Communications in Combinatorics and Optimization
Vol. 1 No. 1, 2016 pp.29-41
DOI: 10.22049/CCO.2016.13543

CCO
Commun. Comb. Optim.

New bounds on proximity and remoteness in graphs

P. Dankelmann

Department of Pure and Applied Mathematics, University of Johannesburg,
PO Box 524 Auckland Park 2006, South Africa
pdankelmann@uj.ac.za

Received: 27 June 2016; Accepted: 6 August 2016;
Available Online: 7 August 2016.

Communicated by ***Alireza Ghaffari-Hadigheh***

Abstract: The average distance of a vertex v of a connected graph G is the arithmetic mean of the distances from v to all other vertices of G. The proximity $\pi(G)$ and the remoteness $\rho(G)$ of G are defined as the minimum and maximum, respectively, average distance of the vertices of G. In this paper we investigate the difference between proximity or remoteness and the classical distance parameters diameter and radius. Among other results we show that in a graph of order n and minimum degree δ the difference between diameter and proximity and the difference between radius and proximity cannot exceed $\frac{9n}{4(\delta+1)}+c_1$ and $\frac{3n}{4(\delta+1)}+c_2$, respectively, for constants c_1 and c_2 which depend on δ but not on n. These bounds improve bounds by Aouchiche and Hansen [3] in terms of order alone by about a factor of $\frac{3}{\delta+1}$. We further give lower bounds on the remoteness in terms of diameter or radius. Finally we show that the average distance of a graph, i.e., the average of the distances between all pairs of vertices, cannot exceed twice the proximity.

Keywords: proximity; remoteness; diameter; radius; average distance; Wiener index.

2010 Mathematics Subject Classification: 05C12

1. Introduction

Let G be a simple onnected finite graph representing a transportation network. An important indicator for the suitability of a vertex v as a location for a facility is the expected distance between v and a randomly chosen vertex, i.e., the average of the distances between v and all other vertices. We define the average distance $\overline{\sigma}(v)$ of v as the arithmetic mean of the distances from v to all other vertices,i.e., if G has n vertices then

$$\overline{\sigma}(v) = \frac{1}{n-1} \sum_{w \in V(G)-\{v\}} d_G(v,w),$$

where $d_G(v,w)$ is the usual distance between v and w. The proximity $\pi(G)$ and remoteness $\rho(G)$ of a connected graph G are defined as the smallest and the largest, respectively, of the average distances of all vertices, i.e.,

$$\pi(G) = \min_{v \in V(G)} \overline{\sigma}(v), \qquad \rho(G) = \max_{v \in V(G)} \overline{\sigma}(v).$$

Informally, the proximity and the remoteness of a graph describe the distance from a most central and from a least central vertex to a typical vertex.
Zelinka [12], and independently Aouchiche and Hansen [3] showed that in a connected graph of order n, the proximity and the remoteness are bounded from above by approximately $\frac{n}{4}$ and $\frac{n}{2}$. Aouchiche and Hansen [3] showed that the difference between remoteness and proximity cannot exceed a value of approximately $\frac{n}{4}$. The author of the present paper showed in [6] that for graphs of minimum degree δ, where $\delta \geq 2$, these three bounds can be improved approximately by a factor of $\frac{3}{\delta+1}$. For more results on proximity and remoteness in graphs see [1, 2, 4, 8, 10, 11].
In this paper, we investigate how proximity and remoteness relate to the classical distance parameters diameter, radius, and average distance. More specifically, we give upper and lower bounds on the difference between diameter and proximity and the difference between radius and proximity in terms of order and minimum degree, thus improving bounds in terms of order alone due to Aouchiche and Hansen [3] by about a factor of $\frac{3}{\delta+1}$. We also present lower bounds on the remoteness in terms of diameter and radius, respectively. Relating the proximity to the average distance, we also show that the average distance of a graph is less than twice its proximity.
The notation in this paper is as follows. By G we always denote a finite, simple, connected graph on $n(G)$ vertices with vertex set $V(G)$. For a vertex v of G, $N_G(v)$ is the neighbourhood of v, i.e., the set of vertices adjacent to v, and $N_G[v]$ is the closed neighbourhood of v, i.e., the set $N_G(v) \cup \{v\}$. For $A \subseteq V(G)$ we define $N[A] = \bigcup_{v \in A} N[v]$. The degree $\deg_G(v)$ of v is the number of vertices in $N_G(v)$, and the minimum degree $\delta(G)$ of G is the smallest of the degrees of the vertices of G. The distance between two vertices u and v, i.e., the minimum length of a (u,v)-path, is denoted by $d_G(u,v)$. The eccentricity $\mathrm{ecc}_G(v)$ of v is the distance from v to a vertex farthest from v. The largest and the smallest of the eccentricities of the vertices of G is the diameter and the radius,

