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Sufficient conditions on the zeroth-order general Randić index for maximally edge-connected digraphs

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Abstract: Let D be a finite and simple digraph with vertex set $V(D)$. For a vertex $v \in V(D)$, the degree of v, denoted by $d(v)$, is defined as the minimum value of its out-degree $d^+(v)$ and its in-degree $d^-(v)$. Now let D be a digraph with minimum degree $\delta \geq 1$ and edge-connectivity λ. If α is real number, then, analogously to graphs, we define the zeroth-order general Randić index by $\sum_{x \in V(D)}(d(x))^\alpha$. A digraph is maximally edge-connected if $\lambda = \delta$. In this paper, we present sufficient conditions for digraphs to be maximally edge-connected in terms of the zeroth-order general Randić index, the order and the minimum degree when $\alpha < 0$, $0 < \alpha < 1$ or $1 < \alpha \leq 2$. Using the associated digraph of a graph, we show that our results include some corresponding known results on graphs.

Keywords: Digraphs, Edge-connectivity, Maximal edge-connected digraphs, zeroth-order general Randić index

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1. Terminology and introduction

Let G be a finite and simple graph with vertex set $V(G)$. The *order* of G is defined by $n = n(G) = |V(G)|$. If $N(v) = N_G(v)$ is the neighborhood of the vertex $v \in V(G)$, then we denote by $d(v) = |N(v)|$ the *degree* of v and by $\delta = \delta(G)$ the *minimum degree* of the graph G. An *edge-cut* of a connected graph G is a set of edges whose removal disconnects G. The *edge connectivity* $\lambda = \lambda(G)$ of a connected graph G is defined as the minimum cardinality of an

edge-cut over all edge-cuts of G. The inequality $\lambda(G) \le \delta(G)$ is immediate. We call a connected graph *maximally edge-connected*, if $\lambda(G) = \delta(G)$.
The *zeroth-order general Randić index* is defined for a connected graph G of order $n \ge 2$ by

$$R^0_\alpha(G) = \sum_{v \in V(G)} (d(v))^\alpha,$$

where α is any real number. In 2005, Li and Zheng [8] proposed this index and named it *first general Zagreb index*. But nowadays, most authors refer to it as to the zeroth-order general Randić index. At this point it is worth mentioning that R^0_2 and $R^0_{-0,5}$ correspond to the first Zagreb index, introduced by Gutman and Trinajstić [5], and zeroth-order Randić index, defined by Kier and Hall [7], respectively. The special case $\alpha = -1$ is known as the *inverse degree*. The inverse degree first attracted attention through conjectures of the computer program Graffiti [3]. In [2], the authors present sufficient conditions for connected graphs to be maximally edge-connected in terms of the inverse degree, the order and the minimum degree.
In this paper, we are concerned with the *zeroth-order general Randić index* for digraphs. Let D be a finite and simple digraph with vertex set $V(D)$. For any vertex v of a digraph D, we denote the set of *out-neighbors* and *in-neighbors* of v be $N^+(v) = N^+_D(v)$ and $N^-(v) = N^-_D(v)$, respectively. For a vertex $v \in V(D)$, the degree of v, denoted by $d(v)$, is defined as the minimum value of its out-degree $d^+(v) = |N^+(v)|$ and its in-degree $d^-(v) = |N^-(v)|$. The *minimum out-degree* and *minimum in-degree* of a digraph D are denoted by $\delta^+(D)$ and $\delta^-(D)$. In addition, let $\delta = \delta(D) = \min\{\delta^+(D), \delta^-(D)\}$ be the *minimum degree* of D. If X and Y are two subsets of $V(D)$, then we denote by (X, Y) the set of arcs with tail in X and head in Y. We write K^*_n for the *complete digraph* of order n. A digraph is *strongly connected* or simply *strong* if for every pair u, v of distinct vertices there exists a directed path from u to v. A digraph D is *k-edge-connected* if for any set S of at most $k-1$ arcs the subdigraph $D - S$ is strong. The *edge-connectivity* $\lambda = \lambda(D)$ of a digraph D is defined as the largest value k such that D is k-edge-connected. The inequality $\lambda(D) \le \delta(D)$ is immediate. We call a digraph D *maximally edge-connected*, if $\lambda(D) = \delta(D)$. Sufficient conditions for graphs or digraphs to be maximally edge-connected were given by several authors, see for example the survey paper by Hellwig and Volkmann [6]. The *associated digraph* $D(G)$ of a graph G is obtained by replacing each edge of G by a pair of mutually opposite oriented arcs. The following observation is simple but useful.

