta = c_{\theta,1}$ and $c_{\theta,2}^2 = (c_{\theta,2} \circ c_{\theta,2})_\theta = c_{\theta,2}$ and $(c_{\theta,1} \circ c_{\theta,2})_\theta = 0$, i.e., $\{c_{\theta,1}, c_{\theta,2}\}$ is a Jordan frame. The cone of squares of the Euclidean Jordan algebra $(R^n, \theta, \circ)$ is the circular cone $\mathcal{Q}_\theta^n$ [3, Theorem 5]. We have $x \in \mathrm{int}\mathcal{Q}_\theta^n$ iff $\lambda_{\theta,\min}(x) > 0$. We say x and s operator commute with respect to θ if they share a Jordan frame, i.e.,

$$x = \lambda_{\theta,\max}(x)c_{\theta,1} + \lambda_{\theta,\min}(x)c_{\theta,2} \quad \text{and} \quad s = \bar{\lambda}_{\theta,\max}(s)c_{\theta,1} + \bar{\lambda}_{\theta,\min}(s)c_{\theta,2},$$

for a Jordan frame $\{c_{\theta,1}, c_{\theta,2}\}$. The Lyapunov transformation $L_\theta(x)$ and the quadratic representation $P_\theta(x)$ associated with $x \in R^n$ with respect to θ are respectively defined in [3]

$$L_\theta(x) := \begin{bmatrix} x_0 & \cot^2(\theta)\bar{x}^T \\ \bar{x} & x_0 I_{n-1} \end{bmatrix},$$

and

$$P_\theta(x) := 2L_\theta(x)^2 - L_\theta(x^2) = \begin{bmatrix} x_0^2 + \cot^2(\theta)\|\bar{x}\|^2 & 2\cot^2(\theta)x_0\bar{x}^T \\ 2x_0\bar{x} & \det_\theta(x)I_{n-1} + 2\cot^2(\theta)\bar{x}\bar{x}^T \end{bmatrix}.$$

Note that $L_\theta(x)e = x, L_\theta(x)x = x^2, P_\theta(x)e = x^2, P_\theta(x)x^{-1} = x$ and using (1) it follows that $(x \circ s)_\theta = L_\theta(x)s$. Using the last equality it is clear that

$$(x \circ (x^2 \circ s)_\theta)_\theta = (x^2 \circ (x \circ s)_\theta)_\theta$$

is equivalent to

$$L_\theta(x)L_\theta(x^2) = L_\theta(x^2)L_\theta(x).$$

We say that two elements x and s are similar, denoted as $x \sim s$, if x and s share the same set of eigenvalues. The Frobenius norm with respect to θ of x is defined as $\|x\|_{\theta,F} := \sqrt{\mathrm{tr}(x^2)} = \sqrt{\lambda^2_{\theta,\max}(x) + \lambda^2_{\theta,\min}(x)}$, and the 2-norm of x with respect to θ is defined as $\|x\|_{\theta,2} := \max\{|\lambda_{\theta,\max}(x)|, |\lambda_{\theta,\min}(x)|\}$. It is clear that $\|x\|_{\theta,2} \leq \|x\|_{\theta,F}$. For each $x \in R^n$, one easily verifies that

$$P_\theta(x^{-1}) = \begin{bmatrix} \dfrac{x_0^2 + \cot^2(\theta)\|\bar{x}\|^2}{\det_\theta(x)^2} & -\dfrac{2\cot^2(\theta)x_0\bar{x}^T}{\det_\theta(x)^2} \\ -\dfrac{2x_0\bar{x}}{\det_\theta(x)^2} & \dfrac{\det_\theta(x)I_{n-1} + 2\cot^2(\theta)\bar{x}\bar{x}^T}{\det_\theta(x)^2} \end{bmatrix}.$$

Furthermore, $P_\theta(x^{-1})P_\theta(x) = P_\theta(x)P_\theta(x^{-1}) = I_n$, i.e., $P_\theta(x^{-1}) = P_\theta(x)^{-1}$. In the sequel, we cite some lemmas and results on the circular cone which are widely used in the subsequent sections. The proofs of Lemmas 1, 2, 3, 4 and Lemma 5 are essentially similar to the proofs of lemmas 4.48, 4.49, 4.55, 4.56 and Lemma 4.58 in [8], respectively, and therefore omitted.

Lemma 1. *For all $x, s \in R^n$, $\|x^2\|_{\theta,F} \leq \|x\|^2_{\theta,F}$.*

Lemma 2. *Let $x, s \in R^n$ and $\langle x, s\rangle_\theta = 0$, then one has*

(i) $-\frac{1}{4}\|x+s\|^2_{\theta,F}e \preceq_{\mathcal{Q}^n_\theta} (x \circ s)_\theta \preceq_{\mathcal{Q}^n_\theta} \frac{1}{4}\|x+s\|^2_{\theta,F}e$.

(ii) $\|(x \circ s)_\theta\|_{\theta,F} \leq \frac{1}{2\sqrt{2}}\|x+s\|^2_{\theta,F}$.