respectively, denoted by $\mathrm{diam}(G)$ and $\mathrm{rad}(G)$. The average distance $\mu(G)$ is defined as $\binom{n}{2}^{-1} \sum_{\{u,v\}\subseteq V} d(u,v)$. The average distance is closely related to the Wiener index, which is defined as $\sum_{\{u,v\}\subseteq V} d(u,v)$. If the graph is understood from the context, then we sometimes omit the argument or subscript G.
For a vertex v and a set $A \subseteq V$, we define $\sigma(v,A)$ as the sum of the distances from v to all vertices of A. Instead of $\sigma(v,V(G))$ we usually write $\sigma(v)$.
By K_n we mean the complete graph on n vertices. For disjoint graphs $G_1, G_2, \dots, G_k$ the sequential sum $G_1 + G_2 + \cdots + G_k$ is the graph obtained from the union of $G_1, G_2, \dots, G_k$ by joining every vertex of G_i to every vertex of G_{i+1} for $i = 1, 2, \dots, k-1$. If in a sequential sum a pattern is repeated ℓ times, then we indicate this with square brackets and the exponent ℓ; for example $G_1 + [G_2 + G_3]^\ell + G_4$ stands for $G_1 + G_2 + G_3 + G_2 + G_3 + \cdots + G_2 + G_3 + G_4$, where the pattern $G_2 + G_3$ appears ℓ times.

2. Results

2.1. Proximity and remoteness vs diameter

Aouchiche and Hansen [3] gave the following bound on the difference between the proximity and the diameter.

Theorem 1. ([3]) *Let G be a connected graph of order n. Then*

$$\mathrm{diam}(G) - \pi(G) \leq \begin{cases} \frac{3n-5}{4} - \frac{1}{4n-4} & \text{if } n \text{ is even,} \\ \frac{3n-5}{4} & \text{if } n \text{ is odd.} \end{cases}$$

Equality holds if and only if G is a path.

The diameter of a graph of order n is at most $n-1$, and the proximity is at most $\frac{n+1}{4}$ for odd n and $\frac{n+1}{4} + \frac{1}{4(n-1)}$ for even n (see [3, 12]). For graphs of given minimum degree both bounds can be improved by about a factor of $\frac{3}{\delta+1}$. This was observed, for example, by Erdös, Pach, Pollack and Tuza [9] for the diameter, and by Dankelmann [6] for the proximity.

Theorem 2. ([9]) *Let G be a connected graph of order n and minimum degree δ, where $\delta \geq 2$. Then*

$$\mathrm{diam}(G) \leq \frac{3n}{\delta+1} - 1,$$

and this bound is sharp, apart from an additive constant.

Theorem 3. ([6]) *Let G be a connected graph of order n and minimum degree δ, where $\delta \geq 2$. Then*

$$\pi(G) \leq \frac{3n}{4(\delta+1)} + 3,$$

and this bound is sharp, apart from an additive constant.

It is reasonable to ask if the bound on the difference between diameter and proximity in Theorem 1 can also be improved by about a factor of $\frac{3}{\delta+1}$. The following theorem shows that this is indeed the case.

Theorem 4. *Let G be a connected graph of order n and minimum degree δ, where $n \geq 20$ and $\delta \geq 2$. Then*

$$\operatorname{diam}(G) - \pi(G) \leq \frac{9}{4(\delta+1)} n + \frac{3}{4}\delta. \tag{1}$$

