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The minus k-domination numbers in graphs

N. Dehgardi

Department of Mathematics and Computer Science, Sirjan University of Technology, Sirjan, I.R. Iran
n.dehgardi@sirjantech.ac.ir

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Abstract: For any integer $k \geq 1$, a minus k-dominating function is a function $f : V \to \{-1, 0, 1\}$ satisfying $\sum_{w \in N[v]} f(w) \geq k$ for every $v \in V(G)$, where $N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$. The minimum of the values of $\sum_{v \in V(G)} f(v)$, taken over all minus k-dominating functions f, is called the minus k-domination number and is denoted by $\gamma_k^-(G)$. In this paper, we introduce the study of minus k-domination in graphs and present several sharp lower bounds on the minus k-domination number for general graphs.

Keywords: Minus k-dominating function, Minus k-domination number.

2010 Mathematics Subject Classification: 05C12

1. Introduction

In this paper, all graphs are finite, simple, and undirected. Let G be a graph. We let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. The integers $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$ are the order and the size of the graph G, respectively. For a vertex $v \in V(G)$, the *open neighborhood* of v, denoted by $N_G(v) = N(v)$, is the set $\{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v, denoted by $N_G[v] = N[v]$, is the set $N_G(v) \cup \{v\}$. The *degree* of $v \in V(G)$, denoted by $d_G(v)$, is defined by $d_G(v) = |N_G(v)|$. The *m*inimum and *m*aximum degrees of G are denoted by $\delta(G) = \delta$ and $\Delta(G) = \Delta$, respectively. For a set S of vertices, we define the open neighborhood $N(S) = \bigcup_{v \in S} N(v)$, and the closed neighborhood $N[S] = N(S) \cup S$. If X and Y are sets of vertices of a graph G, we denote by $E(X, Y)$ the set of edges with one end in X and the other in Y. The *c*omplement of G is denoted by $\overline{G}$. We let P_n, C_n and K_n denote the *p*ath, the *c*ycle and the complete graph of order n, respectively. For a real-valued function $f : V(G) \longrightarrow \mathbb{R}$ the weight of f is $w(f) = \sum_{v\in V} f(v)$, and for $S \subseteq V$, we define $f(S) = \sum_{v\in S} f(v)$, so $w(f) = f(V)$. Consult [8, 13] for notation and terminology that are not defined here.

Let $k \geq 1$ be an integer and G be a graph of minimum degree at least $k - 1$. A function $f : V(G) \to \{-1, 1\}$ is called a *s*igned k-dominating function of G if $f(N_G[v]) \geq k$ for all $v \in V(G)$. The *s*igned k-domination number of G, denoted by $\gamma_{ks}(G)$, is the minimum weight of a signed k-dominating function of G. The concept of signed k-domination number has been introduced in [12]. This parameter has been extensively studied in the literature; see e.g. [1, 11, 12] and the references therein. This parameter has also been studied in [4]. In the special case $k = 1$, the signed 1-domination number is exactly the signed domination number [2, 3, 6].

A *m*inus k-dominating function (briefly MkDF) is a function of the form $f : V \to \{-1, 0, 1\}$ such that the sum of its function values over any closed neighborhood is at least k. The *m*inus k-domination number of a graph G is defined as

$$\gamma_k^-(G) = \min\{\omega(f) \mid f \text{ is a minus } k\text{-dominating function on } G\}.$$

As the assumption $\delta(G) \geq k - 1$ is clearly necessary, we always assume that when we discuss $\gamma_k^-(G)$, all graphs involved satisfy $\delta(G) \geq k - 1$ and thus $n(G) \geq k$. A minus k-dominating function $f : V(G) \to \{-1, 0, 1\}$ can be represented by the ordered partition (V_{-1}, V_1, V_2) of $V(G)$ where $V_i = V_i^f = \{v \in V(G) \mid f(v) = i\}$ for $i = -1, 0, 1$. In the special case $k = 1$, the minus 1-domination number is the usual minus domination number [5].