Lemma 3. *Given $x \in \mathrm{int}\mathcal{Q}^n_\theta$, we have*

$$\|x - x^{-1}\|_{\theta,F} \leq \frac{\|x^2 - e\|_{\theta,F}}{\lambda_{\theta,\min}(x)}.$$

Lemma 4. *Let $x, s \in \mathrm{int}\mathcal{Q}^n_\theta$, then*

$$\|P_\theta(x^{\frac{1}{2}})s - \mu_\theta e\|_{\theta,F} \leq \|(x \circ s)_\theta - \mu_\theta e\|_{\theta,F}.$$

Lemma 5. *Let $x, s \in \mathrm{int}\mathcal{Q}_\theta^n$, then* $\lambda_{\theta,\min}\big(P_\theta(x^{\frac{1}{2}})s\big) \geq \lambda_{\theta,\min}\big((x \circ s)_\theta\big)$.

Lemma 6. *For all $x, s \in R^n$, one has* $\det_\theta\big((x \circ s)_\theta\big) \leq \det_\theta(x)\det_\theta(s)$.

Proof. We have

$$\begin{aligned}\det_\theta\big((x \circ s)_\theta\big) &= \lambda_{\theta,\min}\big((x \circ s)_\theta\big)\lambda_{\theta,\max}\big((x \circ s)_\theta\big)\\ &= \big(x_0 s_0 + \cot^2(\theta)\bar{x}^T\bar{s}\big)^2 - \cot^2(\theta)\|x_0\bar{s} + s_0\bar{x}\|^2,\end{aligned}$$

$$\begin{aligned}\det_\theta(x)\det_\theta(s) &= \big(\lambda_{\theta,\min}(x)\lambda_{\theta,\max}(x)\big)\big(\lambda_{\theta,\min}(s)\lambda_{\theta,\max}(s)\big)\\ &= \big(x_0^2 - \cot^2(\theta)\|\bar{x}\|^2\big)\big(s_0^2 - \cot^2(\theta)\|\bar{s}\|^2\big).\end{aligned}$$

Therefore

$$\det_\theta(x)\det_\theta(s) - \det_\theta\big((x \circ s)_\theta\big) = \cot^4(\theta)\big(\|\bar{x}\|^2\|\bar{s}\|^2 - (\bar{x}^T\bar{s})^2\big) \geq 0.$$

This completes the proof. □

In the sequel, we generalize the above definitions to the case when $N > 1$, that is, the circular cone underlying $\mathcal{Q}_\theta^n$ is the Cartesian product of N circular cones $\mathcal{Q}_{\theta_j}^{n_j}$. For any $x \in R^n$, the algebra $(R^n, \theta, \circ)$ is defined as a direct product of the Jordan algebras $(R^{n_j}, \theta_j, \circ)$ as

$$(x \circ s)_\theta = \big((x^1 \circ s^1)_{\theta_1}; \ldots; (x^N \circ s^N)_{\theta_N}\big).$$

Obviously, if e^j is the identity element in the Jordan algebra for the jth circular cone, then

$$e = (e^1; \ldots; e^N)$$

is the identity element in $(R^n, \theta, \circ)$. Moreover, $\mathrm{tr}(e) = 2N$. The arrow-shaped matrix $L_\theta(x)$ and the quadratic representation $P_\theta(x)$ of $(R^n, \theta, \circ)$ with respect to θ can be respectively adjusted to

$$L_\theta(x) := \mathrm{diag}\big(L_{\theta_1}(x^1), \ldots, L_{\theta_N}(x^N)\big), \quad P_\theta(x) := \mathrm{diag}\big(P_{\theta_1}(x^1), \ldots, P_{\theta_N}(x^N)\big).$$

Furthermore

$$\lambda_{\theta,\max}(x) = \max_{1 \leq j \leq N} \lambda_{\theta_j,\max}(x^j), \quad \lambda_{\theta,\min}(x) = \min_{1 \leq j \leq N} \lambda_{\theta_j,\min}(x^j),$$

and

$$\|x\|_{\theta,F}^2 = \sum_{j=1}^N \|x^j\|_{\theta_j,F}^2, \quad \mathrm{tr}(x) = \sum_{j=1}^N \mathrm{tr}(x^j).$$

3. A feasible full NT-step algorithm

In this section, we present a feasible full NT-step IPM for CCO and its analysis.

3.1. The central path for CCO

By [5, cf.Theorem 2.4.1], strong duality holds, and primal and dual problems are solvable if both (P) and (D) satisfy the interior-point condition (IPC), i.e., there exists (x^0, y^0, s^0) such that $(Ax^0)_\theta = b, x^0 \in \mathrm{int}\mathcal{Q}^n_\theta, A^T y^0 + s^0 = c$ and $s^0 \in \mathrm{int}\mathcal{Q}^n_\theta$. This can be achieved via the so-called homogeneous self-dual embedding (see [20]). Under the IPC, the optimality conditions for (P) and (D) are given as follows [4]:

$$\begin{aligned} (Ax)_\theta = b, &\quad x \in \mathcal{Q}^n_\theta, \\ A^T y + s = c, &\quad s \in \mathcal{Q}^n_\theta, \\ (x \circ s)_\theta = 0. & \end{aligned} \tag{3}$$