Proof: Let $d = \operatorname{diam}(G)$, let v_0 and v_d be two vertices of G at distance d, and let $P : v_0, v_1, \ldots, v_d$ be a shortest path between them. For $i = 0, 1, \ldots, d$ let $A_i = \{w_i^1, w_i^2, \ldots, w_i^\delta\}$ be a set of δ neighbours of v_i, and for $i = 0, 1, \ldots, \lceil \frac{d}{2} \rceil$ let $B_i = \{v_i, v_{d-i}\} \cup A_i \cup A_{d-i}$. We now fix a vertex v with $\overline{\sigma}(v) = \pi(G)$. By the triangle-inequality we get $d(v, v_i) + d(v, v_{d-i}) \geq d(v_i, v_{d-i}) = d - 2i$, and thus $d(v, w_i) + d(v, w_{d-i}) \geq d(v_i, v_{d-i}) - 2 = d - 2i - 2$, for every $w_i \in N_G(v_i)$ and $w_{d-i} \in N_G(v_{d-i})$. Hence,

$$\begin{aligned} \sigma(v, B_i) &= d(v, v_i) + d(v, v_{d-i}) + \sum_{j=1}^{\delta} \big(d(v, w_i^j) + d(v, w_{d-i}^j) \big) \\ &\geq (\delta+1)(d - 2i - 2) + 2. \end{aligned} \tag{2}$$

Clearly, the sets A_i and A_j are disjoint whenever $|i - j| \geq 3$ since otherwise there exists a path of length at most two between v_i and v_j, and replacing the $(v_i - v_j)$-section of P with this path yields a shorter (x, y)-path, a contradiction. From now on we may assume that $d \geq 8$ since for $\delta \geq 2$ and $n \geq 20$ the right hand side of (1) is at least 7. Let $d = 6a + b$, where $a, b \in \mathbb{N}_0$ with $3 \leq b \leq 8$. Then the sets B_{3i}, $i = 0, 1, 2, \ldots, a$ are pairwise disjoint. Therefore,

$$\begin{aligned} \sigma(v) &\geq \sum_{i=0}^{a} \sigma(v, B_{3i}) \\ &\geq \sum_{i=0}^{a} \Big((\delta+1)(d - 6i - 2) + 2 \Big) \\ &= (a+1)(\delta+1)(d - 3a - 2) + 2(a+1). \end{aligned}$$

Now $\frac{d-8}{6} \le a \le \frac{d-3}{6}$. Straightforward calculus shows that the function $f(a) = (a+1)(\delta+1)(d-3a-2)+2(a+1)$ attains its minimum in the interval $[\frac{d-8}{6}, \frac{d-3}{6}]$ for $a = \frac{d-8}{6}$. Substituting this value for a yields

$$\sigma(v) \ge \frac{1}{12}(\delta+1)(d-2)(d+4) + \frac{1}{3}(d-2).$$

Now $\pi(G) = \overline{\sigma}(v) = \frac{\sigma(v)}{n-1}$. Replacing $(d-2)(d+4)$ by $(d+1)^2 - 9$ yields

$$\begin{aligned} \pi(G) &\ge \frac{\delta+1}{12(n-1)}(d+1)^2 - \frac{9}{12}(\delta+1) + \frac{1}{3(n-1)}(d-2) \\ &> \frac{\delta+1}{12n}(d+1)^2 - \frac{3}{4}(\delta+1) + \frac{d-2}{3n}. \end{aligned}$$

The difference between the diameter and the proximity is therefore bounded as follows.

$$d - \pi(G) < d - \frac{\delta+1}{12n}(d+1)^2 + \frac{3}{4}(\delta+1) - \frac{d-2}{3n}. \tag{3}$$

Denote the right hand side of (3) by $f(d)$. Making use of the fact that Theorem 2 implies that $(d+1)(\delta+1) \le 3n$, we obtain

$$f'(d) = 1 - \frac{(\delta+1)(d+1)}{6n} - \frac{1}{3n} \ge 1 - \frac{3n}{6n} - \frac{1}{3n} = \frac{3n-2}{6n} > 0.$$

Hence the right hand side of (3) is increasing in d. By Theorem 2, d is at most $\frac{3n}{\delta+1} - 1$. Substituting this value for d, we obtain, after simplification,

$$d - \pi(G) \le \frac{9}{4(\delta+1)}n + \frac{3}{4}(\delta+1) - 1 - \frac{1}{\delta+1} + \frac{1}{n} \le \frac{9}{4(\delta+1)}n + \frac{3}{4}\delta,$$

as desired. □

In the following example we construct an infinite family of graphs which shows that Theorem 1 and some of the bounds presented later in this paper are sharp, apart from an additive constant.