Observation 1. If G is a graph and $D(G)$ its associated digraph, then $\lambda(G) = \lambda(D(G))$.

Now let D be a digraph with minimimum degree $\delta \geq 1$. If α is real number, then, analogously to graphs, we define the *zeroth-order general Randić index* of D by

$$R_\alpha^0(D) = \sum_{x \in V(D)} (d(x))^\alpha.$$

Inspired by the results in [2, 10, 11], we give in this paper sufficient conditions for strongly connected digraphs to be maximally edge-connected in terms of the zeroth-order general Randić index, the order and the minimum degree when $\alpha < 0$, $0 < \alpha < 1$ or $1 < \alpha \leq 2$. Examples will show that these conditions are best possible. Using Observation 1, we show that our results include some corresponding known results on graphs.

2. Preliminary results

In this section we present some basic lemmas, which we use in the proof of our main results. The first one is easy to prove and can be found in [9].

Lemma 1. If $x - 2 \geq y \geq 1$ and $t < 0$ or $t > 1$, then

$$(x-1)^t + (y+1)^t < x^t + y^t.$$

Lemma 2. Let $\alpha < 0$ or $1 < \alpha$ be a real number, and let $a_1, a_2, \ldots, a_p$ and A be positive reals such that $\sum_{i=1}^p a_i \leq A$. If in addition, $a_1, a_2, \ldots, a_p, A$ are positive integers, and a, b are integers with $A = ap + b$ and $0 \leq b < p$, then

$$\sum_{i=1}^p a_i^\alpha \geq (p-b)a^\alpha + b(a+1)^\alpha.$$

Proof. We can assume that the a_i are chosen such that $\sum_{i=1}^p a_i^\alpha$ is minimum. If no of the a_i differ by more than 1, then $p - b$ of the a_i are equal to a and the remaining b of the a_i are equal to $a + 1$. I this case the desired inequality is immediate. So assume that two of the a_i, say a_1 and a_2, differ by more than 1. Assume, without loss of generality, that $a_1 > a_2$. Let $b_1 = a_1 - 1$, $b_2 = a_2 + 1$ and $b_i = a_i$ for $i \geq 3$. Then Lemma 1 implies

$$\sum_{i=1}^p b_i^\alpha - \sum_{i=1}^p a_i^\alpha = (a_1 - 1)^\alpha + (a_2 + 1)^\alpha - a_1^\alpha - a_2^\alpha < 0,$$

a contradiction to the choice of the a_i. □

The next one can be found in [10].

Lemma 3. Let $0 < \alpha < 1$ be a real number, and let $a_1, a_2, \ldots, a_p$ and A be positive reals such that $\sum_{i=1}^{p} a_i \leq A$. If in addition, $a_1, a_2, \ldots, a_p, A$ are positive integers, and a, b are integers with $A = ap + b$ and $0 \leq b < p$, then

$$\sum_{i=1}^{p} a_i^\alpha \leq (p-b)a^\alpha + b(a+1)^\alpha.$$

The next two lemmas follow from the definitions of convex and concave functions and can be found in [2, 10].

Lemma 4. Let f be a convex function on an interval $[L, R]$. If $\ell, r \in [L, R]$ with $\ell + r = L + R$, then

$$f(L) + f(R) \geq f(\ell) + f(r).$$

Lemma 5. Let f be a concave function on an interval $[L, R]$. If $\ell, r \in [L, R]$ with $\ell + r = L + R$, then

$$f(L) + f(R) \leq f(\ell) + f(r).$$

3. Main results

Theorem 1. Let D be a strongly connected digraph of order $n \geq 3$, minimum degree δ and edge-connectivity λ, and let α be a real number. If

$$\begin{aligned} R_\alpha^0(D) \; < \; & 2\delta^\alpha + \delta^{\alpha+1} + (\delta-1)(n-\delta-1)^\alpha \\ & +(\delta-1)(\delta+1)^\alpha - (\delta-2)(n-\delta-2)^\alpha \end{aligned}$$