The main idea of primal-dual IPMs is to replace the third equation in (3), the so-called complementarity condition for (P) and (D), by the parameterized equation $(x \circ s)_\theta = \mu_\theta e$, with $\mu_\theta > 0$. Thus we consider the following system

$$\begin{aligned} (Ax)_\theta = b, &\quad x \in \mathcal{Q}^n_\theta, \\ A^T y + s = c, &\quad s \in \mathcal{Q}^n_\theta, \\ (x \circ s)_\theta = \mu_\theta e. & \end{aligned} \tag{4}$$

For any $\mu_\theta > 0$ the parameterized system (4) has a unique solution $x(\mu_\theta)$ and $(y(\mu_\theta), s(\mu_\theta))$, as μ_θ-centers of (P) and (D), respectively. The set of μ_θ-centers gives a homotopy path, which is called the central path of (P) and (D). Note that at the μ_θ-center we have

$$x_0(\mu_\theta)s_0(\mu_\theta) + \cot^2(\theta)\bar{x}(\mu_\theta)^T\bar{s}(\mu_\theta) = \frac{1}{2}\mathrm{tr}((x(\mu_\theta) \circ s(\mu_\theta))_\theta) = \frac{1}{2}\mathrm{tr}(\mu_\theta e) = \mu_\theta N.$$

Then we can derive the duality gap as follows

$$\begin{aligned} \mathrm{gap} &= \langle c, x(\mu_\theta)\rangle_\theta - b^T y(\mu_\theta) \\ &= c^T I^2_{\theta,n} x(\mu_\theta) - y(\mu_\theta)^T b - y(\mu_\theta)^T A I^2_{\theta,n} x(\mu_\theta) + y(\mu_\theta)^T A I^2_{\theta,n} x(\mu_\theta) \\ &= y(\mu_\theta)^T \left(A I^2_{\theta,n} x(\mu_\theta) - b \right) - x(\mu_\theta)^T I^2_{\theta,n} \left(A^T y(\mu_\theta) - c \right) \\ &= y(\mu_\theta)^T \left(\left(Ax(\mu_\theta) \right)_\theta - b \right) + x(\mu_\theta)^T I^2_{\theta,n} s(\mu_\theta) \\ &= x(\mu_\theta)^T I^2_{\theta,n} s(\mu_\theta) = x_0(\mu_\theta)s_0(\mu_\theta) + \cot^2(\theta)\bar{x}(\mu_\theta)^T\bar{s}(\mu_\theta) = \mu_\theta N. \end{aligned}$$

If μ_θ tends to zero, then from the above equality it follows that $(x \circ s)_\theta = 0$, i.e., (4) becomes (3). This means that the central path converges to the optimal solution of the problem.

3.2. The Nesterov-Todd search direction

In this section we will propose the search directions for CCO. To this end, we apply Newton's method to system (4) to get the following system:

$$\begin{aligned} AI_{\theta,n}^2\Delta x &= 0, \\ A^T\Delta y + \Delta s &= 0, \\ (\Delta x \circ s)_\theta + (x \circ \Delta s)_\theta &= \mu_\theta e - (x \circ s)_\theta. \end{aligned} \tag{5}$$

The system does not always have a unique solution, due to fact that x and s do not operator commute with respect to θ in general, i.e., $L_\theta(x)L_\theta(s) \neq L_\theta(s)L_\theta(x)$ (see [4]). It is well known that this difficulty can be solved by applying a scaling scheme. This is achieved as follows.
The proof of the following lemma is similar to the proof of Lemma 28 in [22] and is therefore omitted.

Lemma 7. *Let $u \in \mathrm{int}Q_\theta^n$. Then $(x \circ s)_\theta = \mu_\theta e \Leftrightarrow \left(P_\theta(u)x \circ P_\theta(u^{-1})s\right)_\theta = \mu_\theta e$.*

Now, replacing the third equation in (4) by $\left(P_\theta(u)x \circ P_\theta(u^{-1})s\right)_\theta = \mu_\theta e$, and then applying Newton's method again, we get the system

$$\begin{aligned} AI_{\theta,n}^2\Delta x &= 0, \\ A^T\Delta y + \Delta s &= 0, \\ (P_\theta(u)\Delta x \circ P_\theta(u^{-1})s)_\theta & \\ +(P_\theta(u)x \circ P_\theta(u^{-1})\Delta s)_\theta &= \mu_\theta e - (P_\theta(u)x \circ P_\theta(u^{-1})s)_\theta. \end{aligned} \tag{6}$$

By choosing u appropriately, this system can be used to define search directions. Here, we choose $u = w^{-\frac{1}{2}}$ where

$$w = P_\theta(x^{\frac{1}{2}})\Big(P_\theta(x^{\frac{1}{2}})s\Big)^{-\frac{1}{2}}\left[= P_\theta(s^{-\frac{1}{2}})\Big(P_\theta(s^{\frac{1}{2}})x\Big)^{\frac{1}{2}}\right].$$