Example 1. For given $k, \delta \in \mathbb{N}$ with $k, \delta \ge 3$ let

$$G_{k,\delta} = K_1 + K_\delta + K_1 + [K_1 + K_{\delta-1} + K_1]^{k-2} + K_1 + K_\delta + K_1.$$

It is easy to verify that $n(G_{k,\delta}) = k(\delta+1)+2$, $\mathrm{diam}(G_{k,\delta}) = 3k-1 = \frac{3}{\delta+1}n - \frac{\delta+7}{\delta+1}$, and $\mathrm{rad}(G_{k,\delta}) = 3\lfloor \frac{k}{2} \rfloor + 1 \geq \frac{3}{2(\delta+1)}n + \frac{\delta-2}{\delta+1}$. Moreover, for large n and constant δ

$$\pi(G_{k,\delta}) = \frac{3}{4(\delta+1)}n + \frac{3}{4(\delta+1)} + o(1).$$

and

$$\rho(G_{k,\delta}) = \frac{3}{2(\delta+1)}n - \frac{\delta+4}{2(\delta+1)} + o(1),$$

where $o(1)$ stands for a term that approaches 0 as n tends to infinity.

For every fixed δ, the graphs $G_{k,\delta}$ for $k,\delta \geq 3$ form an infinite family of graphs for which $\mathrm{diam}(G_{k,\delta}) - \pi(G_{k,\delta})$ is within an additive constant of the bound in Theorem 1. Indeed,

$$\begin{aligned} \mathrm{diam}(G_{k,\delta}) - \pi(G_{k,\delta}) &= \frac{3}{\delta+1}n - \frac{\delta+7}{\delta+1} - \frac{3}{4(\delta+1)}n - \frac{3}{4(\delta+1)} + o(1) \\ &= \frac{9n}{4(\delta+1)} - \frac{4\delta+31}{4\delta+4} + o(1), \end{aligned}$$

as desired.

We now determine a sharp lower bound on the remoteness in terms of the diameter. We use this bound to derive a generalisation of a bound on the difference between diameter and remoteness in [3].

Proposition 1. *Let G be a connected graph of order n and diameter d. Then*

$$\rho(G) \geq \frac{n}{n-1}\frac{d}{2},$$

and this bound is sharp for all n and d with $n \geq d+1 \geq 3$ for which nd is even.

Proof: Let $u, w \in V$ be two vertices at distance d. Then $d(u,w) \leq d(u,v) + d(v,w)$ for every vertex v of G. Hence

$$\sigma(u) + \sigma(w) = \sum_{v \in V} \big(d(u,v) + d(w,v)\big) \geq nd(u,w) = nd.$$

Hence

$$\rho(G) \geq \max\{\overline{\sigma}(u), \overline{\sigma}(w)\} \geq \frac{1}{2(n-1)}\big(\sigma(u)+\sigma(w)\big) \geq \frac{n}{n-1}\frac{d}{2}.$$

and the desired bound follows.
The following graphs show that this bound is sharp. If d is even and $n \geq d+1$, then let

$$H_{n,d} = [K_1]^{d/2} + K_{n-d} + [K_1]^{d/2},$$

and if d is odd and $n \geq d+1$ and n even, then let

$$H_{n,d} = [K_1]^{(d-1)/2} + K_{(n-d+1)/2} + K_{(n-d+1)/2} + [K_1]^{(d-1)/2}.$$

It is easy to verify that

$$\rho(H_{n,d}) = \frac{n}{n-1}\frac{d}{2},$$

and so the bound is sharp. □

Corollary 1. *Let G be a connected graph of diameter d. Then*

$$\rho(G) > \frac{1}{2}d,$$

and the coefficient $\frac{1}{2}$ of d is best possible.

Since for a connected graph of order n the diameter is at most $n-1$, Corollary 1 implies that the difference $\mathrm{diam}(G) - \rho(G)$ cannot exceed $\frac{n-2}{2}$, which was observed first by Aouchiche and Hansen [3].
We note that the bound in Corollary 1 cannot be improved significantly for graphs of given minimum degree δ. For example it is easy to verify that the graphs $G_{k,\delta}$ attain equality in the bound of Proposition 1 for any k and δ with $k, \delta \geq 3$.

2.2. Proximity and remoteness vs radius

Aouchiche and Hansen [3] gave the following bound on the difference between the proximity and the radius.