for $-1 \leq \alpha < 0$, then $\lambda = \delta$. If

$$\begin{aligned} R_\alpha^0(D) \; < \; & 2\delta^\alpha - \delta^{\alpha+1} + 2(n-\delta-2)^{\alpha+1} + (\delta-1)(n-\delta-1)^\alpha \\ & +(\delta-1)(\delta+1)^\alpha - (\delta-2)(n-\delta-2)^\alpha \end{aligned}$$

for $\alpha \leq -1$, then $\lambda = \delta$. If

$$\begin{aligned} R_\alpha^0(D) \; < \; & 3\delta^\alpha + \delta^{\alpha+1} + (\delta-1)(n-\delta-1)^\alpha \\ & +(\delta-1)(\delta+1)^\alpha - (\delta-1)(n-\delta-2)^\alpha \end{aligned}$$

for $1 < \alpha \leq 2$, then $\lambda = \delta$.

Proof. If $\delta = 1$, then $\lambda = \delta$ in every case. Thus assume in the following that $\delta \geq 2$. Suppose to the contrary that $\lambda \leq \delta - 1$. Then there exist two disjoint sets $X, Y \subset V(D)$ such that $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$. We first show that $\delta+1 \le |X|, |Y| \le n-\delta-1$. Suppose that X contains at most δ vertices. Since every vertex in X has at most $|X|-1$ out-neighbors in X and there are at most λ arcs from X to Y, we obtain the contradiction

$$\delta|X| \le \sum_{x\in X} d^+(x) \le |X|(|X|-1)+\lambda \le \delta(|X|-1)+\delta-1 = \delta|X|-1.$$

Therefore $|X| \ge \delta+1$. Similarly one can show that $|Y| \ge \delta+1$.
The digraph D contains a vertex v of minimum degree. Assume, without loss of generality, that $v \in X$. As above, we see that

$$\sum_{y\in Y} d(y) \le \sum_{y\in Y} d^-(y) \le |Y|(|Y|-1)+\lambda.$$

Applying Lemma 2, we deduce that

$$\begin{aligned}\sum_{y\in Y}(d(y))^\alpha &\ge (|Y|-\lambda)(|Y|-1)^\alpha + \lambda|Y|^\alpha \\ &= (|Y|-1)^\alpha + (|Y|-1)^{\alpha+1} - \lambda[(|Y|-1)^\alpha - |Y|^\alpha].\end{aligned}$$

Analogously, we observe that

$$\sum_{x\in X-\{v\}} d(x) \le \sum_{x\in X-\{v\}} d^+(x) \le (|X|-1)^2+\lambda.$$

In view of Lemma 2, we conclude that

$$\begin{aligned}\sum_{x\in X}(d(x))^\alpha &\ge \delta^\alpha + (|X|-\lambda-1)(|X|-1)^\alpha + \lambda|X|^\alpha \\ &= \delta^\alpha + (|X|-1)^{\alpha+1} - \lambda[(|X|-1)^\alpha - |X|^\alpha].\end{aligned}$$

Adding the inequalities above, we obtain

$$\begin{aligned}R_\alpha^0(D) &= \sum_{y\in Y}(d(y))^\alpha + \sum_{x\in X}(d(x))^\alpha \\ &\ge \delta^\alpha + (|Y|-1)^\alpha + (|Y|-1)^{\alpha+1} + (|X|-1)^{\alpha+1} \\ &\quad -\lambda[(|Y|-1)^\alpha - |Y|^\alpha + (|X|-1)^\alpha - |X|^\alpha]. \end{aligned} \tag{1}$$

If $-1 \le \alpha < 0$, then $(|X|-1)^{\alpha+1}, (|Y|-1)^{\alpha+1} \ge \delta^{\alpha+1}$ and $(|Y|-1)^\alpha \ge (n-\delta-2)^\alpha$ and therefore it follows from (1) that

$$\begin{aligned}R_\alpha^0(D) &\ge \delta^\alpha + (n-\delta-2)^\alpha + 2\delta^{\alpha+1} \\ &\quad -\lambda[(|Y|-1)^\alpha - |Y|^\alpha + (|X|-1)^\alpha - |X|^\alpha]. \end{aligned} \tag{2}$$