This choice of directions was introduced by Nesterov and Todd [18, 19] and is known as the NT-direction. We define

$$v := \frac{P_\theta(w^{-\frac{1}{2}})x}{\sqrt{\mu_\theta}} \left[= \frac{P_\theta(w^{\frac{1}{2}})s}{\sqrt{\mu_\theta}}\right], \tag{7}$$

and

$$d_x := \frac{P_\theta(w^{-\frac{1}{2}})\Delta x}{\sqrt{\mu_\theta}}, \qquad d_s := \frac{P_\theta(w^{\frac{1}{2}})\Delta s}{\sqrt{\mu_\theta}}. \tag{8}$$

This enables us to rewrite the system (6) as follows:

$$\begin{aligned} AI_{\theta,n}^2 P_\theta(w^{\frac{1}{2}})d_x &= 0,\\ I_{\theta,n}^2 P_\theta(w^{\frac{1}{2}})A^T\frac{\Delta y}{\sqrt{\mu_\theta}} + I_{\theta,n}^2 d_s &= 0,\\ L_\theta(v)(d_x+d_s) &= e - L_\theta(v)v. \end{aligned}$$

By using

$$L_\theta(v^{-1})L_\theta(v) = I_n, L_\theta(v^{-1})e = v^{-1},$$

and

$$I_{\theta,n}^2 P_\theta(w^{\frac{1}{2}}) = \big(I_{\theta,n}^2 P_\theta(w^{\frac{1}{2}})\big)^T,$$

the above system can be written as follows:

$$\begin{aligned} AI_{\theta,n}^2 P_\theta(w^{\frac{1}{2}})d_x &= 0,\\ \Big(A\big(I_{\theta,n}^2 P_\theta(w^{\frac{1}{2}})\big)\Big)^T\frac{\Delta y}{\sqrt{\mu_\theta}} + I_{\theta,n}^2 d_s &= 0,\\ d_x + d_s &= v^{-1} - v. \end{aligned} \tag{9}$$

It easily follows that the system (9) has a unique solution. Since the first equation in (9) requires that d_x belongs to the null space of $AI_{\theta,n}^2 P_\theta(w^{\frac{1}{2}})$, and the second equation that $I_{\theta,n}^2 d_s$ belongs to the row space of $AI_{\theta,n}^2 P_\theta(w^{\frac{1}{2}})$, it follows that

$$d_x^T I_{\theta,n}^2 d_s = \langle d_x, d_s\rangle_\theta = 0. \tag{10}$$

From the third equation of (9) and (10) we obtain

$$\|d_x + d_s\|_{\theta,F}^2 = \|d_x\|_{\theta,F}^2 + \|d_s\|_{\theta,F}^2 = \|v^{-1} - v\|_{\theta,F}^2. \tag{11}$$

The search directions d_x and d_s are obtained by solving (9) so that Δx and Δs are computed via (8). The new iterates are obtained by taking a full step, as follows:

$$x_+ = x + \Delta x, \quad y_+ = y + \Delta y, \quad s_+ = s + \Delta s. \tag{12}$$

3.3. Proximity measure

In the analysis of the algorithm we need a measure for the distance of the iterates (x, y, s) to the current μ_θ-center $(x(\mu_\theta), y(\mu_\theta), s(\mu_\theta))$. To this end, we define a norm-based proximity measure $\delta(x, s; \mu_\theta)$ as follows

$$\delta(x, s; \mu_\theta) := \delta(v) := \frac{1}{2}\|v^{-1} - v\|_{\theta,F}, \tag{13}$$

where v is defined in (7). Using (1) we have

$$\begin{aligned}
\operatorname{tr}(v^2) &= \sum_{j=1}^{N} \operatorname{tr}\big((v^j \circ v^j)_\theta\big) \\
&= \sum_{j=1}^{N} \operatorname{tr}\big((v_0^j v_0^j + \cot^2(\theta)(\bar{v}^j)^T(\bar{v})^j; v_0^j(\bar{v})^j + v_0^j(\bar{v})^j)\big) \\
&= 2\sum_{j=1}^{N} \big(v_0^j v_0^j + \cot^2(\theta)(\bar{v})^{j^T}(\bar{v})^j\big) \\
&= \sum_{j=1}^{N} \big(v_0^j + \cot(\theta)\|(\bar{v})^j\|\big)^2 + \sum_{j=1}^{N} \big(v_0^j - \cot(\theta)\|(\bar{v})^j\|\big)^2 \\
&= \sum_{j=1}^{N} \big[\lambda_{\theta,\max}(v^j)^2 + \lambda_{\theta,\min}(v^j)^2\big] = \|v\|_{\theta,F}^2.
\end{aligned}$$

Therefore, from (13) it follows that

$$4\delta(v)^2 = \|v^{-1} - v\|_{\theta,F}^2 = \operatorname{tr}(v^2) + \operatorname{tr}(v^{-2}) - 2\operatorname{tr}(e). \tag{14}$$

3.4. The algorithm

Here, we summarize the above discussion and outline the algorithm. At the start of algorithm, we choose a strictly feasible pair (x_0, s_0) and $\mu_\theta^0 = \frac{\langle x_0, s_0\rangle_\theta}{N}$ such that $\delta(x_0, s_0; \mu_\theta^0) \le \tau$ with $0 < \tau < 1$. Then, μ_θ is reduced by the factor $1-\beta$ with $0 < \beta < 1$ and the search directions are computed by solving (9). The new iterates (x_+, s_+) are obtained by taking a full-NT step. The appropriate choice of the values for τ and β guarantees that (x_+, s_+) is strictly feasible and

$\delta(x_+, s_+; \mu_\theta^+) \leq \tau$. This process is repeated until $N\mu_\theta \leq \epsilon$.