Theorem 5. *Let G be a connected graph of order n. Then*

$$\mathrm{rad}(G) - \pi(G) \leq \begin{cases} \frac{n-1}{4} - \frac{1}{4(n-1)} & \text{if } n \text{ is even,} \\ \frac{n-1}{4} - \frac{1}{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

and this bound is sharp for all n.

The radius of a connected graph of order n is bounded by $\lfloor \frac{n}{2} \rfloor$, and for graphs of minimum degree δ this bound can be improved by about a factor of $\frac{3}{\delta+1}$. This was first observed by Erdös, Pach, Pollack and Tuza [9], who showed that $\mathrm{rad}(G) \leq \frac{3n-9}{2(\delta+1)} + 5$. Their bound was improved slightly by Dlamini [7], see also [5].

Theorem 6. ([7]) *Let G be a connected graph of order n and minimum degree δ. Then*

$$\mathrm{rad}(G) \leq \frac{3n}{2(\delta+1)} + 1,$$

and this bound is sharp apart from an additive constant.

This suggests that, like the bound on $\mathrm{diam}(G) - \pi(G)$, possibly also the bound on $\mathrm{rad}(G) - \pi(G)$ in Theorem 5 can be improved by about a factor of $\frac{3}{\delta+1}$. The following theorem shows that this is indeed the case.

Theorem 7. *Let G be a connected graph of order n and minimum degree δ, where $\delta < \frac{n}{4} - 1$. Then*

$$\mathrm{rad}(G) - \pi(G) \leq \frac{3}{4(\delta+1)} n + \frac{8\delta + 5}{4(\delta+1)},$$

and this bound is best possible, apart from an additive constant.

Proof: Let $r = \mathrm{rad}(G)$. If $r \leq 4$ then the left hand side of the inequality in the theorem is at most 3, while by $\delta < \frac{n}{4} - 1$ its right hand side is at least 3, and the theorem holds in this case. Hence we may assume that $r \geq 5$.
Fix a median vertex v_0 of G. We show that

$$\sigma(v_0) \geq \frac{\delta+1}{3}\big((r-2)^2 + 1\big). \tag{4}$$

Let $R = \mathrm{ecc}(v_0)$. For $i = 0, 1, \ldots, R$, let N_i be the set of vertices at distance i from v_0. For each $i \in \{1, 2, \ldots, R\}$ we choose a set $A_i \subseteq N_i$ of maximum cardinality with the property that any two vertices of A_i are at distance at least three in G. For every vertex $x \in N[A_i]$ we have $d(v_0, x) \geq i - 1$. Since $|N[A_i]| \geq (\delta+1)|A_i|$, we have $\sum_{x \in N[A_i]} d(v_0, x) \geq (\delta+1)(i-1)$. Since every vertex of G is in at most three of the sets $N[A_i]$, $i = 1, 2, \ldots, R$, we have

$$3\sigma(v_0) \geq \sum_{i=1}^{R} (\delta+1)|A_i|.$$

Case 1: $R \geq \frac{1}{2}(3r-5)$.
Since $|A_i| \geq 1$ for $i = 1, 2, \ldots, R$, we have

$$\begin{aligned} 3\sigma(v) &\geq \sum_{i=1}^{R}(\delta+1)(i-1) = \frac{\delta+1}{2}(R^2-R) \\ &\geq (\delta+1)(\frac{9}{8}r^2 - \frac{9}{2}r + \frac{35}{8}) > (\delta+1)\big((r-2)^2+1\big), \end{aligned}$$

and (4) follows in Case 1.

Case 2: $R \leq \frac{1}{2}(3r-6)$.
We first show that $|A_i| \geq 2$, provided that i is not too close to 1 or R.

$$|A_j| \geq 2 \text{ for all } j \text{ with } R-r+3 \leq j \leq 2r-R-3. \tag{5}$$

Suppose, to the contrary, that there exists a j with $R-r+3 \leq j \leq 2r-R-3$ such that $|A_j| = 1$. Fix a vertex $v_R \in N_R$, and let $P : v_0, v_1, \ldots, v_R$ be a shortest (v_0, v_R)-path in G. Consider v_{R-r+3}. In order to prove (5) it suffices to obtain the contradiction $\mathrm{ecc}(v_{R-r+3}) \leq r-1 < \mathrm{rad}(G)$. Indeed, let x be a vertex of G with $d(v_{R-r+3}, x) = \mathrm{ecc}(x)$. If $d(v_0, x) \geq j$, then let Q be a shortest (v_0, x)-path, and let w_j be the unique vertex of Q in N_j. Then