If $\alpha \leq -1$, then $(|X|-1)^{\alpha+1}, (|Y|-1)^{\alpha+1} \geq (n-\delta-2)^{\alpha+1}$ and $(|Y|-1)^{\alpha} \geq (n-\delta-2)^{\alpha}$ and so (1) leads to

$$\begin{aligned} R_\alpha^0(D) \ &\geq \ \delta^\alpha + (n-\delta-2)^\alpha + 2(n-\delta-2)^{\alpha+1} \\ &\qquad -\lambda[(|Y|-1)^\alpha - |Y|^\alpha + (|X|-1)^\alpha - |X|^\alpha]. \end{aligned} \tag{3}$$

If $1 < \alpha \leq 2$, then $(|X|-1)^{\alpha+1}, (|Y|-1)^{\alpha+1} \geq \delta^{\alpha+1}$ and $(|Y|-1)^{\alpha} \geq \delta^{\alpha}$ and thus (1) yields

$$R_\alpha^0(D) \geq 2\delta^\alpha + 2\delta^{\alpha+1} - \lambda[(|Y|-1)^\alpha - |Y|^\alpha + (|X|-1)^\alpha - |X|^\alpha]. \tag{4}$$

To minimize the right hand side of the inequalities (2, 3) or (4), consider the function $g(t) = (t-1)^\alpha - t^\alpha$ for $t > 1$. It is easy to verify that $g''(t) > 0$, and so g is convex when $\alpha < 0$ or $1 < \alpha \leq 2$. Because of $\delta + 1 \leq |X|, |Y| \leq n - \delta - 1$, $|X| + |Y| = n$ and Lemma 4 applied to the function g, we obtain

$$(|Y|-1)^\alpha - |Y|^\alpha + (|X|-1)^\alpha - |X|^\alpha \leq \delta^\alpha - (\delta+1)^\alpha + (n-\delta-2)^\alpha - (n-\delta-1)^\alpha.$$

Applying this to (2) if $-1 \leq \alpha < 0$, in conjunction with $\lambda \leq \delta - 1$, we deduce that

$$\begin{aligned} R_\alpha^0(D) \ &\geq \ \delta^\alpha + (n-\delta-2)^\alpha + 2\delta^{\alpha+1} \\ &\qquad -(\delta-1)[\delta^\alpha - (\delta+1)^\alpha + (n-\delta-2)^\alpha - (n-\delta-1)^\alpha] \\ &= \ 2\delta^\alpha + \delta^{\alpha+1} + (\delta-1)(n-\delta-1)^\alpha \\ &\qquad +(\delta-1)(\delta+1)^\alpha - (\delta-2)(n-\delta-2)^\alpha, \end{aligned}$$

a contradiction to the hypothesis. Applying this to (3) if $\alpha \leq -1$, in conjunction with $\lambda \leq \delta - 1$, we conclude that

$$\begin{aligned} R_\alpha^0(D) \ &\geq \ \delta^\alpha + (n-\delta-2)^\alpha + 2(n-\delta-2)^{\alpha+1} \\ &\qquad -(\delta-1)[\delta^\alpha - (\delta+1)^\alpha + (n-\delta-2)^\alpha - (n-\delta-1)^\alpha] \\ &= \ 2\delta^\alpha - \delta^{\alpha+1} + 2(n-\delta-2)^{\alpha+1} + (\delta-1)(n-\delta-1)^\alpha \\ &\qquad +(\delta-1)(\delta+1)^\alpha - (\delta-2)(n-\delta-2)^\alpha, \end{aligned}$$

a contradiction to the hypothesis. Applying this to (4) if $1 < \alpha \leq 2$, in conjunction with $\lambda \leq \delta - 1$, we have

$$\begin{aligned} R_\alpha^0(D) \ &\geq \ 2\delta^\alpha + 2\delta^{\alpha+1} - (\delta-1)[\delta^\alpha - (\delta+1)^\alpha + (n-\delta-2)^\alpha - (n-\delta-1)^\alpha] \\ &= \ 3\delta^\alpha + \delta^{\alpha+1} + (\delta-1)(n-\delta-1)^\alpha \\ &\qquad +(\delta-1)(\delta+1)^\alpha - (\delta-1)(n-\delta-2)^\alpha, \end{aligned}$$

a contradiction to the hypothesis. Therefore $\lambda = \delta$ in all cases. □

The next example will show that the sufficient conditions in Theorem 1 are best possible.