Clearly, a signed k-dominating function is a minus k-dominating function. Hence, the signed k-domination and the minus k-domination number of a graph are related as follows.

Observation 1. *For a graph G, $\gamma_k^-(G) \leq \gamma_{ks}(G)$.*

Our purpose in this paper is to initiate the study of minus k-domination number in graphs. In particular, we present some sharp bounds on the minus k-domination number in graphs and we determine this parameter for some classes of graphs.

We close this section by showing that the minus k-domination number can be arbitrarily small. For this purpose, we need the following observation proved by Henning [9].

Observation 2. *If k and n are integers with $k < n$, and n is even, then we can construct a k-regular graph on n vertices.*

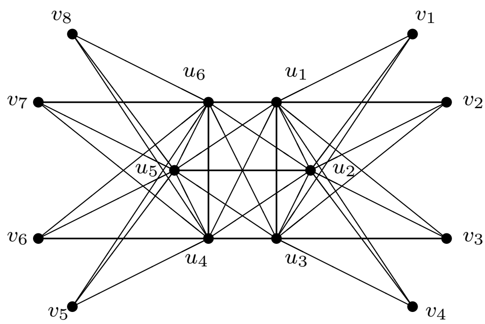

Figure 1. **Graph G with $\gamma_2^-(G) \leq -2$**

Proposition 1. *For an integer $k \geq 2$, there is a connected graph G such that $\gamma_k^-(G) \leq -k$.*

Proof. By Observation 2, there exists a $(2k+1)$-regular graph H on $n = k(k+1)$ vertices. Let $V(H) = \{u_1, u_2, \ldots, u_{k(k+1)}\}$ and G be the graph obtained from H as follows. The vertex set of G is $V(G) = V(H) \cup \{v_1, v_2, \ldots, v_{k(k+2)}\}$, and the edge set of G is

$$E(G) = E(H) \cup \{u_{i+s(k+1)}v_{j+s(k+2)} \mid 1 \leq i \leq k+1, 1 \leq j \leq k+2, 0 \leq s \leq k-1\}.$$

Now define $f : V(G) \to \{-1, 0, 1\}$ by $f(v) = 1$ if $v \in V(H)$, and $f(v) = -1$ otherwise. If $v \in V(H)$, then $f(N[v]) = 1 + 2k + 1 - k - 2 = k$ and if $v \in V(G) - V(H)$, then $f(N[v]) = -1 + k + 1 = k$. Thus f is a MkDF of G with weight

$$\omega(f) = k(k+1) - k(k+2) = -k.$$

The proof is complete. □

2. Bounds on the Minus k-Domination Number

In this section, we provide some sharp bounds on the minus k-domination number. We start with some preliminary results.

Observation 3. *Let G be a graph of order n with $\delta(G) \geq k-1$, and $f = (V_{-1}, V_0, V_1)$ be a $\gamma_k^-(G)$-function. Then*

1. $n = |V_{-1}| + |V_1| + |V_0|$.
2. $\omega(f) = |V_1| - |V_{-1}|$.

Theorem 1. *Let G be a connected graph of order n with maximum degree Δ and minimum degree δ. If f is a MkDF of G, then*

(a) $\frac{\Delta-k+1}{k}|V_1| \geq \frac{\delta+k+1}{k}|V_{-1}| + |V_0|$.

(b) $(\Delta+\delta+2)|V_1| + (\delta+1)|V_0| \geq (\delta+k+1)n$.

(c) $(\delta+1)\omega(f) \geq (\delta-\Delta)|V_1| + kn$.

(d) $\omega(f) \geq \frac{k-\Delta-1}{\Delta+\delta+2}n + |V_1|$.