$$\begin{aligned} d(v_{R-r+3}, x) &\leq d(v_{R-r+3}, v_j) + d(v_j, w_j) + d(w_j, x) \\ &\leq (j-R+r-3) + 2 + (R-j) \\ &= r-1, \end{aligned}$$

which is a contradiction. If $d(v_0, x) \leq j-1$, then

$$\begin{aligned} d(v_{R-r+3}, x) &\leq d(v_{R-r+3}, v_0) + d(v_0, x) \\ &\leq (R-r+3) + j - 1 \\ &\leq R-r+3+2r-R-4 \\ &= r-1, \end{aligned}$$

again a contradiction. Hence (5) holds.

We now proceed to show that (4) holds in Case 2. Clearly, $|A_i| \geq 1$ for $i = 1, 2, \ldots, R$. By (5) we have $|A_i| \geq 2$ for all $i \in \mathbb{N}$ with $R-r+3 \leq i \leq 2r-R-3$.

Hence

$$\begin{aligned} 3\sigma(v_0) &\geq \sum_{i=1}^{R}(\delta+1)(i-1) + \sum_{R-r+3}^{2r-R-3}(\delta+1)(i-1) \\ &= \frac{\delta+1}{2}\Big[2r^2+(R-r)^2+3R-11r+10\Big] \\ &\geq (\delta+1)\Big[(r-2)^2+1\Big], \end{aligned}$$

and (4) follows also in Case 2.

We are now ready to complete the proof of the bound in Theorem 7. From (4) we obtain the following lower bound on the proximity.

$$\pi(G) = \frac{1}{n-1}\sigma(v_0) \geq \frac{\delta+1}{3(n-1)}[(r-2)^2+1].$$

Therefore,

$$\text{rad}(G) - \pi(G) \leq r - \frac{\delta+1}{3(n-1)}[(r-2)^2+1].$$

Denote the right hand side of the last equation by f. Making use of Theorem 6 we obtain

$$\frac{\partial f}{\partial r} = 1 - \frac{2(\delta+1)}{3(n-1)}(r-2) \geq 1 - \frac{2(\delta+1)}{3(n-1)}\Big(\frac{3n}{2(\delta+1)}-1\Big) = \frac{2\delta-1}{3(n-1)} > 0.$$

Hence f is increasing in r. Substituting $r = \frac{3n}{2(\delta+1)}+1$ into f we obtain after straightforward calculations,

$$\text{rad}(G) - \pi(G) \leq \frac{3}{4(\delta+1)}n + \frac{8\delta+5}{4(\delta+1)} - \frac{(\delta+1)(8\delta+20)+9}{23(\delta+1)(n-1)},$$

and thus

$$\text{rad}(G) - \pi(G) \leq \frac{3}{4(\delta+1)}n + \frac{8\delta+5}{4(\delta+1)},$$

which is the desired bound.

To see that the bound in Theorem 7 is sharp apart from an additive constant consider the graph $G_{k,\delta}$ in Example 1 for even k. For these graphs we have for constant δ and large n,

$$\begin{aligned} \text{rad}(G_{k,\delta}) - \pi(G_{k,\delta}) &= \frac{3}{2(\delta+1)}n + \frac{\delta-2}{\delta+1} - \frac{3}{4(\delta+1)}n - \frac{3}{4(\delta+1)} + o(1) \\ &= \frac{3}{4(\delta+1)}n + \frac{4\delta-11}{4(\delta+1)} + o(1), \end{aligned}$$

as desired. □

From Corollary 1 and the inequality $\mathrm{rad}(G) \le \mathrm{diam}(G)$ we get the following lower bound on the remoteness in terms of radius.

Corollary 2. *Let G be a connected graph. Then*

$$\rho(G) > \frac{1}{2}\mathrm{rad}(G).$$

The coefficient $\frac{1}{2}$ cannot be improved, even for graphs of arbitrarily large minimum degree. To see this consider for given $r, b \in \mathbb{N}$ the strong product of a cycle of length $2r+1$ and a complete graph on b vertices. It is easy to verify that this graph has radius r and remoteness $\frac{r^2+r+1}{2r+1} + \frac{r(r-1)}{(2r+1)(n-1)}$, where n is the order of the graph, i.e., $n = (2r+1)b$. If r and b tend to infinity, then $\rho(G) - \frac{1}{2}\mathrm{rad}(G)$ tends to $-\frac{3}{4}$.