Example 1. Let n and δ be integers such that $n = 2\delta + 2 \geq 6$. Furthermore, let $H_1 = K^*_{\delta+1}$ with vertex set $V(H_1) = \{x_1, x_2, \ldots, x_{\delta+1}\}$, and let $H_2 = K^*_{\delta+1}$ with vertex set $V(H_2) = \{y_1, y_2, \ldots, y_{\delta+1}\}$. Define the digraph H by the union of H_1 and H_2 together with the $2\delta - 2$ arcs $x_1y_1, x_2y_2, \ldots, x_{\delta-1}y_{\delta-1}$ as well as $y_1x_1, y_2x_2, \ldots, y_{\delta-1}x_{\delta-1}$. Then $n(H) = n$, $\delta(H) = \delta$ and

$$R^0_\alpha(H) = 4\delta^\alpha + (2\delta - 2)(\delta + 1)^\alpha.$$

Therefore

$$\begin{aligned} R^0_\alpha(H) &= 2\delta^\alpha + \delta^{\alpha+1} + (\delta - 1)(n - \delta - 1)^\alpha \\ &\quad +(\delta - 1)(\delta + 1)^\alpha - (\delta - 2)(n - \delta - 2)^\alpha, \end{aligned}$$

when $-1 \leq \alpha < 0$,

$$\begin{aligned} R^0_\alpha(H) &= 2\delta^\alpha - \delta^{\alpha+1} + 2(n - \delta - 2)^{\alpha+1} + (\delta - 1)(n - \delta - 1)^\alpha \\ &\quad +(\delta - 1)(\delta + 1)^\alpha - (\delta - 2)(n - \delta - 2)^\alpha, \end{aligned}$$

when $\alpha \leq -1$ and

$$\begin{aligned} R^0_\alpha(H) &= 3\delta^\alpha + \delta^{\alpha+1} + (\delta - 1)(n - \delta - 1)^\alpha \\ &\quad +(\delta - 1)(\delta + 1)^\alpha - (\delta - 1)(n - \delta - 2)^\alpha, \end{aligned}$$

when $1 < \alpha \leq 2$. But it is easy to see that $\lambda(H) = \delta(H) - 1$.

Corollary 1. Let G be a connected graph of order $n \geq 3$, minimum degree δ and edge-connectivity λ, and let α be a real number. If

$$\begin{aligned} R^0_\alpha(G) &< 2\delta^\alpha + \delta^{\alpha+1} + (\delta - 1)(n - \delta - 1)^\alpha \\ &\quad +(\delta - 1)(\delta + 1)^\alpha - (\delta - 2)(n - \delta - 2)^\alpha, \end{aligned}$$

for $-1 \leq \alpha < 0$, then $\lambda = \delta$. If

$$\begin{aligned} R^0_\alpha(G) &< 2\delta^\alpha - \delta^{\alpha+1} + 2(n - \delta - 2)^{\alpha+1} + (\delta - 1)(n - \delta - 1)^\alpha \\ &\quad +(\delta - 1)(\delta + 1)^\alpha - (\delta - 2)(n - \delta - 2)^\alpha, \end{aligned}$$

for $\alpha \leq -1$, then $\lambda = \delta$. If

$$\begin{aligned} R^0_\alpha(G) &< 3\delta^\alpha + \delta^{\alpha+1} + (\delta - 1)(n - \delta - 1)^\alpha \\ &\quad +(\delta - 1)(\delta + 1)^\alpha - (\delta - 1)(n - \delta - 2)^\alpha, \end{aligned}$$

for $1 < \alpha \leq 2$, then $\lambda = \delta$.

Proof. Since $N_G(v) = N^+_{D(G)}(v) = N^-_{D(G)}(v)$, for each vertex $v \in V(G) = V(D(G))$, we observe that $n(G) = n(D(G))$, $\delta(G) = \delta(D(G))$ and $R^0_\alpha(G) = R^0_\alpha(D(G))$. Thus Theorem 1 and Observation 1 imply the desired results. $\square$

If $\alpha \le -1$, then Corollary 1 can be found in [11], and the special case $\alpha = -1$ is one of the main results in [2].