Proof. (a) It follows from Observation 3-1 that

$$\begin{aligned} k(|V_1| + |V_{-1}| + |V_0|) &= nk \\ &\leq \sum_{v\in V} \sum_{x\in N[v]} f(x) = \sum_{v\in V} (d_G(v)+1)f(v) \\ &= \sum_{v\in V_1} (d_G(v)+1) - \sum_{v\in V_{-1}} (d_G(v)+1) \\ &\leq (\Delta+1)|V_1| - (\delta+1)|V_{-1}|. \end{aligned}$$

This inequality chain yields to the desired bound in (a).
(b) Using Observation 3-1, and Part (a), we arrive at (b).
(c) Applying Observation 3 and Part (b), we obtain Part (c) as follows

$$\omega(f) = 2|V_1| - n + |V_0|,$$

and

$$\begin{aligned} (\delta+1)\omega(f) &= (\delta+1)(2|V_1| - n + |V_0|) \\ &= (\Delta+\delta+2)|V_1| + (\delta-\Delta)|V_1| - (\delta+1)n + (\delta+1)|V_0| \\ &\geq (\delta-\Delta)|V_1| - (\delta+1)n + (\delta+k+1)n \\ &= (\delta-\Delta)|V_1| + kn. \end{aligned}$$

(d) The inequality chain in the proof of Part (a), and Observation 3-1 show that

$$\begin{aligned} nk &\leq (\Delta+1)|V_1 \cup V_0| - (\delta+1)(n - |V_1 \cup V_0|) \\ &= (\Delta+\delta+2)|V_1 \cup V_0| - (\delta+1)n, \end{aligned}$$

and thus

$$|V_1 \cup V_0| \geq \frac{\delta + k + 1}{\Delta + \delta + 2} n.$$

Using this inequality and Observation 3, we obtain

$$\begin{aligned} \omega(f) &= |V_1| - n + |V_1 \cup V_0| \geq \tfrac{\delta+k+1}{\Delta+\delta+2} n - n + |V_1| \\ &= \tfrac{k-\Delta-1}{\Delta+\delta+2} n + |V_1|. \end{aligned}$$

This is the bound in Part (d), and the proof is complete. □

Corollary 1. *If G is a connected graph of order n, then*

$$\gamma_k^-(G) \geq \frac{2k - \Delta + \delta}{\Delta + \delta + 2} n.$$

Proof. If G is an r-regular graph, then result is an immediate consequence of Theorem 1-(c). Hence let G be a non-regular graph. Multiplying both sides of the inequality in Theorem 1-(d) by $(\Delta - \delta)$ and adding it to the inequality in Theorem 1-(c), we obtain the desired lower bound. □

Theorem 2. *Let G be a graph of order n and t a non-negative integer. If $\delta(G) \geq k + t - 1$, then $\gamma_k^-(G) \leq n - t$.*

Proof. If $t = 0$, then the result is trivial. Let $t \geq 1$ and $A = \{u_1, u_2, \ldots, u_t\}$ be a set of vertices of G. Define the function $g : V(G) \to \{-1, 0, 1\}$ by $g(u_i) = 0$ for $1 \leq i \leq t$, and $g(x) = 1$ otherwise. Obviously, g is a MkDF on G of weight $n - t$ and $\gamma_k^-(G) \leq n - t$. □

Next result is an immediate consequence of Corollary 1 and Theorem 2.

Corollary 2. *For two positive integers $n \geq k$, $\gamma_k^-(K_n) = k$.*

Corollary 2 shows that the bound in Theorem 2 is sharp.

Theorem 3. *Let G be a graph of order n and size m. Then*

$$\gamma_k^-(G) \geq \frac{2k}{k+1} n - \frac{2}{k+1} m.$$

Proof. Let $f = (V_{-1}, V_0, V_1)$ be a $\gamma_k^-(G)$-function. Since for each $v \in V$, $f(N[v]) \geq k$, we have $|N(v) \cap V_1| \geq k + 1$ for every $v \in V_{-1}$, $|N(v) \cap V_1| \geq k$ for $v \in V_0$, and $|N(v) \cap V_{-1}| \leq |N(v) \cap V_1| - (k-1)$ for each $v \in V_1$. It follows that $|E(V_1, V_{-1})| \geq (k+1)|V_{-1}|$, $|E(V_1, V_0)| \geq k|V_0|$ and

$$|E(V_1, V_{-1})| \leq 2|E(V_1, V_1)| - (k-1)|V_1|.$$

Therefore

$$\begin{aligned} m &\geq |E(V_1, V_1)| + |E(V_1, V_{-1})| + |E(V_1, V_0)| \\ &\geq \frac{k-1}{2}|V_1| + \frac{k+1}{2}|V_{-1}| + (k+1)|V_{-1}| + k|V_0| \\ &\geq \frac{k-1}{2}n + \frac{k+1}{2}(2|V_{-1}| + |V_0|). \end{aligned}$$