2.3. Proximity vs average distance

We present an upper bound on the average distance in terms of remoteness which is reminiscent of the well-known inequality $\mathrm{diam}(G) \le 2\mathrm{rad}(G)$. Although it is likely that this inequality is known, we were unable to find a reference for it.

Proposition 2. *(i) Let G be a connected graph. Then*

$$\mu(G) \le 2\pi(G).$$

(ii) There exists no $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$ such that $\mu(G) \le 2\pi(G) - \varepsilon$ for all connected graphs G.

Proof: (i) Let v be a median vertex of G, i.e., a vertex with $\overline{\sigma}(v) = \pi(G)$. Then

$$\begin{aligned}
\sigma(G) &= \sum_{x\in V}\sum_{y\in V} d(x,y) \\
&\le \sum_{x\in V}\sum_{y\in V} \big(d(x,v) + d(v,y)\big) \\
&= n\sum_{x\in V} d(x,v) + n\sum_{y\in V} d(y,v) \\
&= 2n\sigma(v).
\end{aligned}$$

Dividing by $n(n-1)$ yields

$$\mu(G) \leq \frac{2}{n-1}\sigma(v) = 2\pi(G),$$

and (i) follows.

(ii) Consider the trees T_k, $k \in \mathbb{N}$, $k \geq 2$, defined as follows. Let T_k be obtained by attaching k vertices $v_1, v_2, \ldots, v_k$ to a vertex v, and then attaching $k-1$ vertices to each v_i for $i = 1, 2, \ldots, k$. Then T_k has k^2+1 vertices, diameter four, and all its vertices have either degree k or degree 1. It is easy to verify that $\pi(T_k) = 2 + \frac{2-3\sqrt{n-1}}{n-1}$ and $\mu(T_k) = 4 - \frac{4\sqrt{n-1}+2}{n}$, where $n = k^2+1$. Hence, for large k,

$$2\pi(T_k) - \mu(T_k) = O(n^{-\frac{1}{2}}),$$

which implies that for every $\varepsilon > 0$ there exists a k_0 such that $2\pi(T_k) - \mu(T_k) > \varepsilon$ for all k with $k > k_0$. □

References

[1] M., Aouchiche, G. Caporossi, and P. Hansen. Variable Neighbourhood Search for Extremal Graphs, 20 Automated Comparison of Graph Invariants. *MATCH Commun. Math. Comput. Chem.* **58** (2007), 365-384.

[2] M. Aouchiche and P. Hansen. Nordhaus-Gaddum relations for proximity and remoteness in graphs. *Comput. Math. Appl.* **59**(no. 8) (2010), 2827-2835.

[3] M. Aouchiche, P. Hansen. Proximity and remoteness in graphs: results and conjectures. *Networks* **58**(no. 2) (2011), 95-102.

[4] C.A. Barefoot, R.C. Entringer, and L.A. Székely. Extremal values for ratios of distances in trees. *Discrete Appl. Math.* **80** (1997), 37-56.

[5] P. Dankelmann,G. Dlamini, and H.C. Swart. Uppper bounds on distance measures in $K_{2,l}$-free graphs. (manuscript)

[6] P. Dankelmann. Proximity, remoteness, and minimum degree. *Discrete Appl. Math.* **184** (2015), 223-228.

[7] G. Dlamini. Aspects of distances in graphs, *Ph.D. Thesis, University of Natal,* Durban, 2003.

[8] R.C. Entringer, D.E. Jackson, and D.A. Snyder. Distance in graphs. *Czechoslovak Math. J.* **26 (101)** no. 2 (1976), 283-296.

[9] P. Erdös, J. Pach,R. Pollack, and Z. Tuza. Radius, diameter, and minimum degree. *J. Combin. Theory Ser. B* **47** (1989) 73-79.

[10] B. Ma, B. Wu, and W. Zhang. Proximity and average eccentricity of a graph. *Inform. Process. Lett.* **112**(no. 10) (2012), 392-395.

[11] B. Wu and W. Zhang. Average distance, radius and remoteness of a graph. *Ars Math. Contemp.* **7** (2014), 441-452.

[12] B. Zelinka. Medians and peripherians of trees. *Arch. Math. (Brno)* **4** (1968), 87-95.