Theorem 2. Let D be a strongly connected digraph of order $n \ge 3$, minimum degree δ and edge-connectivity λ, and let $0 < \alpha < 1$ be a real number. If

$$R^0_\alpha(D) > 2\delta^\alpha + (\delta-1)(\delta+1)^\alpha + (\delta-1)(n-\delta-1)^\alpha + (n-2\delta)(n-\delta-2)^\alpha,$$

then $\lambda = \delta$.

Proof. If $\delta = 1$, then $\lambda = \delta$ in every case. Thus assume in the following that $\delta \ge 2$. Suppose to the contrary that $\lambda \le \delta - 1$. Then there exist two disjoint sets $X, Y \subset V(D)$ such that $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$. As we have seen in the proof of Theorem 1, the inequalities $\delta + 1 \le |X|, |Y| \le n - \delta - 1$ are valid.
As in the proof of Theorem 1, we observe that

$$\sum_{x\in X} d(x) \le \sum_{x\in X} d^+(x) \le |X|(|X|-1) + \lambda$$

and

$$\sum_{y\in Y} d(y) \le \sum_{y\in Y} d^-(y) \le |Y|(|Y|-1) + \lambda.$$

Applying Lemma 3, we deduce that

$$\begin{aligned}\sum_{x\in X}(d(x))^\alpha &\le (|X|-\lambda)(|X|-1)^\alpha + \lambda|X|^\alpha \\ &= [(|X|-1) + (1-\lambda)](|X|-1)^\alpha + \lambda|X|^\alpha \\ &= (|X|-1)^{\alpha+1} + (|X|-1)^\alpha + \lambda[|X|^\alpha - (|X|-1)^\alpha],\end{aligned}$$

and

$$\begin{aligned}\sum_{y\in Y}(d(y))^\alpha &\le (|Y|-\lambda)(|Y|-1)^\alpha + \lambda|Y|^\alpha \\ &= [(|Y|-1) + (1-\lambda)](|Y|-1)^\alpha + \lambda|Y|^\alpha \\ &= (|Y|-1)^{\alpha+1} + (|Y|-1)^\alpha + \lambda[|Y|^\alpha - (|Y|-1)^\alpha].\end{aligned}$$

Adding these two inequalities, we obtain

$$\begin{aligned} R^0_\alpha(D) &= \sum_{x\in X}(d(x))^\alpha + \sum_{y\in Y}(d(y))^\alpha \\ &\leq (|X|-1)^{\alpha+1} + (|X|-1)^\alpha + (|Y|-1)^{\alpha+1} + (|Y|-1)^\alpha \\ &\qquad +\lambda[|X|^\alpha - (|X|-1)^\alpha + |Y|^\alpha - (|Y|-1)^\alpha]. \end{aligned} \tag{5}$$

To maximize the right side of the last inequality, we consider the functions $g(t) = t^\alpha - (t-1)^\alpha$ and $h(t) = (t-1)^{\alpha+1} + (t-1)^\alpha$. It is easy to verify that $g''(t) > 0$ and $h''(t) > 0$ for $0 < \alpha < 1$ and $t \geq 2$, and thus g and h are convex. Using $\delta + 1 \leq |X|, |Y| \leq n - \delta - 1$, $|X| + |Y| = n$ and Lemma 4, we obtain

$$|X|^\alpha-(|X|-1)^\alpha+|Y|^\alpha-(|Y|-1)^\alpha \leq (\delta+1)^\alpha-\delta^\alpha+(n-\delta-1)^\alpha-(n-\delta-2)^\alpha, \tag{6}$$

and

$$\begin{aligned} &(|X|-1)^{\alpha+1} + (|X|-1)^\alpha \\ &\quad +(|Y|-1)^{\alpha+1} + (|Y|-1)^\alpha \leq \delta^{\alpha+1} + \delta^\alpha \\ &\qquad\qquad +(n-\delta-2)^{\alpha+1} + (n-\delta-2)^\alpha. \end{aligned} \tag{7}$$

Noting that $\lambda \leq \delta - 1$, the inequalities (5-7) lead to

$$\begin{aligned} R^0_\alpha(D) &\leq \delta^{\alpha+1} + \delta^\alpha + (n-\delta-2)^{\alpha+1} + (n-\delta-2)^\alpha \\ &\quad +(\delta-1)[(\delta+1)^\alpha - \delta^\alpha + (n-\delta-1)^\alpha - (n-\delta-2)^\alpha] \\ &= 2\delta^\alpha + (\delta-1)(\delta+1)^\alpha \\ &\quad +(\delta-1)(n-\delta-1)^\alpha + (n-2\delta)(n-\delta-2)^\alpha, \end{aligned}$$

a contradiction to the hypothesis. Therefore $\lambda = \delta$. □

The next example will demonstrate that Theorem 2 is sharp.