Hence, we have

$$\gamma_k^-(G) = |V_1| - |V_{-1}| = n - (2|V_{-1}| + |V_0|) \geq n + \frac{k-1}{k+1}n - \frac{2}{k+1}m.$$

□

Proposition 2. *Let G be a graph of order n. Then $\gamma_k^-(G) = n$ if and only if for each vertex $v \in V(G)$ there is a vertex $u \in N[v]$ such that $d_G(u) \leq k-1$.*

Proof. One side is clear. Let $\gamma_k^-(G) = n$. If there is a vertex v such that $d_G(u) \geq k$ for each $u \in N[v]$, then the function $g : V(G) \to \{-1, 0, 1\}$ defined by $g(v) = 0$ and $g(x) = 1$ otherwise, is a MkDF on G of weight $n-1$, a contradiction. Thus for each vertex $v \in G$, there is a vertex $u \in N[v]$ such that $d_G(u) \leq k-1$, and the proof is complete. □

Dunbar et al. [5] showed that the minus domination number of a graph with maximum degree at most five is non-negative. Next proposition generalizes their result.

Proposition 3. *Let k be a positive integer and G be a graph with $\Delta \leq 3k+2$. Then $\gamma_k^-(G) \geq 0$.*

Proof. Let $f = (V_{-1}, V_0, V_1)$ be a $\gamma_k^-(G)$-function. If $V_{-1} = \varnothing$, then we are done. Assume $V_{-1} \neq \varnothing$. For each $v \in V_{-1}$, it follows from $f(N[v]) \geq k$ that $|N(v) \cap V_1| \geq k+1$. This implies that

$$|E(V_{-1}, V_1)| \geq (k+1)|V_{-1}|. \tag{1}$$

Similarly, for each $v \in V_1$, we have $|N(v) \cap V_1| \geq |N(v) \cap V_{-1}| + k - 1$, and

$$3k + 2 \geq d_G(v) \geq |N(v) \cap V_1| + |N(v) \cap V_{-1}| \geq 2|N(v) \cap V_{-1}| + k - 1.$$

Thus, $k + 1 \geq |N(v) \cap V_{-1}|$ for each $v \in V_1$, and

$$|E(V_{-1}, V_1)| \leq (k + 1)|V_1|. \tag{2}$$

Combining (1) and (2), we obtain $\gamma_k^-(G) = |V_1| - |V_{-1}| \geq 0$. □

Theorem 4. *Let G be a connected graph of order n and minimum degree of $\delta \geq k - 1$. Then*

$$\gamma_k^-(G) \geq -1 + \sqrt{1 + 4(k + 1)n} - n.$$

Proof. If $\gamma_k^-(G) = n$, the result is trivial. So we may assume that $\gamma_k^-(G) < n$. Let $f = (V_{-1}, V_0, V_1)$ be a $\gamma_k^-(G)$-function. Since $f(N[v]) \geq k$ for each $v \in V$, each vertex in V_{-1} has at least $k + 1$ neighbors in V_1. We conclude from the Pigeonhole Principle that at least one vertex $v \in V_1$ has at least $\lceil \frac{(k+1)|V_{-1}|}{|V_1|} \rceil$ neighbors in V_{-1}. It implies that

$$k \leq f(N[v]) = |N(v) \cap V_1| - |N(v) \cap V_{-1}| + 1 \leq (|V_1| - 1) - \lceil \frac{|V_{-1}|(k + 1)}{|V_1|} \rceil + 1,$$

and $|V_1|^2 - k|V_1| - |V_{-1}|(k + 1) \geq 0$. Hence, we have $|V_1|^2 + |V_1| + (k + 1)(|V_0| - n) \geq 0$. Thus

$$|V_1| \geq \frac{-1 + \sqrt{1 + 4(k + 1)(n - |V_0|)}}{2},$$

and

$$\gamma_k^-(G) = 2|V_1| + |V_0| - n \geq -1 + \sqrt{1 + 4(k + 1)(n - |V_0|)} + |V_0| - n.$$