Algorithm : *Primal – dual feasible IPM*

Input :

accuracy parameter $\epsilon > 0$;

barrier update parameter β, $0 < \beta < 1$;

threshold parameter $0 < \tau < 1$;

strictly feasible pair (x^0, s^0) and $\mu_\theta^0 > 0$ such that $\langle x_0, s_0 \rangle_\theta = N\mu_\theta^0$ and $\delta(x_0, s_0; \mu_\theta^0) \leq \tau$.

begin

$x := x^0; y := y^0; s := s^0; \mu := \mu^0$;

while $N\mu_\theta > \epsilon$

$(x, y, s) := (x + \Delta x, y + \Delta y, s + \Delta s)$;

$\mu_\theta := (1 - \beta)\mu_\theta$;

endwhile

end

---

4. Analysis of the algorithm

In this section, we first propose the feasibility condition of the full-NT step. Then, we establish the local quadratic convergence of the full-NT step. Finally, the global convergence of the algorithm is proved.

4.1. Feasibility of the full-NT step

Using (7), (8) and (12), we obtain

$$x_+ = \sqrt{\mu_\theta} P_\theta(w^{\frac{1}{2}})(v + d_x), \quad s_+ = \sqrt{\mu_\theta} P_\theta(w^{-\frac{1}{2}})(v + d_s). \tag{15}$$

Since $P_\theta(w^{\frac{1}{2}})$ and its inverse $P_\theta(w^{-\frac{1}{2}})$ are automorphisms of $\mathcal{Q}_\theta^n$, x_+ and s_+ will belong to $\mathrm{int}\mathcal{Q}_\theta^n$ if and only if $v + d_x$ and $v + d_s$ belong to $\mathrm{int}\mathcal{Q}_\theta^n$. To find a feasibility condition, which is Lemma 9, we need the following lemma.

Lemma 8. *If $\delta(v) < 1$ then $e + (d_x \circ d_s)_\theta \in \mathrm{int}\mathcal{Q}_\theta^n$.*

Proof. Since $\langle d_x, d_s \rangle_\theta = 0$, from Lemma 2(i) it follows that

$$-\frac{1}{4}\|d_x + d_s\|_{\theta,F}^2 e \preceq_{\mathcal{Q}_\theta^n} (d_x \circ d_s)_\theta \preceq_{\mathcal{Q}_\theta^n} \frac{1}{4}\|d_x + d_s\|_{\theta,F}^2 e,$$

or equivalently we have

$$\frac{1}{4}\|d_x+d_s\|_{\theta,F}^2 e+(d_x\circ d_s)_\theta \succeq_{\mathcal{Q}_\theta^n} 0,\quad \frac{1}{4}\|d_x+d_s\|_{\theta,F}^2 e-(d_x\circ d_s)_\theta \succeq_{\mathcal{Q}_\theta^n} 0.$$

These expressions mean that

$$\begin{aligned}\lambda_{\theta,\min}\Big(\frac{1}{4}\|d_x+d_s\|_{\theta,F}^2 e+(d_x\circ d_s)_\theta\Big) &= \lambda_{\theta,\min}\big((d_x\circ d_s)_\theta\big)\\ &\quad+\frac{1}{4}\|d_x+d_s\|_{\theta,F}^2\geq 0,\end{aligned}\tag{16}$$

and

$$\begin{aligned}\lambda_{\theta,\min}\Big(\frac{1}{4}\|d_x+d_s\|_{\theta,F}^2 e-(d_x\circ d_s)_\theta\Big) &= \frac{1}{4}\|d_x+d_s\|_{\theta,F}^2\\ &\quad-\lambda_{\theta,\min}\big((d_x\circ d_s)_\theta\big)\geq 0.\end{aligned}\tag{17}$$

The inequalities (16) and (17) imply that

$$\big|\lambda_{\theta,\min}(d_x\circ d_s)_\theta\big|\leq\frac{1}{4}\|d_x+d_s\|_{\theta,F}^2.$$

On the other hand, from (11) and (13) it follows that

$$\frac{1}{4}\|d_x+d_s\|_{\theta,F}^2=\frac{1}{4}\|v^{-1}-v\|_{\theta,F}^2=\delta(v)^2.$$

Hence, if $\delta(v)<1$ then $-1<\lambda_{\theta,\min}(d_x\circ d_s)_\theta<1$, and therefore $e+(d_x\circ d_s)_\theta\in \operatorname{int}\mathcal{Q}_\theta^n$. This proves the lemma. $\square$

The following lemma gives the strict feasibility of the full NT-step.