Example 2. Let n and δ be integers such that $n \geq 2\delta + 2 \geq 6$. Furthermore, let $H_1 = K^*_{\delta+1}$ with vertex set $V(H_1) = \{x_1, x_2, \ldots, x_{\delta+1}\}$, and let $H_2 = K^*_{n-\delta-1}$ with vertex set $V(H_2) = \{y_1, y_2, \ldots, y_{n-\delta-1}\}$. Define the digraph H by the union of H_1 and H_2 together with the $2\delta - 2$ arcs $x_1y_1, x_2y_2, \ldots x_{\delta-1}y_{\delta-1}$ as well as $y_1x_1, y_2x_2, \ldots y_{\delta-1}x_{\delta-1}$. Then $n(H) = n$, $\delta(H) = \delta$ and

$$R^0_\alpha(H) = 2\delta^\alpha + (\delta-1)(\delta+1)^\alpha + (\delta-1)(n-\delta-1)^\alpha + (n-2\delta)(n-\delta-2)^\alpha,$$

and therefore equality in the inequality of Theorem 2. However, $\lambda(H) = \delta(H) - 1$.

Using Theorem 2 and Observation 1, we obtain the following sufficient condition for graphs to be maximally edge-connected.

Corollary 2. Let G be a connected graph of order $n \geq 3$, minimum degree δ and edge-connectivity λ, and let $0 < \alpha < 1$ be a real number. If

$$R_\alpha^0(G) > 2\delta^\alpha + (\delta-1)(\delta+1)^\alpha + (\delta-1)(n-\delta-1)^\alpha + (n-2\delta)(n-\delta-2)^\alpha,$$

then $\lambda = \delta$.

A classical result of Chartrand [1] says that $\lambda(G) = \delta(G)$ when $n(G) \leq 2\delta(G)+1$. However, in the remaining case $n(G) \geq 2\delta(G)+2$, Corollary 2 is an improvement of the following result, given by Su, Xiong and Su [10] in 2014.

Theorem 3. *([10])* Let G be a connected graph of order $n \geq 3$, minimum degree δ and edge-connectivity λ, and let $0 < \alpha < 1$ be a real number. If

$$R_\alpha^0(G) > 2\delta^\alpha - \delta^{\alpha+1} + (\delta-1)(\delta+1)^\alpha + (\delta-1)(n-\delta-1)^\alpha + (2n-3\delta-2)(n-\delta-2)^\alpha,$$

then $\lambda = \delta$.

Using the method of the proof of Theorem 2, we will improve Theorem 1 for $-\frac{1}{3} \leq \alpha < 0$ in the interesting case $n(D) \geq 2\delta(D)+2$. Note that the first 7 lines of the proof of Theorem 1 show that $\lambda(D) = \delta(D)$ when $n(D) \leq 2\delta(D)+1$, which was first proved by Geller and Harray [4].

Theorem 4. Let D be a strongly connected digraph of order $n \geq 3$, minimum degree δ and edge-connectivity λ, and let $-\frac{1}{3} \leq \alpha < 0$ be a real number. If

$$R_\alpha^0(D) < 2\delta^\alpha + (\delta-1)(\delta+1)^\alpha + (\delta-1)(n-\delta-1)^\alpha + (n-2\delta)(n-\delta-2)^\alpha,$$

then $\lambda = \delta$.

Proof. If $\delta = 1$, then $\lambda = \delta$ in every case. Thus assume in the following that $\delta \geq 2$. Suppose to the contrary that $\lambda \leq \delta - 1$. Then there exist two disjoint sets $X, Y \subset V(D)$ such that $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$. As we have seen in the proof of Theorem 1, the inequalities $\delta + 1 \leq |X|, |Y| \leq n - \delta - 1$ are valid.