Let $g(x) = -1 + \sqrt{1 + 4(k + 1)x} - x$. Then $g'(x) = \frac{2(k+1)}{\sqrt{1+4(k+1)x}} - 1$, thus $g'(x) < 0$ for $x \geq k + 1$. Hence $g(x)$ is a decreasing function when $x \geq k + 1$. Furthermore, since $|V_1| \geq k + 1$, we have $n - |V_0| = |V_1| + |V_{-1}| \geq k + 1$. Therefore, $g(n - |V_0|) \geq g(n)$. Consequently,

$$\gamma_k^-(G) \geq -1 + \sqrt{1 + 4(k + 1)(n - |V_0|)} + |V_0| - n \geq -1 + \sqrt{1 + 4(k + 1)n} - n.$$

□

A set $S \subseteq V(G)$ is a 2-packing of G if $N[u] \cap N[v] = \varnothing$ for any two distinct vertices $u, v \in S$. The 2-packing number $\rho(G)$ of G is defined as

$$\rho(G) = \max\{|S| \mid S \text{ is a 2-packing of } G\}.$$

In the sequel, we present some bounds on the minus k-domination number in terms of the order and the 2-packing number.

Theorem 5. *Let k be an integer, and G be a connected graph of order n with $\delta(G) \geq k+1$. Then the following hold.*

(a) $\gamma_k^-(G) \leq n - 2\rho(G)$.

(b) $\gamma_{k+1}^-(G) \leq n - \rho(G)$.

(c) *If $V(G) = \cup_{s \in S} N[s]$, then $k\rho(G) \leq \gamma_k^-(G)$.*

Proof. Let S be a 2-packing of G. To prove (a), define the function $f : V(G) \to \{-1, 0, 1\}$ by $f(x) = -1$ for $x \in S$ and $f(x) = 1$ otherwise. Clearly f is a minus k-dominating function of G of weight $n - 2\rho(G)$, which implies $\gamma_k^-(G) \leq n - 2\rho(G)$. To prove (b), define the function $f : V(G) \to \{-1, 0, 1\}$ by $f(x) = 0$ for $x \in S$, and $f(x) = 1$ otherwise. Obviously f is a minus $(k+1)$-dominating function of G of weight $n - \rho(G)$, which implies $\gamma_{k+1}^-(G) \leq n - \rho(G)$. Now we prove (c). Let f be a $\gamma_k^-(G)$-function. By definition, we have

$$\gamma_k^-(G) = \sum_{s \in S} f(N[s]) \geq k\rho(G).$$

□

Corollary 3. *For $n \geq 3$, $\gamma_2^-(C_n) = \lceil \frac{2n}{3} \rceil$.*

Proof. Since $\gamma_2^-(C_n)$ is an integer, it follows from Corollary 1 that $\gamma_2^-(C_n) \geq \lceil \frac{2n}{3} \rceil$. On the other hand, since $\rho(C_n) = \lfloor \frac{n}{3} \rfloor$, we conclude from Theorem 5 that $\gamma_2^-(C_n) \leq n - \rho(C_n) = \lceil \frac{2n}{3} \rceil$, and $\gamma_2^-(C_n) = \lceil \frac{2n}{3} \rceil$. □

Applying Theorem 5 and the following result due to Favaron [7], we obtain bounds on the cubic graphs.