Lemma 9. *Let $\delta:=\delta(x,s;\mu_\theta)<1$. Then the full NT-step is strictly feasible.*

Proof. Introduce a step length α with $0\leq\alpha\leq 1$, and define

$$v_x(\alpha):=v+\alpha d_x,\qquad v_s(\alpha):=v+\alpha d_s.$$

We then have $v_x(0)=v, v_x(1)=v+d_x$, and similarly $v_s(0)=v, v_s(1)=v+d_s$. By using the third equation of (9) we have

$$\begin{aligned}\big(v_x(\alpha)\circ v_s(\alpha)\big)_\theta &= \big((v+\alpha d_x)\circ(v+\alpha d_s)\big)_\theta\\ &=(v\circ v)_\theta+\alpha(v\circ(d_x+d_s))_\theta+\alpha^2(d_x\circ d_s)_\theta\\ &=(1-\alpha)(v\circ v)_\theta+\alpha e+\alpha^2\big(d_x\circ d_s\big)_\theta.\end{aligned}$$

Since $\delta < 1$, Lemma 8 implies that $(d_x \circ d_s)_\theta \succ_{\mathcal{Q}_\theta^n} -e$. Substitution gives

$$\big(v_x(\alpha) \circ v_s(\alpha)\big)_\theta \succ_{\mathcal{Q}_\theta^n} (1-\alpha)(v \circ v)_\theta + \alpha e - \alpha^2 e = (1-\alpha)(v^2 + \alpha e).$$

If $0 \le \alpha \le 1$, then $\big(v_x(\alpha) \circ v_s(\alpha)\big)_\theta \succ_{\mathcal{Q}_\theta^n} 0$. Hence we have $\det\big(\big(v_x(\alpha) \circ v_s(\alpha)\big)_\theta\big) > 0$. By Lemma 6, this implies that $\det(v_x(\alpha))\det(v_s(\alpha)) > 0$, for each $\alpha \in [0,1]$. It follows that $\det(v_x(\alpha))$ and $\det(v_s(\alpha))$ do not vanish for $\alpha \in [0,1]$. Since we have $\det(v_x(0)) = \det(v_s(0)) = \det(v) > 0$. By continuity, $\det(v_x(\alpha)) > 0$ and $\det(v_s(\alpha)) > 0$ for each $\alpha \in [0,1]$, whence $v + d_x \succ_{\mathcal{Q}_\theta^n} 0$ and $v + d_s \succ_{\mathcal{Q}_\theta^n} 0$. This completes the proof. □

4.2. Local quadratic convergence

Here, we prove quadratic convergence to the target point $(x(\mu_\theta), s(\mu_\theta))$ when taking full NT-steps. According to (7), the v-vector after the step is given by

$$v_+ := \frac{P_\theta(w_+^{-\frac{1}{2}})x_+}{\sqrt{\mu_\theta}} \ \Big[= \frac{P_\theta(w_+^{\frac{1}{2}})s_+}{\sqrt{\mu_\theta}} \Big], \tag{18}$$

where w_+ is the scaling point of x_+ and s_+.
The proof of the next lemma is similar to the proof of Proposition 3.2.4 in [23] and is therefore omitted.

Lemma 10. *Let $x, s \in \operatorname{int}\mathcal{Q}_\theta^n$. If w is the scaling point of x and s, then $\big(P_\theta(w^{\frac{1}{2}})s\big)^2 \sim P_\theta(x^{\frac{1}{2}})s$.*

Lemma 11. *One has $v_+^2 \sim P_\theta\big((v+d_x)^{\frac{1}{2}}\big)(v+d_s)$.*

Proof. It follows from (18) and Lemma 10 that

$$\mu_\theta v_+^2 = \Big(P_\theta(w_+^{\frac{1}{2}})s_+\Big)^2 \sim P_\theta(x_+^{\frac{1}{2}})s_+.$$

Using (15) and item ii of [22, Proposition 21] for $P_\theta(x)$ we obtain

$$\begin{aligned} P_\theta(x_+^{\frac{1}{2}})s_+ &= P_\theta\Big(\big(\sqrt{\mu_\theta}P_\theta(w^{\frac{1}{2}})(v+d_x)\big)^{\frac{1}{2}}\Big)\sqrt{\mu_\theta}P_\theta(w^{-\frac{1}{2}})(v+d_s) \\ &= \mu_\theta P_\theta\Big(\big(P_\theta(w^{\frac{1}{2}})(v+d_x)\big)^{\frac{1}{2}}\Big)P_\theta(w^{-\frac{1}{2}})(v+d_s) \\ &\sim \mu_\theta P_\theta\big((v+d_x)^{\frac{1}{2}}\big)(v+d_s). \end{aligned}$$

From this the lemma follows. □

Lemma 12. *Let $\delta = \delta(v) < 1$, then the full NT-step is strictly feasible and*

$$\delta(v_+) \leq \frac{\delta^2}{\sqrt{2(1-\delta^2)}}.$$

Proof. Since $\delta < 1$, from Lemma 9 and its proof, it follows that $v + d_x, v + d_s$ and $(v + d_x) \circ (v + d_s)$ belong to the $\mathrm{int}\,\mathcal{Q}_\theta^n$. By applying Lemma 3, we obtain

$$2\delta(v_+) = \|v_+ - (v_+)^{-1}\|_{\theta,F} \leq \frac{\|(v_+)^2 - e\|_{\theta,F}}{\lambda_{\theta,\min}(v_+)^{\frac{1}{2}}}.$$