As in the proof of Theorem 1, we observe that

$$\sum_{x\in X} d(x) \le \sum_{x\in X} d^+(x) \le |X|(|X|-1)+\delta-1$$

and

$$\sum_{y\in Y} d(y) \le \sum_{y\in Y} d^-(y) \le |Y|(|Y|-1)+\delta-1.$$

Applying Lemma 2, we deduce that

$$\begin{aligned}\sum_{x\in X}(d(x))^\alpha &\ge (|X|-(\delta-1))(|X|-1)^\alpha+(\delta-1)|X|^\alpha\\ &= (|X|-1)^{\alpha+1}+(|X|-1)^\alpha+(\delta-1)[|X|^\alpha-(|X|-1)^\alpha],\end{aligned}$$

and

$$\begin{aligned}\sum_{y\in Y}(d(y))^\alpha &\ge (|Y|-(\delta-1))(|Y|-1)^\alpha+(\delta-1)|Y|^\alpha\\ &= (|Y|-1)^{\alpha+1}+(|Y|-1)^\alpha+(\delta-1)[|Y|^\alpha-(|Y|-1)^\alpha].\end{aligned}$$

Adding these two inequalities, we obtain

$$\begin{aligned}R_\alpha^0(D) &= \sum_{x\in X}(d(x))^\alpha+\sum_{y\in Y}(d(y))^\alpha\\ &\ge (|X|-1)^{\alpha+1}+(|X|-1)^\alpha+(|Y|-1)^{\alpha+1}+(|Y|-1)^\alpha\\ &\quad +(\delta-1)[|X|^\alpha-(|X|-1)^\alpha+|Y|^\alpha-(|Y|-1)^\alpha]. \qquad (8)\end{aligned}$$

To minimize the right side of the last inequality, we consider the functions $g(t)=t^\alpha-(t-1)^\alpha$ and $h(t)=(t-1)^{\alpha+1}+(t-1)^\alpha$. It is easy to verify that $g''(t)<0$ and $h''(t)\le 0$ for $-\frac{1}{3}\le\alpha<0$ and $t\ge 3$, and thus g and h are concave. Using $3\le\delta+1\le|X|,|Y|\le n-\delta-1$, $|X|+|Y|=n$ and Lemma 5, we obtain

$$\begin{aligned}&|X|^\alpha-(|X|-1)^\alpha+|Y|^\alpha\\ &\qquad -(|Y|-1)^\alpha \ge (\delta+1)^\alpha-\delta^\alpha\\ &\qquad\qquad +(n-\delta-1)^\alpha-(n-\delta-2)^\alpha, \qquad (9)\end{aligned}$$

and

$$\begin{aligned}&(|X|-1)^{\alpha+1}+(|X|-1)^\alpha\\ &\qquad +(|Y|-1)^{\alpha+1}+(|Y|-1)^\alpha \ge \delta^{\alpha+1}+\delta^\alpha \qquad (10)\\ &+(n-\delta-2)^{\alpha+1}+(n-\delta-2)^\alpha.\end{aligned}$$

The inequalities (8-10) lead to

$$\begin{aligned} R^0_\alpha(D) &\geq \delta^{\alpha+1} + \delta^\alpha + (n-\delta-2)^{\alpha+1} + (n-\delta-2)^\alpha \\ &\quad +(\delta-1)[(\delta+1)^\alpha - \delta^\alpha + (n-\delta-1)^\alpha - (n-\delta-2)^\alpha] \\ &= 2\delta^\alpha + (\delta-1)(\delta+1)^\alpha \\ &\quad +(\delta-1)(n-\delta-1)^\alpha + (n-2\delta)(n-\delta-2)^\alpha, \end{aligned}$$

a contradiction to the hypothesis. Therefore $\lambda = \delta$. □

Example 2 also shows the sharpness of Theorem 4. If $\delta \geq 3$, then we can improve Theorem 1 analogously to the proof of Theorem 4 for a greater interval of α when $n(D) \geq 2\delta(D) + 2$. We omit the proof.

Theorem 5. Let D be a strongly connected digraph of order n, minimum degree $\delta \geq 3$ and edge-connectivity λ, and let $-\frac{\delta-1}{\delta+1} \leq \alpha < 0$ be a real number. If

$$R^0_\alpha(D) < 2\delta^\alpha + (\delta-1)(\delta+1)^\alpha + (\delta-1)(n-\delta-1)^\alpha + (n-2\delta)(n-\delta-2)^\alpha,$$

then $\lambda = \delta$.

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