Proposition 4. *[7] If G is a connected cubic graph G of order n, then $\rho(G) \geq n/8$, unless G is the Petersen graph which in this case $\rho(G) = (n-2)/8 = 1$.*

Corollary 4. *If G is a connected cubic graph of order n different from the Petersen graph, then*

(i) $\frac{n}{2} \leq \gamma_2^-(G) \leq \frac{3n}{4}$.

(ii) $\frac{3n}{4} \leq \gamma_3^-(G) \leq \frac{7n}{8}$, *the upper bound satisfies if G is not the Petersen graph which in this case* $\gamma_3^-(G) = 9$.

Now we present a so called Nordhaus-Gaddum type inequality for the minus k-domination number of regular graphs.

Theorem 6. *Let G be an r-regular graph of order n where $r \geq k-1$ and $n-r \geq k$. Then*

$$\gamma_k^-(G) + \gamma_k^-(\overline{G}) \geq \begin{cases} \frac{4kn}{n+1} & \text{if} \quad n \text{ is odd} \\ \frac{4k(n+1)}{n+2} & \text{if} \quad n \text{ is even.} \end{cases}$$

Proof. Since G is r-regular, the complement $\overline{G}$ is $(n-r-1)$-regular. It follows from Corollary 1 that

$$\gamma_k^-(G) + \gamma_k^-(\overline{G}) \geq kn(\frac{1}{r+1} + \frac{1}{n-r}).$$

The conditions $r \geq k-1$ and $n-r \geq k$ imply that $k-1 \leq r \leq n-k$. Consider the function $f(x) = \frac{1}{x+1} + \frac{1}{n-x}$ on the interval $[k-1, n-k] \cap \mathbb{Z}$. If n is odd, then the function f gets its minimum at $x = \frac{n-1}{2}$, and we have

$$\gamma_k^-(G) + \gamma_k^-(\overline{G}) \geq kn(\frac{1}{r+1} + \frac{1}{n-r}) \geq kn(\frac{2}{n+1} + \frac{2}{n+1}) = \frac{4kn}{n+1}.$$

If n is even, then the function f gets its minimum at $r = x = \frac{n-2}{2}$ or $r = x = \frac{n}{2}$, since r is an integer. This implies that

$$\gamma_k^-(G) + \gamma_k^-(\overline{G}) \geq kn(\frac{1}{r+1} + \frac{1}{n-r}) \geq kn(\frac{2}{n} + \frac{2}{n+2}) = \frac{4k(n+1)}{n+2},$$

and the proof is complete. □

3. t-Partite Graphs

In this section, we present a lower bound on the minus k-domination number of t-partite graphs. The proof of the following result can be found in [10].

Proposition 5. *For non-negative integers $p_1, p_2, \ldots, p_t$ $(t \geq 2)$,*

$$\sqrt{(2+\frac{2}{t-1})\sum_{i=1}^{t-1}\sum_{j=i+1}^{t} p_i p_j} \leq \sum_{i=1}^{t} p_i.$$

The proof of the next result is essentially similar to the proof of Theorem 2 in [10].

Theorem 7. *Let $k \geq 1$, $t \geq 2$ and $G = (V, E)$ be a t-partite graph of order $n \geq \frac{t(k+1)}{t-1}$ with partite sets $X_1, X_2, \ldots, X_t$. Then*

$$\gamma_k^-(G) \geq \frac{(k+1)t}{(t-1)}\left(-1+\sqrt{\frac{4}{(k+1)^2}+\frac{4(t-1)}{t(k+1)}n}\right) - n + \frac{(k-1)t}{t-1}(1-\frac{k-1}{2(k+1)}).$$

Proof. Let f be a $\gamma_k^-(G)$-function and

$P_i = X_i \cap V_1, M_i = X_i \cap V_{-1}, Q_i = X_i \cap V_0, p_i = |P_i|, m_i = |M_i|, q_i = |Q_i|, i = 1, \ldots, t.$