Due to Lemmas 11, 5, 4 and 2, we get

$$\begin{aligned}
2\delta(v_+) &\leq \frac{\|P_\theta\big((v+d_x)^{\frac{1}{2}}\big)(v+d_s) - e\|_{\theta,F}}{\Big(\lambda_{\theta,\min}\big(P_\theta\big((v+d_x)^{\frac{1}{2}}\big)(v+d_s)\big)\Big)^{\frac{1}{2}}} \\
&\leq \frac{\|\big((v+d_x)\circ(v+d_s)\big)_\theta - e\|_{\theta,F}}{\lambda_{\theta,\min}\big((v+d_x)\circ(v+d_s)\big)_\theta^{\frac{1}{2}}} \\
&= \frac{\big\|\big(d_x \circ d_s\big)_\theta\big\|_{\theta,F}}{\big(1 + \lambda_{\theta,\min}\big(d_x \circ d_s\big)_\theta\big)^{\frac{1}{2}}} \\
&\leq \frac{\frac{1}{2\sqrt{2}}\|d_x + d_s\|_{\theta,F}^2}{\sqrt{1-\delta^2}} \leq \frac{\sqrt{2}\delta^2}{\sqrt{1-\delta^2}}.
\end{aligned}$$

This proves the lemma. □

As an immediate consequence, we have the following simple result.

Corollary 1. *If $\delta \leq \frac{1}{\sqrt{2}}$, then the full NT-step is strictly feasible and $\delta(v_+) \leq \delta^2$, which means that the full NT-step ensures local quadratic convergence of the proximity measure.*

4.3. Global convergence of the algorithm

The next lemma shows that the target duality gap is attained after a full NT-step.

Lemma 13. *Let $(x, s) \in \mathcal{Q}_\theta^n$ and $\mu_\theta > 0$. Then $\langle x_+, s_+\rangle_\theta = \mu_\theta N$.*

Proof. Due to (15), $I_{\theta,n}^2 P_\theta(x) = P_\theta(x)^T I_{\theta,n}^2$ and $P_\theta(x^{-1}) = P_\theta(x)^{-1}$ we may write

$$\begin{aligned}
\langle x_+, s_+\rangle_\theta &= x_+^T I_{\theta,n}^2 s_+ = \mu_\theta(v+d_x)^T P_\theta(w^{\frac{1}{2}})^T I_{\theta,n}^2 P_\theta(w^{-\frac{1}{2}})(v+d_s)\\
&= \mu_\theta(v+d_x)^T I_{\theta,n}^2 P_\theta(w^{\frac{1}{2}}) P_\theta(w^{\frac{1}{2}})^{-1}(v+d_s)\\
&= \mu_\theta(v+d_x)^T I_{\theta,n}^2 (v+d_s) = \mu_\theta\langle v+d_x, v+d_s\rangle_\theta.
\end{aligned}$$

Using the third equation of (9) we obtain

$$\begin{aligned}
\langle v+d_x, v+d_s\rangle_\theta &= \langle v,v\rangle_\theta + \langle v, d_x+d_s\rangle_\theta + \langle d_x,d_s\rangle_\theta\\
&= \langle v,v\rangle_\theta + \langle v, v^{-1}-v\rangle_\theta + \langle d_x,d_s\rangle_\theta\\
&= \langle v, v^{-1}\rangle_\theta + \langle d_x,d_s\rangle_\theta\\
&= \frac{1}{2}\mathrm{tr}\big((v\circ v^{-1})_\theta\big) + \langle d_x,d_s\rangle_\theta\\
&= \frac{1}{2}\mathrm{tr}(e) + \langle d_x,d_s\rangle_\theta.
\end{aligned}$$

Using (10) and $\mathrm{tr}(e) = 2N$, the lemma follows. $\square$

In next lemma, we establish an important relation for the proximity measure just before and after a μ_θ-update.

Lemma 14. *Let $\delta := \delta(x,s;\mu_\theta) < 1$ and $\mu_\theta^+ := (1-\beta)\mu_\theta$ for $0<\beta<1$, then*

$$\delta(x_+, s_+; \mu_\theta^+)^2 = (1-\beta)\delta(v_+)^2 + \frac{\beta^2 N}{2(1-\beta)}.$$

Proof. After updating $\mu_\theta^+ = (1-\beta)\mu_\theta$, the vector v_+ is divided to the factor $\sqrt{1-\beta}$. Using (13), it follows that

$$\begin{aligned}
4\delta(x_+, s_+; \mu_\theta^+)^2 &= \left\|\sqrt{1-\beta}\,v_+^{-1} - \frac{v_+}{\sqrt{1-\beta}}\right\|_{\theta,F}^2\\
&= \left\|\sqrt{1-\beta}\big(v_+^{-1} - v_+\big) - \frac{\beta v_+}{\sqrt{1-\beta}}\right\|_{\theta,F}^2\\
&= (1-\beta)\big\|v_+^{-1} - v_+\big\|_{\theta,F}^2 + \frac{\beta^2}{1-\beta}\|v_+\|_{\theta,F}^2\\
&\quad -2\beta\,\mathrm{tr}\Big(\big((v_+^{-1}-v_+)\circ v_+\big)_\theta\Big)
\end{aligned}$$