Thus

$$n = \sum_{i=1}^{t} p_i + \sum_{i=1}^{t} m_i + \sum_{i=1}^{t} q_i. \tag{3}$$

Since for each $v \in V$, $f(N[v]) \geq k$, we have $|N(v) \cap V_1| \geq k+1$ for every $v \in M_i$, $i = 1, \ldots, t$. So

$$|E(V_1, V_{-1})| \geq (k+1)\sum_{i=1}^{t} m_i. \tag{4}$$

For each $v \in P_i$, we have $|N(v) \cap V_{-1}| \leq |N(v) \cap V_1| - (k-1)$. Thus

$$\begin{aligned}
|E(V_1, V_{-1})| &= \sum_{i=1}^{t}\sum_{v\in P_i} |N(v) \cap V_{-1}| \\
&\leq \sum_{i=1}^{t}\sum_{v\in P_i} (|N(v) \cap V_1| - (k-1)) \\
&\leq \sum_{i=1}^{t} p_i(\sum_{j=1, j\neq i}^{t} p_j) - (k-1)\sum_{i=1}^{t} p_i \\
&= 2\sum_{i=1}^{t-1}\sum_{j=i+1}^{t} p_i p_j - (k-1)\sum_{i=1}^{t} p_i \\
&\leq 2\sum_{i=1}^{t-1}\sum_{j=i+1}^{t} p_i p_j - (k-1)\sqrt{(2+\frac{2}{t-1})\sum_{i=1}^{t-1}\sum_{j=i+1}^{t} p_i p_j}. \qquad (5)
\end{aligned}$$

Combining (4) and (5), we have

$$(k+1)\sum_{i=1}^{t} m_i \leq 2\sum_{i=1}^{t-1}\sum_{j=i+1}^{t} p_i p_j - (k-1)\sqrt{(2+\frac{2}{t-1})\sum_{i=1}^{t-1}\sum_{j=i+1}^{t} p_i p_j}. \tag{6}$$

Define the function

$$f(x) = 2x^2 - (k-1)\sqrt{(2+\frac{2}{t-1})}x - (k+1)\sum_{i=1}^{t} m_i,$$

where $f(x) \geq 0$ and $x = \sqrt{\sum_{i=1}^{t-1}\sum_{j=i+1}^{t} p_i p_j}$. We have

$$x \geq \frac{(k-1)\sqrt{(2+\frac{2}{t-1})} + \sqrt{(k-1)^2(2+\frac{2}{t-1}) + 8(k+1)\sum_{i=1}^{t} m_i}}{4},$$

and

$$\begin{aligned}\sqrt{(2+\frac{2}{t-1})\sum_{i=1}^{t-1}\sum_{j=i+1}^{t} p_i p_j} \;&\geq\; \frac{t(k-1)}{2(t-1)} \\ &+\sqrt{(k-1)^2(\frac{t}{2(t-1)})^2 + (k+1)\frac{t}{t-1}\sum_{i=1}^{t} m_i} \\ &= \frac{t}{2(t-1)}\left(k-1+\sqrt{(k-1)^2+(k+1)\frac{4(t-1)}{t}\sum_{i=1}^{t} m_i}\right). \end{aligned} \tag{7}$$

By (3) and Proposition 5, we obtain

$$\sqrt{(2+\frac{2}{t-1})\sum_{i=1}^{t-1}\sum_{j=i+1}^{t} p_i p_j} + \sum_{i=1}^{t} m_i + \sum_{i=1}^{t} q_i \leq n. \tag{8}$$

Using (7) and (8), we have

$$\frac{t}{2(t-1)}\left(k-1+\sqrt{(k-1)^2+(k+1)\frac{4(t-1)}{t}\sum_{i=1}^{t} m_i}\right) + \sum_{i=1}^{t} m_i + \sum_{i=1}^{t} q_i \leq n, \tag{9}$$

and

$$\frac{t}{2(t-1)}\left(k-1+\sqrt{(k-1)^2+(k+1)\frac{4(t-1)}{t}\sum_{i=1}^{t}m_i}\right)+\sum_{i=1}^{t}m_i\leq n. \quad (10)$$