$$
\begin{aligned}
&= 4(1-\beta)\delta(v_+)^2 + \frac{2\beta^2 N}{1-\beta} \\
&\quad -2\beta \mathrm{tr}\big((v_+^{-1} \circ v_+)_\theta\big) + 2\beta \mathrm{tr}\big((v_+ \circ v_+)_\theta\big) \\
&= 4(1-\beta)\delta(v_+)^2 + \frac{2\beta^2 N}{1-\beta} - 2\beta \mathrm{tr}(e) + 4\beta \langle v_+, v_+\rangle_\theta \\
&= 4(1-\beta)\delta(v_+)^2 + \frac{2\beta^2 N}{1-\beta} - 4\beta N + 4\beta N \\
&= 4(1-\beta)\delta(v_+)^2 + \frac{2\beta^2 N}{1-\beta}.
\end{aligned}
$$

The inequality follows from the following fact

$$
\|v_+\|_{\theta,F}^2 = 2\langle v_+, v_+\rangle_\theta = 2\Big\langle \frac{P_\theta(w_+^{-\frac{1}{2}})x_+}{\sqrt{\mu_\theta}}, \frac{P_\theta(w_+^{\frac{1}{2}})s_+}{\sqrt{\mu_\theta}} \Big\rangle_\theta = \frac{2}{\mu_\theta}\langle x_+, s_+\rangle_\theta = 2N.
$$

Therefore, the proof is complete. $\square$

Corollary 2. *Let $\delta := \delta(x, s; \mu_\theta) \leq \frac{1}{\sqrt{2}}$ and $\beta = \frac{1}{\sqrt{3N}}$ with $N \geq 2$. Then*

$$
\delta(x_+, s_+; \mu_\theta^+) < \frac{1}{\sqrt{2}}.
$$

Proof. From Corollary 1 we obtain

$$
\delta(x_+, s_+; \mu_\theta) \leq \delta^2 \leq \frac{1}{2}.
$$

Then, after the barrier parameter is updated to $\mu_\theta^+ = (1-\beta)\mu_\theta$ with $\beta = \frac{1}{\sqrt{3N}}$, it follows from Lemma 14 that

$$
\delta(x_+, s_+; \mu_\theta^+)^2 \leq \frac{1-\beta}{4} + \frac{1}{6(1-\beta)}. \tag{19}
$$

Note that $0 \leq \beta \leq \frac{1}{\sqrt{6}}$ for $N \geq 2$. One easily verifies the right-hand side expression in (19) is monotonically increasing with respect to β. Consequently, we have

$$
\frac{1-\beta}{4} + \frac{1}{6(1-\beta)} \leq \frac{1-\frac{1}{\sqrt{6}}}{4} + \frac{1}{6(1-\frac{1}{\sqrt{6}})} \approx 0.4296 < \frac{1}{2}.
$$

Then,

$$
\delta(x_+, s_+; \mu_\theta^+) < \frac{1}{\sqrt{2}}.
$$

This completes the proof of the corollary. $\square$

5. Complexity of the algorithm

In this section, we derive the upper bound on the number of iterations needed by the algorithm to find an ϵ-approximate solution of CCP.

Lemma 15. *Suppose that x^0 and s^0 are strictly feasible, $\mu_\theta^0 N = \langle x^0, s^0\rangle_\theta$ and $\delta(x^0, s^0; \mu_\theta^0) \leq \frac{1}{\sqrt{2}}$. Moreover, let x^k and s^k be the iterates obtained after k iterations. Then, the inequality $\langle x^k, s^k\rangle_\theta \leq \epsilon$ is satisfied for*

$$k \geq \frac{1}{\beta} \log \frac{\langle x^0, s^0\rangle_\theta}{\epsilon}.$$

Proof. From Lemma 13 it follows that $\langle x^k, s^k\rangle_\theta = \mu_\theta^k N = (1-\beta)^k \langle x^0, s^0\rangle_\theta$. Then, the inequality $\langle x^k, s^k\rangle_\theta \leq \epsilon$ holds if $(1-\beta)^k \langle x^0, s^0\rangle_\theta \leq \epsilon$. Taking logarithms, we obtain $k \log(1-\beta) + \log\langle x^0, s^0\rangle_\theta \leq \log \epsilon$. Using $\log(1-\beta) \leq -\beta$, we observe that the above inequality holds if $-\beta k + \log\langle x^0, s^0\rangle_\theta \leq \log \epsilon$, which implies the inequality in the lemma. □

The following theorem gives an upper bound for the total number of iterations produced by Algorithm.

Theorem 1. *Let $\beta = \frac{1}{\sqrt{3N}}$ with $N \geq 2$. Then, Algorithm requires at most*

$$\mathcal{O}\Big(\sqrt{N} \log \frac{\langle x^0, s^0\rangle_\theta}{\epsilon}\Big)$$

iterations. The output is a primal-dual pair (x, s) satisfying $\langle x, s\rangle_\theta \leq \epsilon$.

Acknowledgements

The author is very grateful to the editors and the anonymous referees for their useful comments and suggestions which helped to improve the presentation of this paper.

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