Since $\gamma_k^-(G)=n-2\sum_{i=1}^{t}m_i-\sum_{i=1}^{t}q_i$, using (9) we have

$$\gamma_k^-(G)\geq\frac{t}{2(t-1)}\left(k-1+\sqrt{(k-1)^2+(k+1)\frac{4(t-1)}{t}\sum_{i=1}^{t}m_i}\right)$$
$$-\sum_{i=1}^{t}m_i. \quad (11)$$

For notational convenience, we write

$$a=\frac{t(k+1)}{2(t-1)}\sqrt{(\frac{k-1}{k+1})^2+\frac{4(t-1)}{t(k+1)}\sum_{i=1}^{t}m_i},$$

and define two functions

$$h(y)=\frac{t-1}{(k+1)t}y^2+y+\frac{(k-1)t}{2(t-1)}-\frac{(k-1)^2t}{4(t-1)(k+1)},\ y\geq\frac{t(k-1)}{2(t-1)},$$

and

$$g(y)=\begin{cases}-\frac{t-1}{(k+1)t}y^2+y+\frac{(k-1)t}{2(t-1)}-\frac{(k-1)^2t}{4(t-1)(k+1)} & y\geq\frac{t(k+1)}{2(t-1)}\\ k & \frac{t(k+1)}{2(t-1)}>y\geq\frac{t(k-1)}{2(t-1)}.\end{cases}$$

Since $\frac{\mathrm{d}g}{\mathrm{d}y}\leq 0$ and $\frac{\mathrm{d}h}{\mathrm{d}y}>0$, $g(y)$ is monotonously decreasing for $y\geq\frac{t(k+1)}{2(t-1)}$ and $h(y)$ is a monotonous increasing function if $y\geq\frac{t(k-1)}{2(t-1)}$. By (10), we obtain

$$h(a)=\frac{t-1}{(k+1)t}a^2+a+\frac{(k-1)t}{2(t-1)}-\frac{(k-1)^2t}{4(t-1)(k+1)}\leq n.$$

Furthermore, we note that when

$$y_0=\frac{(k+1)t}{2(t-1)}\left(-1+\sqrt{\frac{4}{(k+1)^2}+\frac{4(t-1)}{t(k+1)}n}\right)$$

it follows that $h(y_0) = n \geq \max\{h(\frac{t(k+1)}{2(t-1)}), h(a)\}$. Thus

$$\max\{\frac{t(k+1)}{2(t-1)}, a\} \leq \frac{(k+1)t}{2(t-1)}\left(-1+\sqrt{\frac{4}{(k+1)^2}+\frac{4(t-1)}{t(k+1)}n}\right).$$

If $|V_{-1}| = 0$, then $a = \frac{t(k-1)}{2(t-1)}$. We show that $\gamma_k^-(G) \geq g(\frac{t(k+1)}{2(t-1)})$. Let $\gamma_k^-(G) = k$. If $t = k$, then $n > k+1$ and this is a contradiction.Hence $t \geq k+1$ and $\gamma_k^-(G) = k \geq g(\frac{t(k+1)}{2(t-1)})$. If $\gamma_k^-(G) \geq k+1$, then $\gamma_k^-(G) \geq k+1 \geq g(\frac{t(k+1)}{2(t-1)})$. Now let $|V_{-1}| \geq 1$. By the monotonicity of $g(y)$ and (11),

$$\begin{aligned}\gamma_k^-(G) \geq g(a) \geq g(y_0) &= \frac{(k+1)t}{(t-1)}\left(-1+\sqrt{\frac{4}{(k+1)^2}+\frac{4(t-1)}{t(k+1)}n}\right) - n \\ &\quad + \frac{(k-1)t}{t-1}(1-\frac{k-1}{2(k+1)}).\end{aligned}$$

The proof is complete. □

Corollary 5. *If G is a bipartite graph of order $n \geq 2(k+1)$, then*

$$\gamma_k^-(G) \geq -k-3+4\sqrt{1+\frac{k+1}{2}n}-n-\frac{(k-1)^2}{k+1}.$$

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